

TECHNICAL NOTE NR. 9

DERIVATION OF THE EULER ANGLE RATE
EQUATIONS USING ROTATION AXES AS A
NON-ORTHOGONAL BASES-WITH TABLES
OF MATRIX EQUATIONS FOR THE
12 ROTATION SEQUENCES

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A B S T R A C T

The Euler angle rates and the Euler angles relating two moving bases are derived in a novel manner in a vector space setting. The relative angular velocity vector is expressed in a non-orthogonal basis (along the rotation axis) with a time-varying metric. The angular velocity vectors are projected onto the dual basis to the non-orthogonal set of base vectors, thus mapping the Gibbs vector representations to transfer matrix equations. Twelve tables of matrix rotations between the base vectors and the Euler angle rates for the six successive and the six repetitive types of rotation sequences are presented as an aid to the design engineer. Examples of application to flight dynamics are shown.

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INTRODUCTION

A distinguishing feature of aero-space vehicle guidance and control problems is the necessity of inter-relating information, the description of which is only available in terms of many different frames of reference. These frames of reference are, in general, rotating relative both to each other and to the reference frame selected for the description of the behaviour of the complete system. Recognition of the need to systematise the process of change of reference frame was one of the reasons which led many engineers to adopt compact matrix representation as a means of handling the equations governing the behaviour of complex dynamical systems. The use of matrix techniques greatly reduces the burden of detailed algebraic manipulation since the process of change of reference frame is reduced to a single matrix operation and repeated change of reference frame is accomplished simply by routine matrix manipulation. In many cases an application of the elegant and powerful theorems of classical matrix analysis enables the general properties of a system to be deduced with a minimum of effort. Furthermore matrix methods allow, in general, a better appreciation of the relationship of various subsystems to the system as a whole.

Fundamental to the synthesis and analysis of flight systems are the flight simulators, a significant portion of which is generally concerned with Euler angle and Euler angle rate computations. In general there are twelve distinct rotation sequences, six of which are repetitive sequences, and six of which are successive sequences. The kinematics of gimballed bodies such as stabilized platforms (on which are mounted gyros, accelerometers, radars, star trackers, or other sensors) are described in terms of gimbal angles and one of the rotation sequences. Thus, depending upon the characteristics of the system under study, one generally has a simulator in which any number of the rotation sequences are being generated simultaneously.

In this report, the transformation matrices relating the inertial angular velocities of two rotating reference frames to the Euler angle rates between the two frames are derived. The derivations differ from the classical approach in that the three rotation axes are considered as a set of non-orthonormal base vectors. As the gimbal angles change the metric for this non-orthonormal space changes, that is the angles between the base vectors are varying.

The connection between two sets of base vectors may be written in matrix form as

$$\begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} = D_{rf} \bar{r} > \quad (1)$$

If the base vectors are not orthonormal, then each set has its corresponding reciprocal set of base vectors, e.g., $\bar{r}^* >$ with the following property:

$$\begin{aligned} \bar{r} > \cdot < \bar{r}^* &= \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} \cdot \begin{pmatrix} \bar{r}_1^* & \bar{r}_2^* & \bar{r}_3^* \end{pmatrix} = \\ & \begin{pmatrix} \bar{r}_1 \cdot \bar{r}_1^* & \bar{r}_1 \cdot \bar{r}_2^* & \bar{r}_1 \cdot \bar{r}_3^* \\ \bar{r}_2 \cdot \bar{r}_1^* & \bar{r}_2 \cdot \bar{r}_2^* & \bar{r}_2 \cdot \bar{r}_3^* \\ \bar{r}_3 \cdot \bar{r}_1^* & \bar{r}_3 \cdot \bar{r}_2^* & \bar{r}_3 \cdot \bar{r}_3^* \end{pmatrix} = I \end{aligned} \quad (2)$$

As will be shown by introducing the scalar dot product (a special inner product) and the reciprocal base vectors (the basis for the dual space) the scalar elements of the matrix of Equation (1) can be evaluated. Operating on Equation (1) with $\cdot < \bar{r}^*$ on the right and using Equation (2) one obtains

$$\bar{r} > \cdot < \bar{r}^* = D_{rf} \quad (3)$$

Matrix methods are extensively used to achieve a change of basis in linear vector space theory. An extension of these methods to a change of bases between time varying bases and a simple systematic method for achieving this is the aim of this paper.

The classical dynamics expression for the angular velocity vectors of two rigid rotating reference frames is

$$\bar{\omega}_r = \bar{\omega}_{rf} + \bar{\omega}_f \quad (4)$$

where $\bar{\omega}_r$ and $\bar{\omega}_f$ are the inertial angular velocity vectors and $\bar{\omega}_{rf}$ is the relative angular velocity of the \bar{r} > basis with respect to the \bar{f} > basis. If \bar{r} > and \bar{f} > are orthonormal bases and \bar{n} > is a non-orthonormal basis with the three unit vectors along the three rotation axes associated with the Euler sequence, then Equation (4) can be written as

$$\langle \bar{r} \omega_r \rangle = \langle \bar{n} \dot{\phi}_{rf} \rangle + \langle \bar{f} \omega_f \rangle , \quad (5)$$

where

$$\bar{\omega}_r = \omega_1 \bar{r}_1 + \omega_2 \bar{r}_2 + \omega_3 \bar{r}_3 \equiv \langle \bar{r} \omega_r \rangle . \quad (6)$$

The above vector equation may be mapped to a matrix of scalar equations by operating with \bar{r} > · , hence

$$\omega_r \rangle = \bar{r} \rangle \cdot \langle \bar{n} \dot{\phi}_{rf} \rangle + \bar{r} \rangle \cdot \langle \bar{f} \omega_f \rangle . \quad (7)$$

The conventional method for solving for the three Euler angle rates is to invert the matrix $\bar{r} \rangle \cdot \langle \bar{n}$ of Equation (7). However, $\dot{\phi}_{rf} \rangle$ can be obtained directly by projecting the vector terms of Equation (5) onto the \bar{n}^* > base vectors, hence

$$\dot{\phi}_{rf} \rangle = \bar{n}^* \rangle \cdot \langle \bar{r} \omega_r \rangle - \bar{n}^* \rangle \cdot \langle \bar{f} \omega_f \rangle . \quad (8)$$

One may solve for $\omega_f \rangle$ in a similar manner, thus

$$\omega_f \rangle = \bar{f} \rangle \cdot \langle \bar{r} \omega_r \rangle - \bar{f} \rangle \cdot \langle \bar{n} \dot{\phi}_{rf} \rangle . \quad (9)$$

In the three equations (7) through (9) there are six square matrices having scalar product elements. Since there are twelve different rotation sequences (Euler sequences) discussed in the paper, one has seventy-two matrices to consider.

The method for deriving the six matrices associated with one rotation sequence is presented in the paper. The six matrices associated with each sequence for all twelve sequences is presented in the tables. These tables are very useful in the design and analysis of aerospace flight systems.

I. TRANSFORMATIONS BETWEEN BASES

Some of the well known properties of transformations between orthonormal bases are derived in this section. The transpose of Equation (1) may be written as

$$\langle \bar{r} = \langle \bar{f} D'_{rf} \quad . \quad (10)$$

The direct product of Equation (1) and Equation (10) yields

$$\bar{r} \rangle \cdot \langle \bar{r} = D_{rf} \bar{f} \rangle \cdot \langle \bar{f} D'_{rf} \quad (11)$$

or

$$M_{rr} = D_{rf} M_{ff} D'_{rf} \quad . \quad (12)$$

If the metric matrix of the two spaces are equal, that is

$$M_{ff} = M_{rr} \quad (13)$$

then M_{rr} is said to be a congruent automorph of M_{ff} , i.e.,

$$M_{ff} = D_{rf} M_{ff} D'_{rf} \quad . \quad (14)$$

For the special case of orthonormal bases

$$M_{ff} = M_{rr} = I = D_{rf} D'_{rf} \quad (15)$$

hence transformation matrices between two orthonormal bases are called orthogonal and their inverses equal their transposes.

When $\bar{f} \rangle$ and $\bar{r} \rangle$ are orthonormal bases the transformation matrix D_{rf} is a matrix of scalar products as shown by Equation (3) because the \bar{f}_i and \bar{f}_i^* are identical (self-reciprocal), hence

$$D_{rf} = \bar{r} \rangle \cdot \langle \bar{f} = \left[\bar{r}_i \cdot \bar{f}_j \right] = \left[\cos (\bar{r}_i, \bar{f}_j) \right] = \left[r_{ij} \right] \quad (16)$$

and the matrix is called the direction cosine matrix.

The matrix of scalar products is designated by M_{rf} to distinguish it from the matrix of field elements D_{rf} . It is only for O.N. bases that they are equal. Using this notation, Equation (16) may be written as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{pmatrix} \quad (17)$$

The transformation matrix orienting one O.N. basis $\bar{r} >$ w.r.t. a second O.N. basis $\bar{f} >$ may be written as

$$\bar{r} > = M_k (\phi_3) M_j (\phi_2) M_i (\phi_1) \bar{f} > = M_{rf} \bar{f} > \quad , \quad (18)$$

or upon transposing

$$\langle \bar{f} = \langle \bar{f} M'_{rf} \quad . \quad (19)$$

The three rotation matrices of Equation (18) are

$$M_1 (\phi_\ell) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{pmatrix} \quad (20)$$

$$M_2 (\phi_\ell) = \begin{pmatrix} c\phi & 0 & -s\phi \\ 0 & 1 & 0 \\ s\phi & 0 & c\phi \end{pmatrix} \quad (21)$$

$$M_3 (\phi_\ell) = \begin{pmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (22)$$

and ϕ_ℓ ($\ell=1, 2, 3$) refers to the first angle of the rotation sequence, the second and the third angle respectively.

It is well known that for any one of the six successive rotation sequences (that is, $i \neq j \neq k$) that for small angles $\Delta\phi_l$

$$\bar{r} > = \begin{pmatrix} 1 & \Delta\phi_3 & -\Delta\phi_2 \\ -\Delta\phi_3 & 1 & \Delta\phi_1 \\ \Delta\phi_2 & -\Delta\phi_1 & 1 \end{pmatrix} \bar{f} > \quad (23)$$

where $\Delta\phi_1$ is the small angular motion about the \bar{r}_1 axis, $\Delta\phi_2$ is the small angular motion about the \bar{r}_2 axis and $\Delta\phi_3$ is the small angular motion about the \bar{r}_3 axis. Equation (23) may be written as

$$\bar{r} > = \left(I + S_{\Delta\phi} \right) \bar{f} > \quad (24)$$

There is an abundance of literature on the rotation sequences and their properties with kinematical applications to the rotating reference frames necessary for missile and satellite design.

One reference frame may be uniquely oriented with respect to a second reference frame through three ordered angles (Euler angles). The Euler angles frequently encountered are of two types:

(1) Successive Euler angles in which none of the rotations are repeated, e.g., $M_3 (\phi_3) M_2 (\phi_2) M_1 (\phi_1)$. The first rotation is about an axis of the one (initial) reference frame, the third of the ordered rotations is about an axis of the other (final) reference frame, and the second rotation is about an axis normal to both the first and third rotation axes.

(2) Repetitive Euler angles as used in classical mechanics in which one of the three angular rotations is repeated and a line of nodes is established, e.g., the three ordered rotations might be first a rotation about the initial \bar{f}_3 unit vector, second a rotation about the new \bar{r}_1 unit vector, and finally a rotation about the new \bar{r}_3 unit vector, i.e., $M_3 (\phi_3) M_1 (\phi_2) M_3 (\phi_1)$. In missile parlance one would say a yaw, roll, yaw rotation sequence.

Example of Successive Euler Angle Transformation Sequence

Consider, initially, that the two O.N. frames $\bar{r} >$ and $\bar{f} >$ are aligned, that is,

$$\bar{r}_0 > = \bar{f} > \quad (25)$$

The $\bar{r} >$ base vectors after the first rotation about (as an example) the \bar{f}_1 axis are given by (linear dependence property)

$$\begin{pmatrix} \bar{n}_1 = \bar{r}_{11} \\ \bar{r}_{21} \\ \bar{r}_{31} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{pmatrix} . \quad (26)$$

The vector \bar{n}_1 is one of the vectors of the set $\bar{n} >$ and are of unit magnitude but not an orthonormal set. They are unit vectors about the rotation axes as shown. They will be defined differently for each Euler sequence.

The elements a_{ij} are the scalar coordinates of the three position vectors \bar{r}_{11} , \bar{r}_{21} , \bar{r}_{31} in the $\bar{f} >$ space as shown in Figure (1).

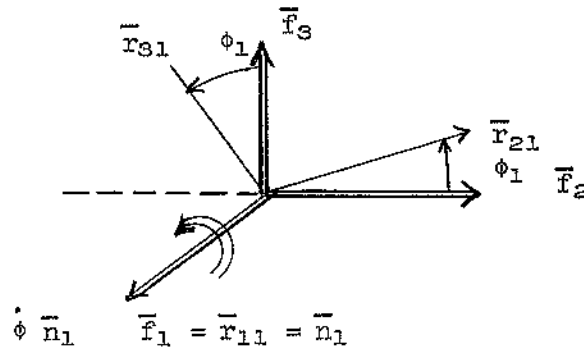


Figure 1

Plane in which Rotation Angle ϕ_2 is Measured
for Rotation about \bar{f}_1 for sequence $M_{321} = M_{rb}$

The vectors \bar{r}_{31} and \bar{r}_{21} lie in the \bar{f}_2, \bar{f}_3 plane and the angle ϕ , is measured in that plane. All angles are positive as conventionally associated with right-handed reference frames.

Operating on equation (26) with $\cdot < \bar{f}$, one obtains

$$\bar{f}_1 > \cdot < \bar{f} = \begin{pmatrix} \bar{r}_{11} \cdot \bar{f}_1 & \bar{r}_{11} \cdot \bar{f}_2 & \bar{r}_{11} \cdot \bar{f}_3 \\ \bar{r}_{21} \cdot \bar{f}_1 & \bar{r}_{21} \cdot \bar{f}_2 & \bar{r}_{21} \cdot \bar{f}_3 \\ \bar{r}_{31} \cdot \bar{f}_1 & \bar{r}_{31} \cdot \bar{f}_2 & \bar{r}_{31} \cdot \bar{f}_3 \end{pmatrix} = A \quad (27)$$

It follows from figure (1) and equation (26) that

$$\bar{r}_1 > = \begin{pmatrix} \bar{r}_{11} = \bar{n}_1 \\ \bar{r}_{21} = \bar{n}_2 \\ \bar{r}_{31} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi_1 & s\phi_1 \\ 0 & -s\phi_1 & c\phi_1 \end{pmatrix} \begin{pmatrix} \bar{f}_1 = \bar{n}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{pmatrix}. \quad (28)$$

If the second rotation is about the number two base vector after the first rotation \bar{r}_{21} , then by similar analysis one obtains

$$\bar{r}_2 > = \begin{pmatrix} \bar{r}_{12} \\ \bar{r}_{22} = \bar{n}_2 \\ \bar{r}_{32} = \bar{n}_3 \end{pmatrix} = \begin{pmatrix} c\phi_2 & 0 & -s\phi_2 \\ 0 & 1 & 0 \\ s\phi_2 & 0 & c\phi_2 \end{pmatrix} \begin{pmatrix} \bar{r}_{11} = \bar{n}_1 \\ \bar{r}_{21} = \bar{n}_2 \\ \bar{r}_{31} \end{pmatrix} \quad (29)$$

where ϕ_2 is shown in figure (2)

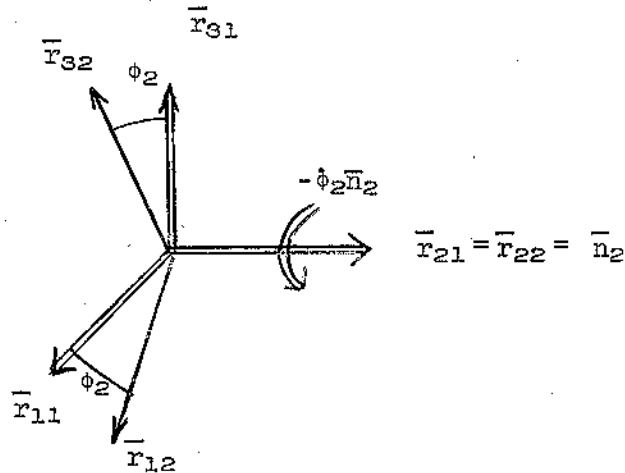


Figure 2

Plane in which Rotation Angle ϕ_2 is
 Measured for Rotation about r_2 , and
 Sequence $M_{321} = M_{rb}$

If the third rotation is about the number three base vector after the second rotation \bar{r}_{32} , then by figure (3)

$$\bar{r}_{32} = \bar{r}_{33} = \bar{n}_3 \quad (30)$$

$$\bar{r}_3 > = \begin{pmatrix} \bar{r}_{13} \\ \bar{r}_{23} \\ \bar{r}_{33} = \bar{n}_3 \end{pmatrix} = \begin{pmatrix} c\phi_3 & s\phi_3 & 0 \\ -s\phi_3 & c\phi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{r}_{12} \\ \bar{r}_{22} = \bar{n}_2 \\ \bar{r}_{32} = \bar{n}_3 \end{pmatrix} \quad (31)$$

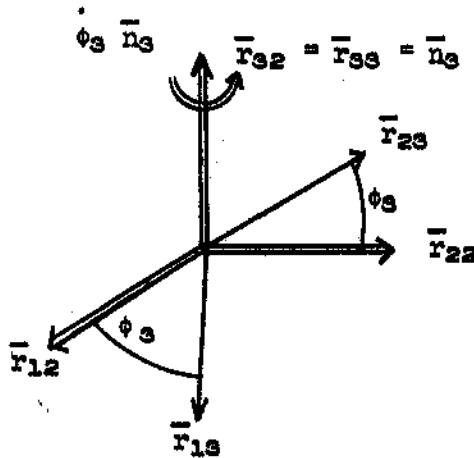


Figure 3

Plane in which ϕ_3 is Measured for
Rotation about \bar{r}_{32} and sequence
 $M_{321} = M_{rb}$

Defining the $\bar{r}_3 >$ bases after the third rotation as the $\bar{r} >$ bases,
that is

$$\bar{r}_3 > = \bar{r} > \quad (32)$$

one obtains the resultant of the three rotations

$$\bar{r} > = M_3 (\phi_3) M_2 (\phi_2) M_1 (\phi_1) \bar{r} > \quad (33)$$

The product of the three matrices of equation (33) for the sequence of Equation (33) is

$$\begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} = \begin{bmatrix} c\phi_2 c\phi_3 & c\phi_3 s\phi_2 s\phi_1 + s\phi_3 c\phi_1 & -s\phi_2 c\phi_1 c\phi_3 + s\phi_3 s\phi_1 \\ -s\phi_3 c\phi_2 & -s\phi_3 s\phi_2 s\phi_1 + c\phi_3 c\phi_1 & s\phi_2 s\phi_3 c\phi_1 + c\phi_3 s\phi_1 \\ s\phi_2 & -s\phi_1 c\phi_2 & c\phi_2 c\phi_1 \end{bmatrix} \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{pmatrix} \quad (34)$$

The matrix orienting the $\bar{n} >$ basis (unit vectors along the non-orthogonal rotation axes) with respect to the $\bar{f} >$ basis may be obtained from the geometry of the figures and defining matrix equations. By Equation (27)

$$\bar{n}_1 = \bar{f}_1 + 0\bar{f}_2 + 0\bar{f}_3 \quad (35)$$

By Equation (29) and Equation (31)

$$\bar{n}_2 = \bar{r}_{21} = \bar{r}_{22} = c\phi_1 \bar{f}_2 + s\phi_1 \bar{f}_3 \quad (36)$$

By equations (31) and (34)

$$\bar{n}_3 = \bar{r}_3 = s\phi_2 \bar{f}_1 - s\phi_1 c\phi_2 \bar{f}_2 + c\phi_2 c\phi_1 \bar{f}_3 \quad (37)$$

In matrix form Equation (35), (36) and Equation (37) become

$$\begin{pmatrix} \bar{n}_1 \\ \bar{n}_2 \\ \bar{n}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi_1 & s\phi_1 \\ s\phi_2 & -s\phi_1 c\phi_2 & c\phi_2 c\phi_1 \end{pmatrix} \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{pmatrix} \quad (38)$$

The total number of the successive type Euler angle transformation matrices is given by the permutation of the three matrices M_1, M_2, M_3 , taken three at a time (no repetitions) or six. Designating the resultant direction cosine matrix of equation (18) as M_{rf} , one may write for the six successive type rotation sequences

$$M_{rf} = M_{123} = M_{132} = M_{231} = M_{213} = M_{321} = M_{312} \quad (39)$$

where

$$M_{rf} = M_{kji} = M_k(\phi_3) M_j(\phi_2) M_i(\phi_1) \quad (i \neq j \neq k) . \quad (40)$$

The angle ϕ_1 is always measured in the plane normal to the first rotation axis \bar{n}_1 , and is not the angle measured about the \bar{f}_1 axis except for the cases in which the \bar{f}_1 axis is the \bar{n}_1 axis. Similar remarks hold for the meaning of ϕ_2 and ϕ_3 . The motivation for this convention arises from the desire to express the components of the relative angular velocity as a triple of numbers in the $\bar{n} >$ basis, i.e.,

$$\bar{\omega}_{rf} = \langle \dot{\phi}_{rf} \bar{n} \rangle = \dot{\phi}_1 \bar{n}_1 + \dot{\phi}_2 \bar{n}_2 + \dot{\phi}_3 \bar{n}_3 . \quad (41)$$

The following convention is adopted. The sequence of equation (40) will be designated as (i, j, k) meaning the first rotation is through an i-type matrix, similarly for j and k. The corresponding matrix M_{kji} uses the reverse ordering on the subscripts to conform with the order of the matrix products.

There are also six repetitive-type Euler sequences given as

$$M_{rf} = M_{121} = M_{131} = M_{212} = M_{232} = M_{313} = M_{323} \quad (42)$$

where

$$M_{rf} = M_{iji} = M_i(\phi_3) M_j(\phi_2) M_i(\phi_1) \quad (43)$$

The twelve rotation sequences given by equations (39) and (42) have twelve different associated non-orthogonal reference frames to be designated by $\bar{n} >$. The $\bar{n} >$ base vectors are unit magnitude vectors lying along the rotation axes. Each different sequence has a different set of rotation axes. For the sake of simplicity of notation, only one successive type sequence is used as an example. No distinction will be made between the twelve $\bar{n} >$ bases from the point of view of notation but it should be remembered that they are different axes. The \bar{n}_1 vector is defined to be the axis about which the first rotation occurs, the \bar{n}_2 vector about the second, etc.

The matrix orienting the $\bar{n} >$ basis with respect to the $\bar{r} >$ basis may be obtained in a similar manner.

By equation (34) and (35)

$$\bar{n}_1 = \bar{f}_1 = c\phi_2 c\phi_3 \bar{r}_1 - s\phi_3 c\phi_2 \bar{r}_2 + s\phi_2 \bar{r}_3 , \quad (44)$$

by equation (31), since $\bar{r}_{13} = \bar{r}_1$ and $\bar{r}_{23} = \bar{r}_2$

$$\bar{n}_2 = s\phi_3\bar{r}_1 + c\phi_3\bar{r}_2, \quad (45)$$

and by equation (37)

$$\bar{n}_3 = \bar{r}_3. \quad (46)$$

Writing equations (44), (45) and (46) in matrix form

$$\begin{pmatrix} \bar{n}_1 \\ \bar{n}_2 \\ \bar{n}_3 \end{pmatrix} = \begin{pmatrix} c\phi_2c\phi_3 & -s\phi_3c\phi_2 & s\phi_2 \\ s\phi_3 & c\phi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix}, \quad (47)$$

or

$$\bar{n} > = D_{nr} \bar{r} >. \quad (48)$$

II. TRANSFORMATIONS FROM DIFFERENTIAL EQUATIONS IN THE DIRECTION COSINES

The direction cosine matrix is usually needed to transform information from one O.N. reference frame to another in physical systems and in system simulations. The elements of the direction cosine matrix may be obtained from a solution of a system of first order differential equations in the direction cosines. Mechanization of such a scheme in a vehicle-borne computer requires integrations and initial conditions and instruments for sensing the components of the inertial angular velocity vector of the two reference frames.

The direction cosine matrix orienting the $\bar{r} >$ frame w.r.t. the $\bar{f} >$ frame is

$$\bar{r} > = M_{rf} \bar{f} > \quad (49)$$

and

$$\bar{f} > = M_{rf}^{-1} \bar{r} >. \quad (50)$$

Taking the time derivative of equation (1) as observed by an inertial observer

$$\dot{\bar{r}}^i > = M_{rf}^i \dot{\bar{r}}^f > + M_{rf}^i \dot{\bar{r}}^f > \quad (51)$$

Since $\bar{r}^i >$ is a basis, the three vectors $\dot{\bar{r}}^i >$ may be expressed as

$$\dot{\bar{r}}^i > = M_i^j \bar{r}^j > \quad (52)$$

To evaluate the elements of the matrix of Equation (52), consider the orientation by the $\bar{r}^i >$ vectors at t plus Δt with respect to the $\bar{r}^i >$ vectors at time t , then by Equation (23), one obtains

$$\bar{b}^i (t + \Delta t) > = (I + S_{\Delta\phi}) \bar{b}^i (t) > \quad (53)$$

Passing to the limit on Δt , Equation (53) yields

$$\begin{pmatrix} \dot{\bar{b}}^1 \\ \dot{\bar{b}}^2 \\ \dot{\bar{b}}^3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \begin{pmatrix} \bar{b}^1 \\ \bar{b}^2 \\ \bar{b}^3 \end{pmatrix} \quad (54)$$

where

$$\omega_i = \lim_{\Delta t \rightarrow 0} \left(\frac{\Delta\phi_i}{\Delta t} \right), \quad (i = 1, 2, 3). \quad (55)$$

Equation (54) may be written as

$$\dot{\bar{r}}^i > = S_{\omega r}^i \bar{r}^i > \quad (56)$$

and similarly for the $\bar{r}^i >$ basis

$$\dot{\bar{r}}^i > = S_{\omega r}^i \bar{r}^i > \quad (57)$$

Using Equation (56) and Equation (57) in Equation (51) one obtains

$$S_{\omega r}^i \bar{r}^i > = M_{rf}^i \bar{r}^f > + M_{rf}^i S_{\omega r}^f \bar{r}^f > \quad (58)$$

Operating on Equation (58) with $\cdot \langle \bar{r}$

$$\dot{M}_{rf} = S_{\omega r} M_{rf} - M_{rf} S_{\omega f} \quad (59)$$

or in open form

$$\begin{pmatrix} \dot{r}_{11} & \dot{r}_{12} & \dot{r}_{13} \\ \dot{r}_{21} & \dot{r}_{22} & \dot{r}_{23} \\ \dot{r}_{31} & \dot{r}_{32} & \dot{r}_{33} \end{pmatrix} = \begin{pmatrix} 0 & \omega_{3r} & -\omega_{2r} \\ -\omega_{3r} & 0 & \omega_{1r} \\ \omega_{2r} & -\omega_{1r} & 0 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \\ - \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} 0 & \omega_{3f} & -\omega_{2f} \\ -\omega_{3f} & 0 & \omega_{1f} \\ \omega_{2f} & -\omega_{1f} & 0 \end{pmatrix} \quad (60)$$

Operating on Equation (58) with $\cdot \langle \bar{r}$ one obtains

$$S_{\omega r} = \dot{M}_{rf} M_{fr} + M_{rf} S_{\omega f} M_{rf}' \quad (61)$$

III. EULER ANGLE RATE EQUATIONS

This section develops the relationships between the inertial angular velocities of two rotating O.N. reference frames and the time rates of the angles orienting the two frames. The derivations are based on the non-orthogonal rotation axes and the reciprocal bases.

The angular velocity of the $\bar{r} >$ frame with respect to an inertial frame may be expressed as the angular velocity of the $\bar{r} >$ frame w.r.t. an inertial frame, i.e.,

$$\bar{\omega}_r = \bar{\omega}_{rf} + \bar{\omega}_f \quad (62)$$

The two inertial angular velocities of equation (62) may be expressed in their respective frames as

$$\bar{\omega}_r = \langle \omega_r \bar{r} \rangle = \langle \bar{r} \omega_r \rangle \quad (63)$$

and

$$\bar{\omega}_f = \langle \omega_f \bar{f} \rangle = \langle \bar{f} \omega_f \rangle \quad (64)$$

The relative angular velocity $\bar{\omega}_{rf}$ may be expressed in a non-orthogonal reference frame $\bar{n} >$ as

$$\bar{\omega}_{rf} = \langle \dot{\phi}_{rf} \bar{n} \rangle = \langle \dot{\bar{n}}_{rf} \rangle, \quad (65)$$

where the unit magnitude vectors $\bar{n} >$ are taken along the rotation axes. The set $\bar{n} >$ is different for each Euler sequence. Consider the example of section I, where the ordered sequence corresponds to a first rotation about the $\bar{f}_1 = \bar{n}_1$ axis, a second rotation about the $\bar{r}_{21} = \bar{n}_2$ (\bar{r}_2 after the first rotation) and a third rotation about the \bar{r}_3 axis.

The three components of the relative angular velocity vector $\bar{\omega}_{rf}$ about the rotation axes $\bar{n} >$ are shown in Figures (1), (2), and (3).

The vector-matrix expression for equation (65) is

$$\langle \omega_r \bar{r} \rangle = \langle \dot{\phi}_{rf} \bar{n} \rangle + \langle \omega_f \bar{f} \rangle. \quad (66)$$

Operating on equation (66) with $\langle \bar{n}^* \rangle$, yields

$$\langle \dot{\phi}_{rf} \rangle = \langle \omega_r \bar{r} \rangle \cdot \langle \bar{n}^* \rangle - \langle \omega_f \bar{f} \rangle \cdot \langle \bar{n}^* \rangle, \quad (67)$$

$$\langle \dot{\phi}_{rf} \rangle = \langle \omega_r M_{rn^*} \rangle - \langle \omega_f M_{fn^*} \rangle. \quad (68)$$

The matrix M_{fn^*} of equation (68) is the inverse of the transformation matrix of equation (38) orienting the $\bar{n} >$ basis w.r.t. the $\bar{f} >$ basis, i.e.,

$$\bar{n} > = D_{nf} \bar{f} >, \quad (69)$$

hence

$$I = D_{nf} \bar{f} > \cdot \langle \bar{n}^* \rangle \quad (70)$$

or

$$D_{nf}^{-1} = \bar{f} > \cdot \langle \bar{n}^* \rangle = M_{fn^*} \quad (71)$$

The inverse matrix D_{nf}^{-1} of equation (71) is, by equation (a-20)

$$D_{nf}^{-1} = \frac{k_v^f}{k_v^n} \text{adj } D_{nf}, \quad (72)$$

but, since the \bar{f} basis is ortho-normal,

$$k_v^f = 1 \quad (73)$$

and by equation (a-14) it can be shown that

$$k_v^n = \cos \phi_2, \quad (74)$$

hence by equation (a-21) for the adjoint matrix

$$D_{nf}^{-1} = \frac{1}{\cos \phi_2} \begin{pmatrix} \cos \phi_2 & 0 & 0 \\ s\phi_1 s\phi_2 & \cos \phi_2 \cos \phi_1 & -s\phi_1 \\ -c\phi_1 s\phi_2 & s\phi_1 \cos \phi_2 & \cos \phi_1 \end{pmatrix} = M_{fn}^* \quad (75)$$

In a similar manner the matrix M_{rn}^* of equation (68) is

$$D_{nr}^{-1} = M_{rn}^* \frac{1}{\cos \phi_2} \begin{bmatrix} \cos \phi_3 & s\phi_3 \cos \phi_2 & -c\phi_3 s\phi_2 \\ -s\phi_3 & \cos \phi_2 \cos \phi_3 & s\phi_2 s\phi_3 \\ 0 & 0 & 0 \end{bmatrix} \quad (76)$$

Utilizing equations (75) and (76) in equation (68), one obtains the matrix equation relating the Euler angle rates and the angular velocities of the r > and f > frames w.r.t. inertial space.

$$\begin{pmatrix} \dot{\phi}_{1rf} \\ \dot{\phi}_{2rf} \\ \dot{\phi}_{3rf} \end{pmatrix} = \frac{1}{c\phi_2} \left\{ \begin{pmatrix} c\phi_3 & -s\phi_3 & 0 \\ s\phi_3 c\phi_2 & c\phi_2 c\phi_3 & 0 \\ -c\phi_3 s\phi_2 & s\phi_2 s\phi_3 & c\phi_2 \end{pmatrix} \begin{pmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{pmatrix} - \begin{pmatrix} c\phi_2 & s\phi_1 s\phi_2 & -c\phi_1 s\phi_2 \\ 0 & c\phi_2 c\phi_1 & s\phi_1 c\phi_2 \\ 0 & -s\phi_1 & c\phi_1 \end{pmatrix} \begin{pmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{pmatrix} \right\} \quad (77)$$

Equation (66) can also be solved for $\langle \omega_r \rangle$ by operating with $\langle \bar{r} \rangle$, noting that $\langle \bar{r} \rangle$ is ortho-normal, i.e.,

$$\langle \omega_r \rangle = \langle \dot{\phi}_{rf} \bar{n} \rangle \cdot \langle \bar{r} \rangle + \langle \omega_f \bar{f} \rangle \cdot \langle \bar{r} \rangle \quad (78)$$

or as a column matrix in the form

$$\omega_r \rangle = \bar{r} \rangle \cdot \langle \bar{n} \dot{\phi}_{rf} \rangle + \bar{r} \rangle \cdot \langle \bar{f} \omega_f \rangle \quad (79)$$

The matrix $\langle \bar{n} \rangle \cdot \langle \bar{r} \rangle$ is the matrix of equation (47) and the matrix $\bar{r} \rangle \cdot \langle \bar{f} \rangle$ is the matrix of equation (38). Substituting these into equation (79)

$$\begin{pmatrix} \omega_{1r} \\ \omega_{2r} \\ \omega_{3r} \end{pmatrix} = \frac{1}{c\phi_2} \begin{pmatrix} c\phi_2 c\phi_3 & s\phi_3 & 0 \\ -s\phi_3 c\phi_2 & c\phi_3 & 0 \\ s\phi_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\phi}_{1rf} \\ \dot{\phi}_{2rf} \\ \dot{\phi}_{3rf} \end{pmatrix} - D_{rf} \omega_f \rangle \quad (80)$$

The singularity occurring when the second angle becomes equal to 90° for the successive sequences is again observed in equations (77) and (80). The angular rates corresponding to equations (77) and (80) are given in Tables (1) through (12) for the twelve rotation sequences.

In system simulation models or in system design it is quite often required to solve equation (77) for the Euler rates assuming that the two sets of inertial angular velocities $\omega_r \rangle$ and $\omega_f \rangle$ are known--measured by gyros or other sensors or computed from the simulated dynamics. Equation (80) is

useful in problems in which one set of the inertial angular rates are measured or computed and the relative-angular rates about the rotation axes are measured with tachometers or other instrumentation.

It is of interest to consider the description of the reciprocal base vectors \bar{n}^* in the \bar{r} and \bar{f} bases respectively. By the linear dependence property

$$\bar{n}^* = D_{n^*f} \bar{f} \quad (81)$$

and

$$\bar{n}^* \cdot \bar{f} = D_{n^*f} = (\bar{f} \cdot \bar{n}^*) \quad (82)$$

By equation (71)

$$D_{n^*f} = (D_{nf}^{-1})' \quad (83)$$

Transposing equation (75) and substituting into equation (81), one obtains

$$\begin{pmatrix} \bar{n}_1^* \\ \bar{n}_2^* \\ \bar{n}_3^* \end{pmatrix} = \begin{pmatrix} c\phi_2 & s\phi_1 s\phi_2 & -c\phi_1 s\phi_2 \\ 0 & c\phi_2 c\phi_1 & s\phi_1 c\phi_2 \\ 0 & -s\phi_1 & c\phi_1 \end{pmatrix} \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{pmatrix} \quad (84)$$

In a similar manner

$$\begin{pmatrix} \bar{n}_1^* \\ \bar{n}_2^* \\ \bar{n}_3^* \end{pmatrix} = \begin{pmatrix} c\phi_3 & -s\phi_3 & -c\phi_3 s\phi_2 \\ s\phi_3 c\phi_2 & c\phi_2 c\phi_3 & s\phi_2 s\phi_3 \\ -c\phi_3 s\phi_2 & 0 & c\phi_2 \end{pmatrix} \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} \quad (85)$$

The well-known vector equation relating the angular velocities of equation (62), i.e.,

$$\bar{\omega}_{rf} = \bar{\omega}_r - \bar{\omega}_f, \quad (86)$$

may be succinctly described for the twelve sequences as

$$\bar{\omega}_{rf} = \dot{\phi}_1 \bar{f}_i + \dot{\phi}_2 \bar{r}_{j1} + \dot{\phi}_3 \bar{r}_k = \langle \dot{\phi} \bar{n} \rangle \quad (87)$$

where i, j, k are

successive: $i = 1, 2, 3, i \neq j \neq k$ (88)

repetitive: $i = 1, 2, 3, i = k \neq j$ (89)

The relative angular velocity vector for the six successive sequences is:

$$\begin{aligned}
 (1, 2, 3) \quad \bar{\omega}_{r/f} &= \dot{\phi}_1 \bar{r}_1 + \dot{\phi}_2 \bar{r}_{21} + \dot{\phi}_3 \bar{r}_3 \\
 (1, 3, 2) &= \dot{\phi}_1 \bar{r}_1 + \dot{\phi}_2 \bar{r}_{31} + \dot{\phi}_3 \bar{r}_2 \\
 (2, 1, 3) &= \dot{\phi}_1 \bar{r}_2 + \dot{\phi}_2 \bar{r}_{11} + \dot{\phi}_3 \bar{r}_3 \\
 (2, 3, 1) &= \dot{\phi}_1 \bar{r}_2 + \dot{\phi}_2 \bar{r}_{31} + \dot{\phi}_3 \bar{r}_1 \\
 (3, 1, 2) &= \dot{\phi}_1 \bar{r}_3 + \dot{\phi}_2 \bar{r}_{11} + \dot{\phi}_3 \bar{r}_2 \\
 (3, 2, 1) &= \dot{\phi}_1 \bar{r}_3 + \dot{\phi}_2 \bar{r}_{21} + \dot{\phi}_3 \bar{r}_1
 \end{aligned} \tag{90}$$

The relative angular velocity vector for the six repetitive sequence is:

$$\begin{aligned}
 (1, 2, 1): \quad \bar{\omega}_{rf} &= \dot{\phi}_1 \bar{r}_1 + \dot{\phi}_2 \bar{r}_{21} + \dot{\phi}_3 \bar{r}_1 \\
 (1, 3, 1): &= \dot{\phi}_1 \bar{r}_1 + \dot{\phi}_2 \bar{r}_{31} + \dot{\phi}_3 \bar{r}_1 \\
 (2, 1, 2): &= \dot{\phi}_1 \bar{r}_2 + \dot{\phi}_2 \bar{r}_{11} + \dot{\phi}_3 \bar{r}_2 \\
 (2, 3, 2): &= \dot{\phi}_1 \bar{r}_2 + \dot{\phi}_2 \bar{r}_{31} + \dot{\phi}_3 \bar{r}_2 \\
 (3, 1, 3): &= \dot{\phi}_1 \bar{r}_3 + \dot{\phi}_2 \bar{r}_{11} + \dot{\phi}_3 \bar{r}_3 \\
 (3, 2, 3): &= \dot{\phi}_1 \bar{r}_3 + \dot{\phi}_2 \bar{r}_{21} + \dot{\phi}_3 \bar{r}_3
 \end{aligned} \tag{91}$$

IV. EXAMPLE OF APPLICATION

A brief example of the application of the vector-matrix methods of this paper to six degrees of freedom rigid body flight dynamics is presented below.

The velocity vector of a flight vehicle expressed in body axes $\bar{b} >$ may be written as

$$\bar{v} = \langle u^{\bar{b}} \rangle \tag{92}$$

and the time derivatives with respect to inertial space is

$$\dot{\bar{v}} = \langle \dot{u}^{\bar{b}} \rangle - \langle u^{\dot{\bar{b}}} \rangle \tag{93}$$

The three translational degrees of freedom of the rigid body are given by

$$m\dot{\bar{V}} = \bar{F} \quad (94)$$

Utilizing Equation (54) and Equation (93) in Equation (94) and transposing that result one obtains

$$\dot{\bar{V}} = \langle \bar{b} \left\{ \dot{u} \right\rangle - S_{\omega_b} u \rangle = \frac{\langle \bar{b} f \rangle}{m} \quad (95)$$

In a similar manner the system angular momentum may be written in body axes $\bar{b} >$ as

$$\bar{H} = \langle \bar{b} h \rangle \quad (96)$$

The rotational dynamics expressed in the $\bar{b} >$ basis is

$$\dot{\bar{H}} = \langle \bar{b} \left\{ \dot{h} \right\rangle - S_{\omega_b} h \rangle = \langle \bar{b} T \rangle \quad (97)$$

Equation (117) may also be written as

$$\dot{u}_b > = I_b^{-1} S_{\omega_b} I_b u_b > + I_b^{-1} T > \quad (98)$$

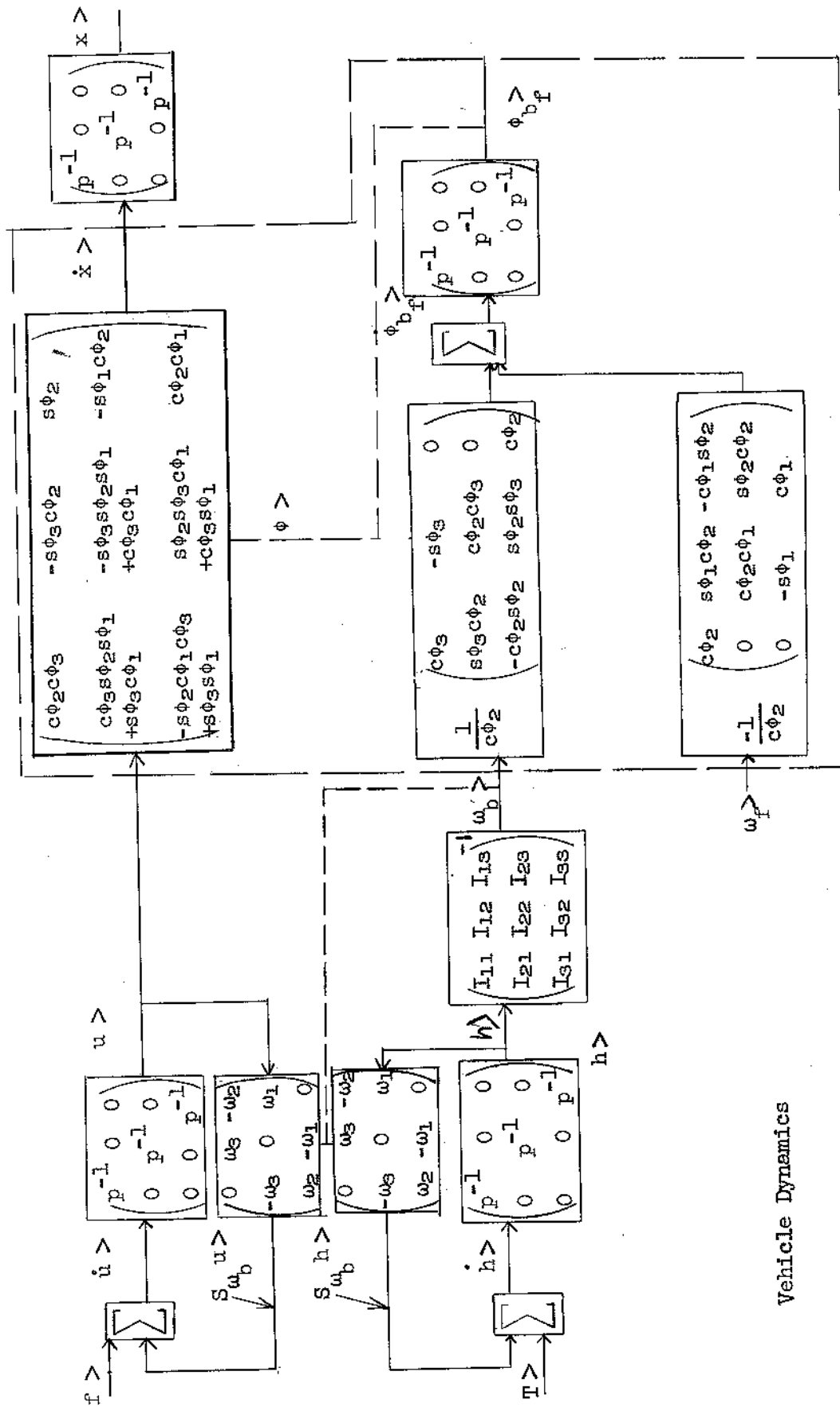
Suppose one desires to obtain the angular orientation of the body axes with respect to some arbitrary rotating frame say $\bar{F} >$. In particular, $\bar{F} >$ may be stable platform axes and the Euler angles and their rates correspond to gimbal angles, or $\bar{F} >$ may be some ground fixed reference frame rotating with the earth, etc.

The Euler angle rate equations for the sequence (1, 2, 3) are given by Equation (77) as

$$\dot{\phi}_{bf} > = \frac{1}{c\phi_2} \begin{pmatrix} c\phi_3 & -s\phi_3 & 0 \\ s\phi_3 c\phi_2 & c\phi_2 c\phi_3 & 0 \\ -c\phi_3 s\phi_2 & s\phi_2 s\phi_3 & c\phi_2 \end{pmatrix} \begin{pmatrix} \omega_{b1} \\ \omega_{b2} \\ \omega_{b3} \end{pmatrix} - \frac{1}{c\phi_2} \begin{pmatrix} c\phi_2 & s\phi_1 s\phi_2 & -c\phi_1 s\phi_2 \\ 0 & c\phi_2 c\phi_1 & s\phi_2 c\phi_2 \\ 0 & -s\phi_1 & c\phi_1 \end{pmatrix} \begin{pmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{pmatrix} \quad (99)$$

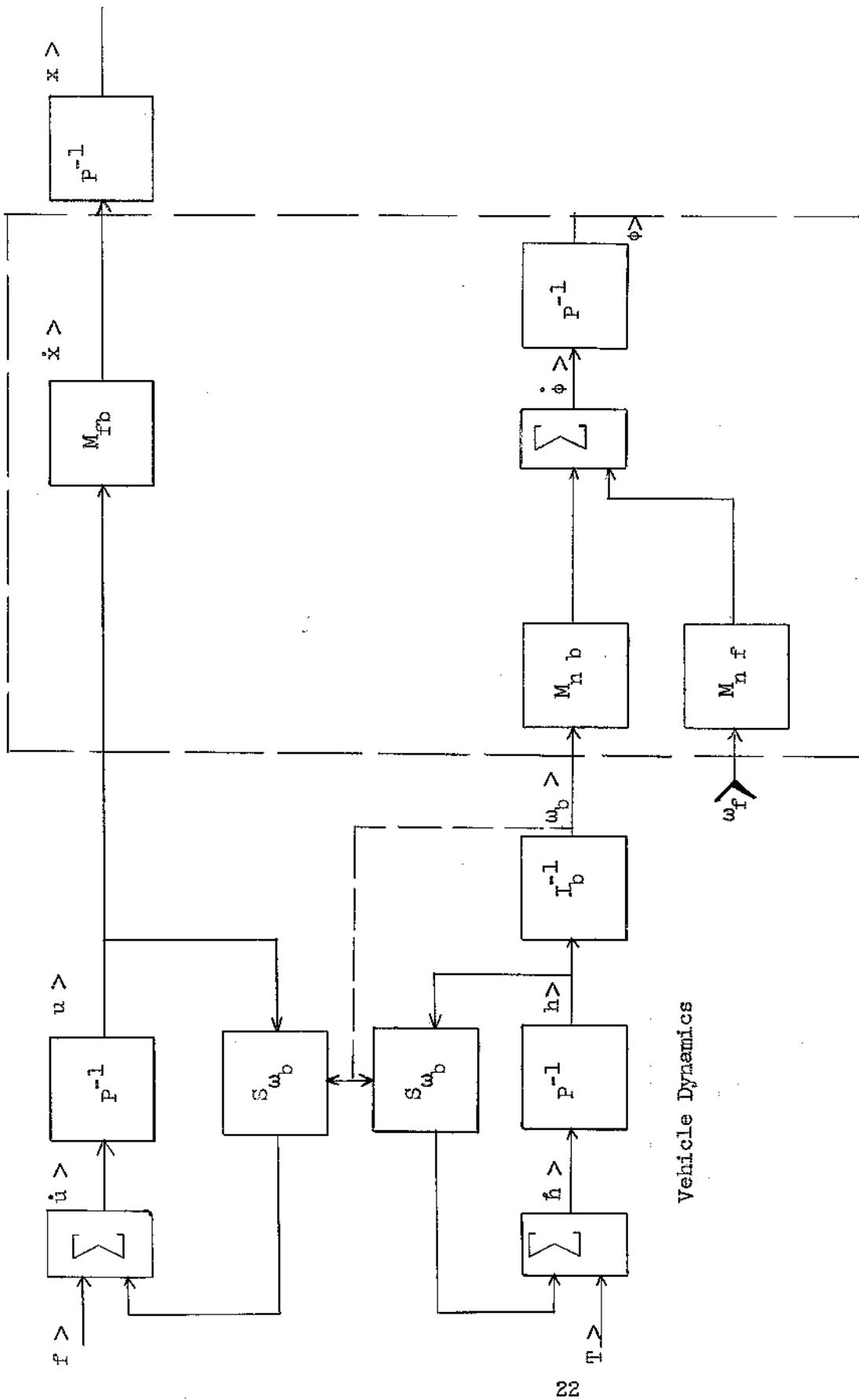
The system block diagram for the sequence of Equation (119) is given in Figure (4).

In general, for any one of the twelve sequences, one has merely to select the sequence from tables (1) through (12) and utilize for the general Euler angle block of Figure (5).



Euler Angle Block

Figure 4
 Non-Linear Transfer Matrix Block Diagram of Six Degree of Freedom Equations of Motion



Euler Angle Block

Figure 5

Non-Linear Transfer Matrix Block Diagram of Six Degree of Freedom Equations of Motion

The three first order matrix equations Equation (95), Equation (98) and Equation (119) may be written in state-vector form as

$$\dot{X} > = M (u >, \omega_b >, \phi_{bf} >, \omega_r >, f >, T >) \quad (100)$$

where $X \equiv$

$$\begin{pmatrix} \dot{u} > \\ \dot{\omega}_b > \\ \dot{\phi}_{bf} > \end{pmatrix} \quad (101)$$

Clearly the above equations are non-linear.

The elegance of these vector-matrix methods can be contrasted with the conventional form for expressing the rotational dynamics of a syngle rigid body. The conventional form is

$$\begin{aligned} I_{11}\dot{\omega}_1 + I_{21}\dot{\omega}_2 + I_{13}\dot{\omega}_3 + \omega_2\omega_3 (I_{33} - I_{22}) + \omega_1\omega_2 I_{31} \\ - \omega_1\omega_3 I_{21} + I_{23} (\omega_2^2 - \omega_3^2) = t_1 \end{aligned}$$

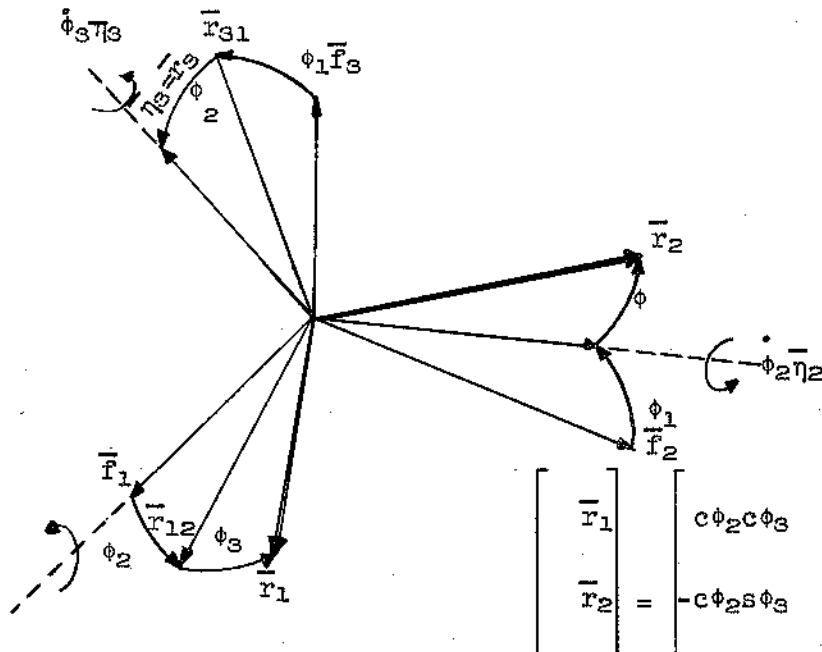
$$\begin{aligned} I_{22}\dot{\omega}_2 + I_{21}\dot{\omega}_1 + I_{23}\dot{\omega}_3 + \omega_1\omega_3 (I_{11} - I_{33}) \\ + \omega_2\omega_3 I_{12} - \omega_2\omega_1 I_{32} + I_{13} (\omega_3^2 - \omega_1^2) = t_2 \end{aligned}$$

$$\begin{aligned} I_{33}\dot{\omega}_3 + I_{31}\dot{\omega}_1 + I_{32}\dot{\omega}_2 + \omega_1\omega_2 (I_{22} - I_{11}) + \omega_3\omega_1 I_{23} \\ - \omega_3\omega_2 I_{13} + I_{21} (\omega_1^2 - \omega_2^2) = t_3 \end{aligned}$$

The vector matrix form is

$$\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} - \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} \quad (102)$$

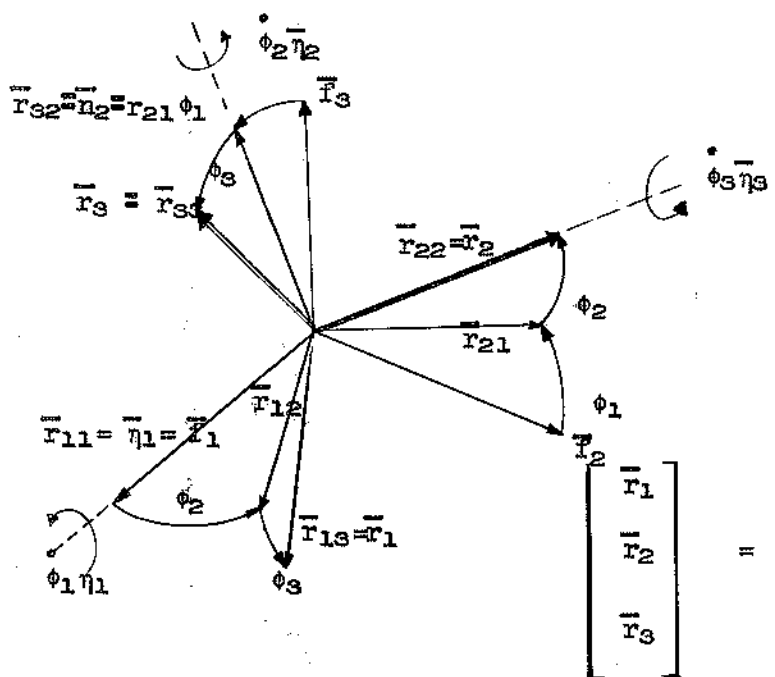
The latter matrix equations retain the system inertia parameters lumped in the inertia matrix. Clearly these matrix methods allow the design engineer to better visualize the functional relationships between his variables. Furthermore, the matrix equations can be easily cast into the powerful state-vector form.



$$\begin{aligned}
 \begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} &= \begin{bmatrix} c\phi_2 c\phi_3 & s\phi_3 & 0 \\ -c\phi_2 s\phi_3 & c\phi_3 & 0 \\ s\phi_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi}_{r1} \\ \dot{\phi}_{r2} \\ \dot{\phi}_{r3} \end{bmatrix} + \begin{bmatrix} c\phi_2 c\phi_3 & c\phi_1 s\phi_3 & (s\phi_1 s\phi_3 - s\phi_1 s\phi_2 c\phi_3) \\ -c\phi_2 s\phi_3 & c\phi_1 c\phi_3 & (s\phi_1 c\phi_3 - s\phi_1 s\phi_2 s\phi_3) \\ s\phi_2 & -s\phi_1 c\phi_2 & c\phi_1 c\phi_2 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix} \\
 \begin{bmatrix} \dot{\phi}_{r1} \\ \dot{\phi}_{r2} \\ \dot{\phi}_{r3} \end{bmatrix} &= \frac{1}{c\phi_2} \begin{bmatrix} c\phi_3 & -s\phi_2 & 0 \\ c\phi_2 s\phi_3 & c\phi_2 c\phi_3 & 0 \\ -s\phi_2 c\phi_3 & s\phi_2 s\phi_3 & c\phi_2 \end{bmatrix} \begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} - \frac{1}{c\phi_2} \begin{bmatrix} c\phi_2 & s\phi_1 s\phi_2 & -c\phi_1 s\phi_2 \\ 0 & c\phi_1 c\phi_2 & s\phi_1 c\phi_2 \\ 0 & -s\phi_1 & c\phi_1 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix} \\
 \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} &= \begin{bmatrix} c\phi_2 c\phi_3 & -c\phi_2 s\phi_3 & s\phi_2 \\ s\phi_3 & c\phi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi_1 & s\phi_1 \\ s\phi_2 & -s\phi_1 c\phi_2 & c\phi_1 c\phi_2 \end{bmatrix} \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{bmatrix} \\
 \begin{bmatrix} \eta_1^* \\ \eta_2^* \\ \eta_3^* \end{bmatrix} &= \frac{1}{c\phi_2} \begin{bmatrix} c\phi_3 & -s\phi_3 & 0 \\ c\phi_2 s\phi_3 & c\phi_2 c\phi_3 & 0 \\ -s\phi_2 c\phi_3 & s\phi_2 s\phi_3 & c\phi_2 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} c\phi_2 & s\phi_1 s\phi_2 & -c\phi_1 s\phi_2 \\ 0 & c\phi_1 c\phi_2 & s\phi_1 c\phi_2 \\ 0 & -s\phi_1 & c\phi_1 \end{bmatrix} \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{bmatrix}
 \end{aligned}$$

$$M_{rf} = M_3(\phi_3)M_2(\phi_2)M_1(\phi_1), \quad (\phi_1, \phi_2, \phi_3) = (\phi, \theta, \psi), \quad \langle \dot{\phi}\eta \rangle = \dot{\phi}_1 \bar{F}_1 + \dot{\phi}_2 \bar{r}_{21} + \dot{\phi}_3 \bar{r}_3$$

Figure 5. Successive Rotation Sequence (1,2,3)



$$\begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} c\phi_2 c\phi_3 & s\phi_1 s\phi_3 & -c\phi_1 s\phi_3 \\ -s\phi_2 & c\phi_1 c\phi_2 & s\phi_1 c\phi_2 \\ c\phi_2 s\phi_3 & -s\phi_1 c\phi_3 & c\phi_1 c\phi_3 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix}$$

$$\begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} = \begin{bmatrix} c\phi_2 c\phi_3 & -s\phi_3 & 0 \\ -s\phi_2 & 0 & 1 \\ c\phi_2 s\phi_3 & c\phi_3 & 0 \end{bmatrix} \begin{bmatrix} \phi_{rf1} \\ \phi_{rf2} \\ \phi_{rf3} \end{bmatrix} + \begin{bmatrix} c\phi_2 c\phi_3 & s\phi_1 s\phi_3 & -c\phi_1 s\phi_3 \\ -s\phi_2 & c\phi_1 c\phi_2 & s\phi_1 c\phi_2 \\ c\phi_2 s\phi_3 & -s\phi_1 c\phi_3 & c\phi_1 c\phi_3 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix}$$

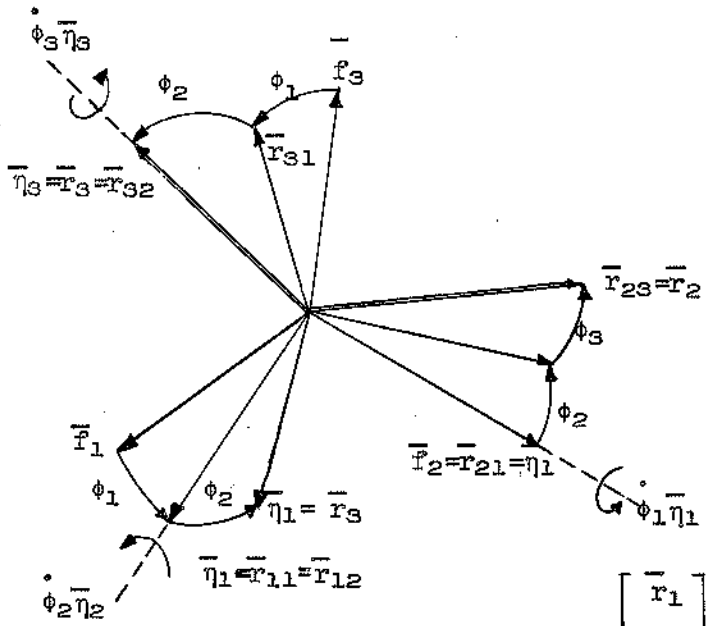
$$\begin{bmatrix} \phi_{rf1} \\ \phi_{rf2} \\ \phi_{rf3} \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} c\phi_3 & 0 & s\phi_3 \\ -c\phi_2 s\phi_3 & 0 & c\phi_2 c\phi_3 \\ -s\phi_2 c\phi_3 & c\phi_2 & s\phi_2 s\phi_3 \end{bmatrix} \begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} c\phi_2 & c\phi_1 s\phi_2 & s\phi_1 s\phi_2 \\ 0 & -s\phi_1 c\phi_2 & c\phi_1 c\phi_2 \\ 0 & c\phi_1 & s\phi_1 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix}$$

$$\begin{bmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \\ \bar{\eta}_3 \end{bmatrix} = \begin{bmatrix} c\phi_2 c\phi_3 & -s\phi_2 & c\phi_2 s\phi_3 \\ -s\phi_3 & 0 & c\phi_3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -s\phi_1 & c\phi_1 \\ s\phi_2 & c\phi_1 c\phi_2 & s\phi_1 c\phi_2 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix}$$

$$\begin{bmatrix} \bar{\eta}_1^* \\ \bar{\eta}_2^* \\ \bar{\eta}_3^* \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} c\phi_3 & 0 & s\phi_3 \\ -c\phi_2 s\phi_3 & 0 & c\phi_2 c\phi_3 \\ s\phi_2 c\phi_3 & c\phi_2 & s\phi_2 s\phi_3 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} c\phi_2 & c\phi_1 s\phi_2 & s\phi_1 s\phi_2 \\ 0 & -s\phi_1 c\phi_2 & c\phi_1 c\phi_2 \\ 0 & c\phi_1 & s\phi_1 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix}$$

$$M_{rf} = M_2(\phi_3)M_3(\phi_2)M_1(\phi_1) \quad (\phi_1, \phi_2, \phi_3) \quad (\phi, \psi, \theta) \quad \langle \phi \eta \rangle = \phi_1 \bar{r}_1 + \phi_2 \bar{r}_{31} + \phi_3 \bar{r}_2$$

Figure 6. Successive Rotation Sequence (1,3,2)



$$\begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} c\phi_1 c\phi_3 & c\phi_2 s\phi_3 & -s\phi_1 c\phi_3 \\ +s\phi_1 s\phi_2 s\phi_3 & & +c\phi_1 s\phi_2 s\phi_3 \\ -c\phi_1 s\phi_3 & c\phi_2 c\phi_3 & s\phi_1 s\phi_3 \\ +s\phi_1 s\phi_2 c\phi_3 & & +c\phi_1 s\phi_2 c\phi_3 \\ s\phi_1 c\phi_2 & -s\phi_2 & c\phi_1 c\phi_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$\begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} = \begin{bmatrix} c\phi_2 s\phi_3 & c\phi_3 & 0 \\ c\phi_2 c\phi_3 & -s\phi_3 & 0 \\ -s\phi_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix} + \begin{bmatrix} c\phi_1 c\phi_3 & c\phi_2 s\phi_3 & -s\phi_1 c\phi_3 \\ +s\phi_1 s\phi_2 s\phi_3 & & +c\phi_1 s\phi_2 s\phi_3 \\ -c\phi_1 s\phi_3 & c\phi_2 c\phi_3 & s\phi_1 s\phi_3 \\ +s\phi_1 s\phi_2 c\phi_3 & & +c\phi_1 s\phi_2 c\phi_3 \\ s\phi_1 c\phi_2 & -s\phi_2 & c\phi_1 c\phi_2 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix}$$

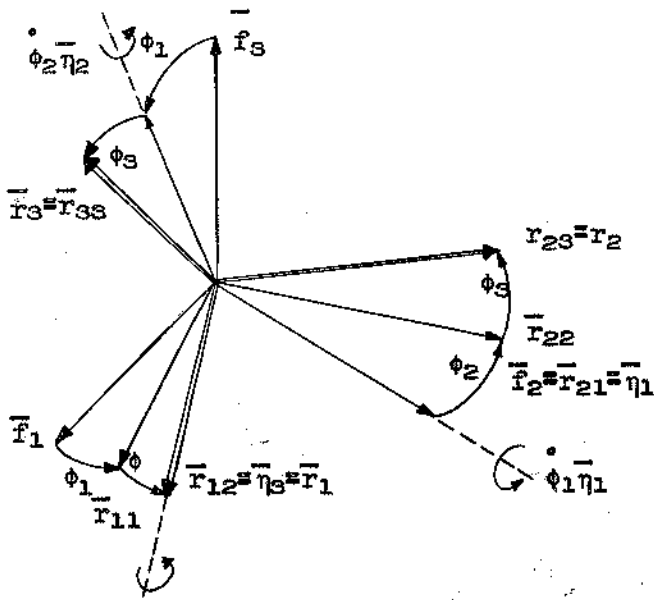
$$\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} s\phi_3 & c\phi_3 & 0 \\ c\phi_2 c\phi_3 & -c\phi_2 s\phi_3 & 0 \\ s\phi_2 s\phi_3 & s\phi_2 c\phi_3 & c\phi_2 \end{bmatrix} \begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} - \frac{1}{c\phi_2} \begin{bmatrix} s\phi_1 s\phi_2 & c\phi_2 & c\phi_1 s\phi_2 \\ c\phi_1 c\phi_2 & 0 & -s\phi_1 c\phi_2 \\ s\phi_1 & 0 & c\phi_1 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix}$$

$$\begin{bmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \\ \bar{\eta}_3 \end{bmatrix} = \begin{bmatrix} c\phi_2 s\phi_3 & c\phi_2 c\phi_3 & -s\phi_2 \\ c\phi_3 & -s\phi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ c\phi_1 & 0 & -s\phi_1 \\ s\phi_1 c\phi_2 & -s\phi_2 & c\phi_1 c\phi_2 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix}$$

$$\begin{bmatrix} \bar{\eta}_1^* \\ \bar{\eta}_2^* \\ \bar{\eta}_3^* \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} s\phi_3 & c\phi_3 & 0 \\ c\phi_2 c\phi_3 & -c\phi_2 s\phi_3 & 0 \\ s\phi_2 s\phi_3 & s\phi_2 c\phi_3 & c\phi_2 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} s\phi_1 s\phi_2 & c\phi_2 & c\phi_1 s\phi_2 \\ c\phi_1 c\phi_2 & 0 & -s\phi_1 c\phi_2 \\ s\phi_1 & 0 & c\phi_1 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix}$$

$$M_{rr} = M_3(\phi_3)M_1(\phi_2)M_2(\phi_1) (\phi_1, \phi_2, \phi_3) \quad (\theta, \phi, \psi) \quad \langle \dot{\phi} \bar{\eta} \rangle = \dot{\phi}_1 \bar{r}_2 + \dot{\phi}_2 \bar{r}_{11} + \dot{\phi}_3 \bar{r}_3$$

Figure 7. Successive Rotation Sequence (2,1,3)
27



$$\begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} c\phi_1 c\phi_2 & s\phi_2 & -s\phi_1 c\phi_2 \\ s\phi_1 s\phi_3 & c\phi_2 c\phi_3 & c\phi_1 s\phi_3 \\ -c\phi_1 s\phi_2 c\phi_3 & c\phi_2 s\phi_3 & +s\phi_1 s\phi_2 c\phi_3 \\ s\phi_1 c\phi_3 & -c\phi_2 s\phi_3 & c\phi_1 c\phi_3 \\ +c\phi_1 s\phi_2 s\phi_3 & -c\phi_2 s\phi_3 & -s\phi_1 s\phi_2 s\phi_3 \end{bmatrix} \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{bmatrix}$$

$$\begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} = \begin{bmatrix} s\phi_2 & 0 & 1 \\ c\phi_2 c\phi_3 & s\phi_3 & 0 \\ -c\phi_2 s\phi_3 & c\phi_3 & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_{rf1} \\ \dot{\phi}_{rf2} \\ \dot{\phi}_{rf3} \end{bmatrix} + \begin{bmatrix} c\phi_1 c\phi_2 & s\phi_2 & -s\phi_1 c\phi_2 \\ s\phi_1 s\phi_3 & c\phi_2 c\phi_3 & c\phi_1 s\phi_3 \\ -c\phi_1 s\phi_2 c\phi_3 & c\phi_2 s\phi_3 & +s\phi_1 s\phi_2 c\phi_3 \\ s\phi_1 c\phi_3 & -c\phi_2 s\phi_3 & c\phi_1 c\phi_3 \\ +c\phi_1 s\phi_2 s\phi_3 & -c\phi_2 s\phi_3 & -s\phi_1 s\phi_2 s\phi_3 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix}$$

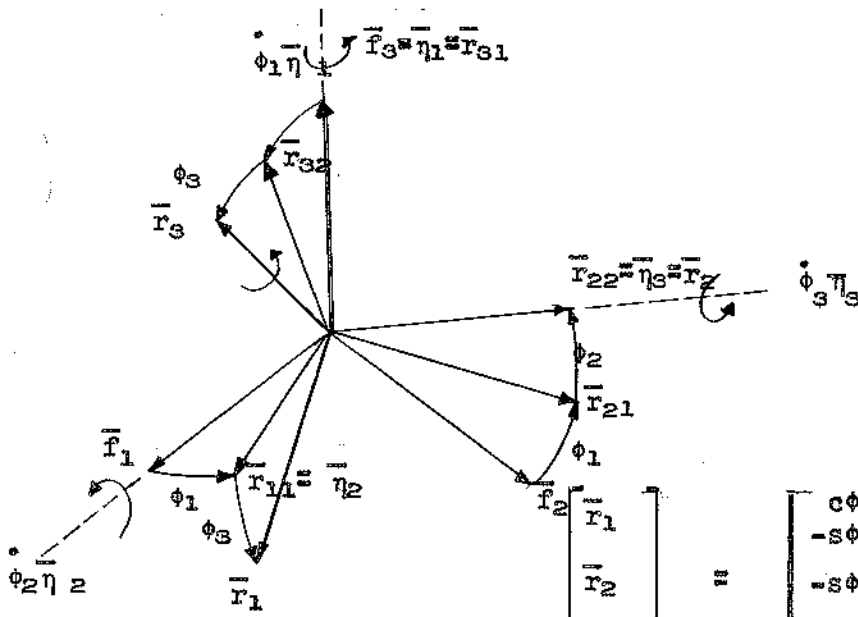
$$\begin{bmatrix} \dot{\phi}_{rf1} \\ \dot{\phi}_{rf2} \\ \dot{\phi}_{rf3} \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} 0 & c\phi_3 & -s\phi_3 \\ 0 & c\phi_2 s\phi_3 & c\phi_2 c\phi_3 \\ c\phi_2 & -s\phi_2 c\phi_3 & s\phi_2 s\phi_3 \end{bmatrix} \begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} - \frac{1}{c\phi_2} \begin{bmatrix} c\phi_1 s\phi_2 & c\phi_2 & s\phi_1 s\phi_2 \\ s\phi_1 c\phi_2 & 0 & c\phi_1 c\phi_2 \\ c\phi_1 & 0 & -s\phi_1 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix}$$

$$\begin{bmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \\ \bar{\eta}_3 \end{bmatrix} = \begin{bmatrix} s\phi_2 & c\phi_2 c\phi_3 & -c\phi_2 s\phi_3 \\ 0 & s\phi_3 & c\phi_3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ s\phi_1 & 0 & c\phi_1 \\ c\phi_1 c\phi_2 & s\phi_2 & -s\phi_1 c\phi_2 \end{bmatrix} \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{bmatrix}$$

$$\begin{bmatrix} \bar{\eta}_1^* \\ \bar{\eta}_2^* \\ \bar{\eta}_3^* \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} 0 & c\phi_3 & -s\phi_3 \\ 0 & c\phi_2 s\phi_3 & c\phi_2 c\phi_3 \\ c\phi_2 & -s\phi_2 c\phi_3 & s\phi_2 s\phi_3 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} -c\phi_1 s\phi_2 & c\phi_2 & s\phi_1 s\phi_2 \\ s\phi_1 c\phi_2 & 0 & c\phi_1 c\phi_2 \\ c\phi_1 & 0 & -s\phi_1 \end{bmatrix} \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{bmatrix}$$

$$M_{rf} = M_1(\phi_3) M_3(\phi_2) M_2(\phi_1) \quad (\phi_1, \phi_2, \phi_3) \quad (\theta, \psi, \phi) \quad \langle \phi \bar{\eta} \rangle = \phi_1 \bar{f}_2 + \phi_2 \bar{r}_{31} + \phi_3 \bar{r}_1$$

Fig 8. Successive Rotation Sequence (2,3,1)



$$\begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} c\phi_1 c\phi_3 & s\phi_1 c\phi_3 & -c\phi_2 s\phi_3 \\ -s\phi_1 s\phi_2 s\phi_3 & +c\phi_1 s\phi_2 s\phi_3 & \\ -s\phi_1 c\phi_2 & c\phi_1 c\phi_2 & s\phi_2 \\ c\phi_1 s\phi_3 & s\phi_1 s\phi_3 & \\ +s\phi_1 s\phi_2 c\phi_3 & -c\phi_1 s\phi_2 c\phi_3 & c\phi_2 c\phi_3 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} \dot{\phi}_1 \eta_1 \\ \dot{\phi}_2 \eta_2 \\ \dot{\phi}_3 \eta_3 \end{bmatrix}$$

$$\begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} = \begin{bmatrix} -c\phi_2 s\phi_3 & c\phi_3 & 0 \\ s\phi_2 & 0 & 1 \\ c\phi_2 c\phi_3 & s\phi_3 & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \eta_1 \\ \dot{\phi}_2 \eta_2 \\ \dot{\phi}_3 \eta_3 \end{bmatrix} + \begin{bmatrix} c\phi_1 c\phi_3 & s\phi_1 c\phi_3 & -c\phi_2 s\phi_3 \\ -s\phi_1 s\phi_2 s\phi_3 & +c\phi_1 s\phi_2 s\phi_3 & \\ -s\phi_1 c\phi_2 & c\phi_1 c\phi_2 & s\phi_2 \\ c\phi_1 s\phi_3 & s\phi_1 s\phi_3 & \\ +s\phi_1 s\phi_2 c\phi_3 & -c\phi_1 s\phi_2 c\phi_3 & c\phi_2 c\phi_3 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix}$$

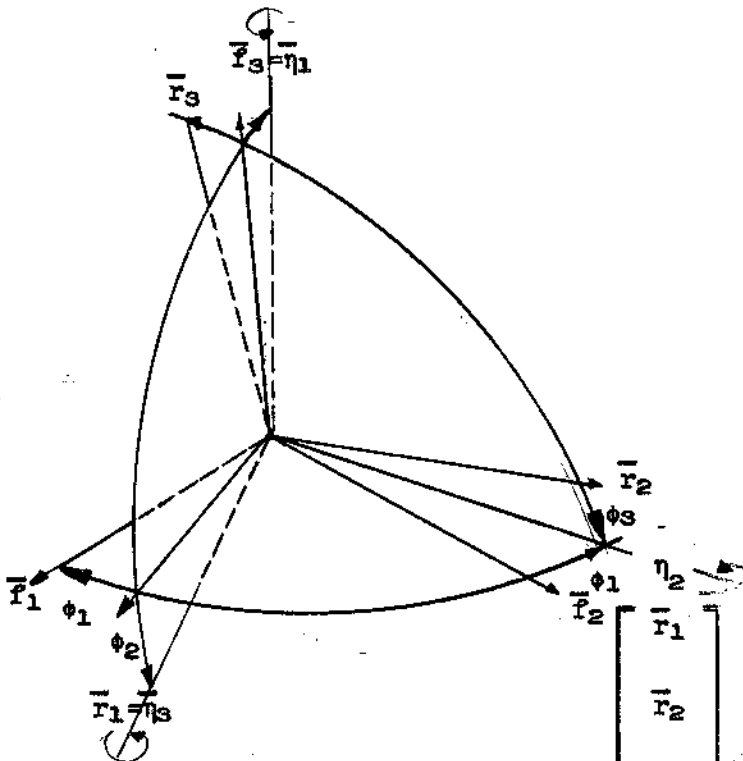
$$\begin{bmatrix} \dot{\phi}_1 \eta_1 \\ \dot{\phi}_2 \eta_2 \\ \dot{\phi}_3 \eta_3 \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} -s\phi_3 & 0 & c\phi_3 \\ c\phi_2 c\phi_3 & 0 & c\phi_2 s\phi_3 \\ s\phi_2 s\phi_3 & c\phi_2 & -s\phi_2 c\phi_3 \end{bmatrix} \begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} - \frac{1}{c\phi_2} \begin{bmatrix} s\phi_1 s\phi_2 & -c\phi_1 s\phi_2 & c\phi_2 \\ c\phi_1 c\phi_2 & s\phi_1 c\phi_2 & 0 \\ -s\phi_1 & c\phi_1 & 0 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix}$$

$$\begin{bmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \\ \bar{\eta}_3 \end{bmatrix} = \begin{bmatrix} -c\phi_2 s\phi_3 & s\phi_2 & c\phi_2 c\phi_3 \\ c\phi_3 & 0 & s\phi_3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ c\phi_1 & s\phi_1 & 0 \\ -s\phi_1 c\phi_2 & c\phi_1 c\phi_2 & s\phi_2 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix}$$

$$\begin{bmatrix} \bar{\eta}_1^* \\ \bar{\eta}_2^* \\ \bar{\eta}_3^* \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} -s\phi_3 & 0 & c\phi_3 \\ c\phi_2 c\phi_3 & 0 & c\phi_2 s\phi_3 \\ (s\phi_2 s\phi_3) & c\phi_2 & (-s\phi_2 c\phi_3) \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} s\phi_1 s\phi_2 & -c\phi_1 s\phi_2 & c\phi_2 \\ c\phi_1 c\phi_2 & s\phi_1 c\phi_2 & 0 \\ -s\phi_1 & c\phi_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix}$$

$$M_{rf} = M_2(\phi_3) M_1(\phi_2) M_3(\phi_1) \quad (\phi_1, \phi_2, \phi_3) = (\psi, \phi, \theta) \quad \langle \dot{\phi} \eta \rangle = \dot{\phi}_1 \bar{r}_3 + \dot{\phi}_2 \bar{r}_{11} + \dot{\phi}_3 \bar{r}_2$$

Figure 9. Successive Rotation Sequence (3,1,2)
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$$\begin{aligned}
 \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} &= \begin{bmatrix} c\phi_1 c\phi_2 & s\phi_1 c\phi_2 & -s\phi_2 \\ -s\phi_1 c\phi_3 & c\phi_1 c\phi_3 & \\ +c\phi_1 s\phi_2 s\phi_3 & +s\phi_1 s\phi_2 s\phi_3 & c\phi_2 s\phi_3 \\ s\phi_1 s\phi_3 & (c\phi_1 s\phi_3) & c\phi_2 c\phi_3 \\ +c\phi_1 s\phi_2 c\phi_3 & +s\phi_1 s\phi_2 c\phi_3 & \end{bmatrix} \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{bmatrix} \\
 \begin{bmatrix} \dot{\omega}_{r1} \\ \dot{\omega}_{r2} \\ \dot{\omega}_{r3} \end{bmatrix} &= \begin{bmatrix} s\phi_2 & 0 & 1 \\ c\phi_2 s\phi_3 & c\phi_3 & 0 \\ c\phi_2 c\phi_3 & (-s\phi_3) & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_{rf1} \\ \dot{\phi}_{rf2} \\ \dot{\phi}_{rf3} \end{bmatrix} + \begin{bmatrix} c\phi_1 c\phi_2 & s\phi_1 c\phi_2 & -s\phi_2 \\ -s\phi_1 c\phi_3 & (c\phi_1 c\phi_3) & \\ +c\phi_1 s\phi_2 s\phi_3 & +s\phi_1 s\phi_2 s\phi_3 & c\phi_2 s\phi_3 \\ s\phi_1 s\phi_3 & (c\phi_1 s\phi_3) & c\phi_2 c\phi_3 \\ +c\phi_1 s\phi_2 c\phi_3 & +s\phi_1 s\phi_2 c\phi_3 & c\phi_2 c\phi_3 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix} \\
 \begin{bmatrix} \dot{\phi}_{rf1} \\ \dot{\phi}_{rf2} \\ \dot{\phi}_{rf3} \end{bmatrix} &= \frac{1}{c\phi_2} \begin{bmatrix} 0 & s\phi_3 & c\phi_3 \\ 0 & c\phi_2 c\phi_3 & -c\phi_2 s\phi_3 \\ c\phi_2 & -s\phi_2 s\phi_3 & -s\phi_2 c\phi_3 \end{bmatrix} \begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} c\phi_1 s\phi_2 & -s\phi_1 s\phi_2 & c\phi_2 \\ -s\phi_1 c\phi_2 & c\phi_1 c\phi_2 & 0 \\ c\phi_1 & s\phi_1 & 0 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix} \\
 \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} &= \begin{bmatrix} -s\phi_2 & c\phi_2 s\phi_3 & c\phi_2 c\phi_3 \\ 0 & c\phi_3 & -s\phi_3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -s\phi_1 & c\phi_1 & 0 \\ c\phi_1 c\phi_2 & s\phi_1 s\phi_2 & -s\phi_2 \end{bmatrix} \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{bmatrix} \\
 \begin{bmatrix} \eta_1^* \\ \eta_2^* \\ \eta_3^* \end{bmatrix} &= \frac{1}{c\phi_2} \begin{bmatrix} 0 & s\phi_3 & c\phi_3 \\ 0 & c\phi_2 c\phi_3 & -c\phi_2 s\phi_3 \\ c\phi_2 & -s\phi_2 s\phi_3 & -s\phi_2 c\phi_3 \end{bmatrix} \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{bmatrix} = \frac{1}{c\phi_2} \begin{bmatrix} s\phi_1 s\phi_2 & -s\phi_1 s\phi_2 & c\phi_2 \\ -s\phi_1 c\phi_2 & c\phi_1 c\phi_2 & 0 \\ c\phi_1 & s\phi_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{bmatrix}
 \end{aligned}$$

$$M_{rf} = M_1(\phi_3) M_2(\phi_2) M_3(\phi_1) \quad (\phi_1, \phi_2, \phi_3) = (\psi, \theta, \phi) \quad \langle \dot{\phi} \eta \rangle = \dot{\phi}_1 \bar{F}_3 + \dot{\phi}_2 \bar{r}_{21} + \dot{\phi}_3 \bar{F}_1$$

Figure 10. Successive Rotation Sequence (3,2,1)

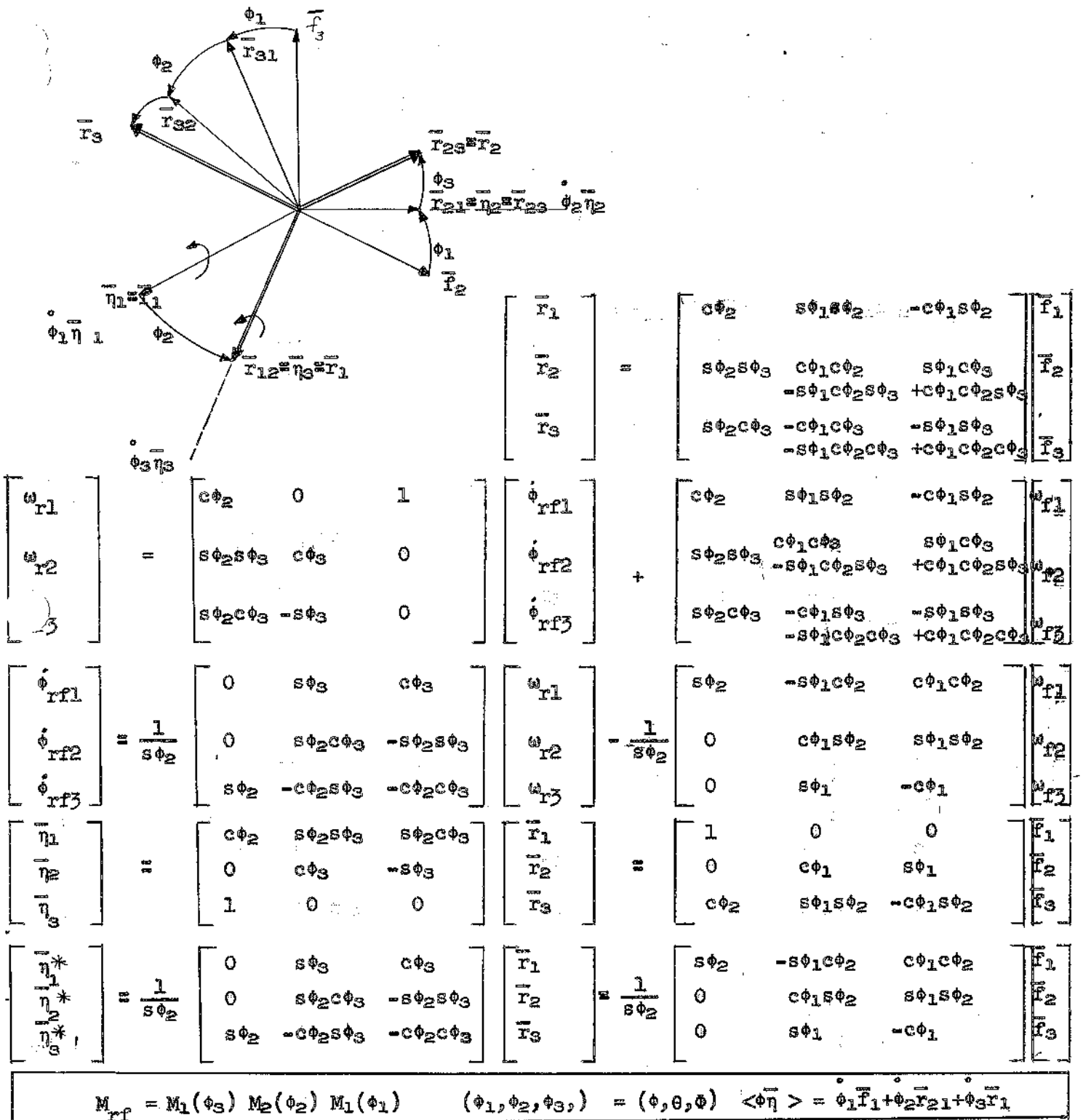
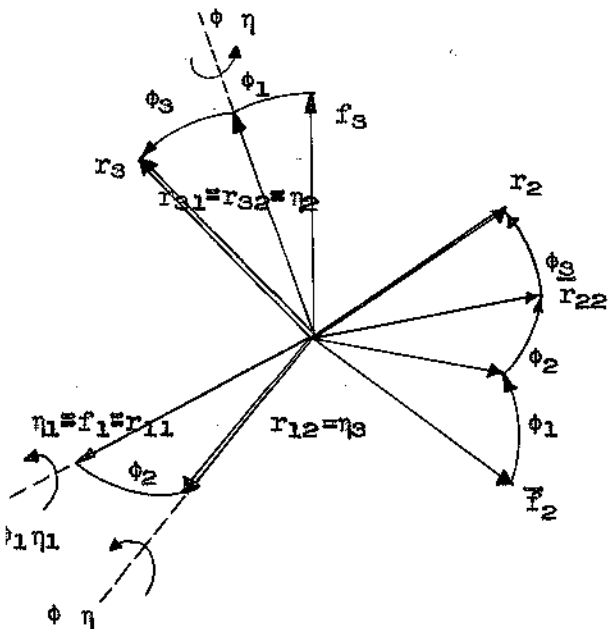


Figure 11. Repetitive Rotation Sequence (1,2,1)



$$\begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} c\phi_2 & c\phi_1 s\phi_2 & s\phi_1 s\phi_2 \\ -s\phi_2 c\phi_3 & -s\phi_1 s\phi_3 & c\phi_1 s\phi_3 \\ s\phi_2 s\phi_3 & (-s\phi_1 c\phi_3) & c\phi_1 c\phi_3 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix}$$

$$\begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} = \begin{bmatrix} c\phi_2 & 0 & 1 \\ -s\phi_2 c\phi_3 & s\phi_3 & 0 \\ s\phi_2 s\phi_3 & c\phi_3 & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_{rf1} \\ \dot{\phi}_{rf2} \\ \dot{\phi}_{rf3} \end{bmatrix} + \begin{bmatrix} c\phi_2 & c\phi_1 s\phi_2 & s\phi_1 s\phi_2 \\ -s\phi_2 c\phi_3 & (-s\phi_1 s\phi_3) & c\phi_1 s\phi_3 \\ s\phi_2 s\phi_3 & (-s\phi_1 c\phi_3) & c\phi_1 c\phi_3 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix}$$

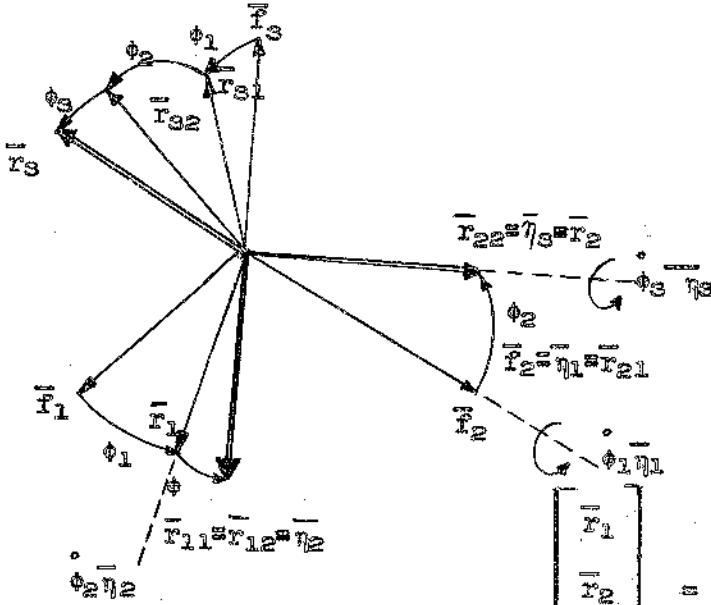
$$\begin{bmatrix} \dot{\phi}_{rf1} \\ \dot{\phi}_{rf2} \\ \dot{\phi}_{rf3} \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} 0 & -c\phi_3 & s\phi_3 \\ 0 & s\phi_2 s\phi_3 & s\phi_2 c\phi_3 \\ s\phi_2 & c\phi_2 c\phi_3 & -c\phi_2 s\phi_3 \end{bmatrix} \begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} - \frac{1}{s\phi_2} \begin{bmatrix} s\phi_2 & -c\phi_1 c\phi_2 & -s\phi_1 c\phi_2 \\ 0 & -s\phi_1 s\phi_2 & c\phi_1 s\phi_2 \\ 0 & c\phi_1 & s\phi_1 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix}$$

$$\begin{bmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \\ \bar{\eta}_3 \end{bmatrix} = \begin{bmatrix} c\phi_2 & -s\phi_2 c\phi_3 & s\phi_2 s\phi_3 \\ 0 & s\phi_3 & c\phi_3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -s\phi_1 & c\phi_1 \\ c\phi_2 & c\phi_1 s\phi_2 & s\phi_1 s\phi_2 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix}$$

$$\begin{bmatrix} \bar{\eta}_1^* \\ \bar{\eta}_2^* \\ \bar{\eta}_3^* \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} 0 & -c\phi_3 & s\phi_3 \\ 0 & s\phi_2 s\phi_3 & s\phi_2 c\phi_3 \\ s\phi_2 & c\phi_2 c\phi_3 & -c\phi_2 s\phi_3 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} s\phi_2 & -c\phi_1 c\phi_2 & -s\phi_1 c\phi_2 \\ 0 & -s\phi_1 s\phi_2 & c\phi_1 c\phi_2 \\ 0 & c\phi_1 & s\phi_1 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix}$$

$$M_{rf} = M_1(\phi_3) M_3(\phi_2) M_1(\phi_1) \quad (\phi_1, \phi_2, \phi_3) = (\phi, \psi, \theta) \quad \langle \dot{\phi}_{\eta} \rangle = \dot{\phi}_1 \bar{r}_1 + \dot{\phi}_2 \bar{r}_2 + \dot{\phi}_3 \bar{r}_3$$

Figure 12. Repetitive Rotation Sequence (1,3,1)



$$\begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} = \begin{bmatrix} s\phi_2 s\phi_3 & c\phi_3 & 0 \\ c\phi_2 & 0 & 1 \\ -s\phi_2 c\phi_3 & s\phi_3 & 0 \end{bmatrix} \begin{bmatrix} \phi_{rf1} \\ \phi_{rf2} \\ \phi_{rf3} \end{bmatrix} + \begin{bmatrix} c\phi_1 c\phi_3 & s\phi_2 s\phi_3 & -s\phi_1 c\phi_3 \\ -s\phi_1 c\phi_2 s\phi_3 & & -c\phi_1 c\phi_2 s\phi_3 \\ s\phi_1 s\phi_2 & c\phi_2 & c\phi_1 s\phi_2 \\ c\phi_1 s\phi_3 & -s\phi_2 c\phi_3 & -s\phi_1 s\phi_3 \\ +s\phi_1 c\phi_2 c\phi_3 & & +c\phi_1 c\phi_2 c\phi_3 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix}$$

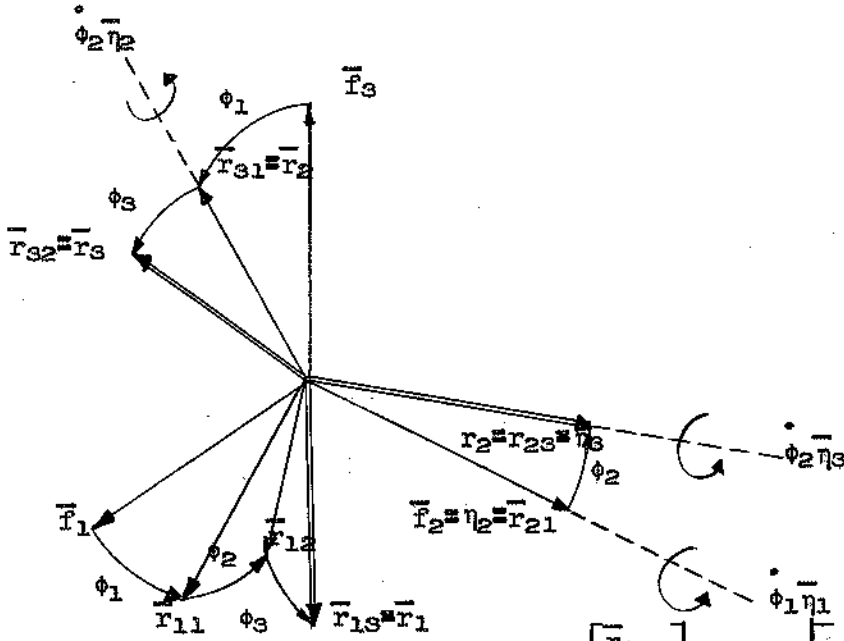
$$\begin{bmatrix} \phi_{rf1} \\ \phi_{rf2} \\ \phi_{rf3} \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} s\phi_3 & 0 & -c\phi_3 \\ s\phi_2 c\phi_3 & 0 & s\phi_2 s\phi_3 \\ -c\phi_2 s\phi_3 & s\phi_2 & c\phi_2 c\phi_3 \end{bmatrix} \begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} - \frac{1}{s\phi_2} \begin{bmatrix} -s\phi_1 c\phi_2 & s\phi_2 & -c\phi_1 c\phi_2 \\ c\phi_1 s\phi_2 & 0 & -s\phi_1 s\phi_2 \\ s\phi_1 & 0 & c\phi_1 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix}$$

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} s\phi_2 s\phi_3 & c\phi_2 & -s\phi_2 c\phi_3 \\ c\phi_3 & 0 & s\phi_3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ c\phi_1 & 0 & -s\phi_1 \\ s\phi_1 s\phi_2 & c\phi_2 & c\phi_1 s\phi_2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$$

$$\begin{bmatrix} \eta_1^* \\ \eta_2^* \\ \eta_3^* \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} s\phi_3 & 0 & -c\phi_3 \\ s\phi_2 c\phi_3 & 0 & s\phi_2 s\phi_3 \\ -c\phi_2 s\phi_3 & s\phi_2 & c\phi_2 c\phi_3 \end{bmatrix} \begin{bmatrix} \eta_1^* \\ \eta_2^* \\ \eta_3^* \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} -s\phi_1 c\phi_2 & s\phi_2 & -c\phi_1 c\phi_2 \\ c\phi_1 s\phi_2 & 0 & -s\phi_1 s\phi_2 \\ s\phi_1 & 0 & c\phi_1 \end{bmatrix} \begin{bmatrix} \eta_1^* \\ \eta_2^* \\ \eta_3^* \end{bmatrix}$$

$$M_{rf} = M_2(\phi_3) M_1(\phi_2) M_2(\phi_1) \quad (\phi_1, \phi_2, \phi_3) = (0, \phi, 0) \quad \langle \phi \eta \rangle = \phi_1 \eta_2 + \phi_2 \eta_1 + \phi_3 \eta_2$$

Figure 13 Repetition Rotation Sequence (2, 1, 2)



$$\begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} -s\phi_1 s\phi_3 & s\phi_2 c\phi_3 & -c\phi_1 s\phi_3 \\ +c\phi_1 c\phi_2 c\phi_3 & & -s\phi_1 c\phi_2 c\phi_3 \\ -c\phi_1 s\phi_2 & c\phi_2 & s\phi_1 s\phi_2 \\ s\phi_1 c\phi_3 & & c\phi_1 c\phi_3 \\ +c\phi_1 c\phi_2 s\phi_3 & s\phi_2 s\phi_3 & -s\phi_1 c\phi_2 s\phi_3 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \end{bmatrix}$$

$$\begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} = \begin{bmatrix} s\phi_2 c\phi_3 & -s\phi_3 & 0 \\ c\phi_2 & 0 & 1 \\ s\phi_2 s\phi_3 & c\phi_3 & 0 \end{bmatrix} \begin{bmatrix} \dot{\eta}_{r1} \\ \dot{\eta}_{r2} \\ \dot{\eta}_{r3} \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} -s\phi_1 s\phi_3 & s\phi_2 c\phi_3 & -c\phi_1 s\phi_3 \\ +c\phi_1 c\phi_2 c\phi_3 & & -s\phi_1 c\phi_2 c\phi_3 \\ -c\phi_1 s\phi_2 & c\phi_2 & s\phi_1 s\phi_2 \\ s\phi_1 c\phi_3 & & c\phi_1 c\phi_3 \\ +c\phi_1 c\phi_2 s\phi_3 & s\phi_2 s\phi_3 & -s\phi_1 c\phi_2 s\phi_3 \end{bmatrix} \begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix}$$

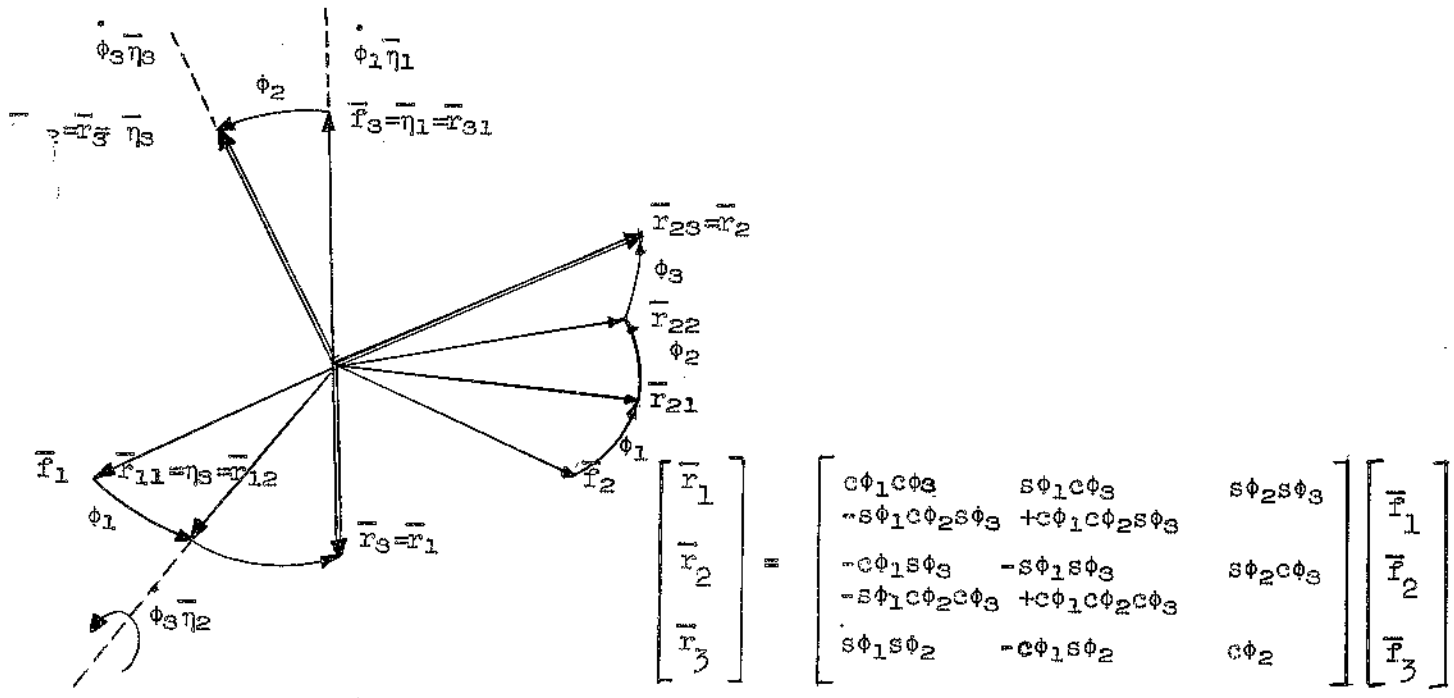
$$\begin{bmatrix} \dot{\eta}_{r1} \\ \dot{\eta}_{r2} \\ \dot{\eta}_{r3} \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} c\phi_3 & 0 & s\phi_3 \\ s\phi_2 s\phi_3 & 0 & s\phi_2 c\phi_3 \\ -c\phi_2 c\phi_3 & s\phi_2 & -c\phi_2 s\phi_3 \end{bmatrix} \begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} c\phi_1 c\phi_2 & s\phi_2 & -s\phi_1 c\phi_2 \\ s\phi_1 s\phi_2 & 0 & c\phi_1 s\phi_2 \\ -c\phi_1 & 0 & s\phi_1 \end{bmatrix} \begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix}$$

$$\begin{bmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \\ \bar{\eta}_3 \end{bmatrix} = \begin{bmatrix} s\phi_2 c\phi_3 & c\phi_2 & s\phi_2 s\phi_3 \\ -s\phi_3 & 0 & c\phi_3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ s\phi_1 & 0 & c\phi_1 \\ -c\phi_1 s\phi_2 & c\phi_2 & s\phi_1 s\phi_2 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix}$$

$$\begin{bmatrix} \bar{\eta}_1^* \\ \bar{\eta}_2^* \\ \bar{\eta}_3^* \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} c\phi_3 & 0 & s\phi_3 \\ -s\phi_2 s\phi_3 & 0 & s\phi_2 c\phi_3 \\ -c\phi_2 c\phi_3 & s\phi_2 & -c\phi_2 s\phi_3 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} c\phi_1 c\phi_2 & s\phi_2 & -s\phi_1 c\phi_2 \\ s\phi_1 s\phi_2 & 0 & c\phi_1 s\phi_2 \\ -c\phi_1 & 0 & s\phi_1 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix}$$

$$M_{rf} = M_2(\phi_3) M_3(\phi_2) M_2(\phi_1) \quad (\phi_1, \phi_2, \phi_3) = (\theta, \psi, \theta) \quad \langle \dot{\eta} \rangle = \dot{\phi}_1 \bar{r}_2 + \dot{\phi}_2 \bar{r}_{31} + \dot{\phi}_3 \bar{r}_2$$

Figure 14. Repetitive Rotation Sequence (2,3,2)



$$\begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} c\phi_1 c\phi_3 & s\phi_1 c\phi_3 & s\phi_2 s\phi_3 \\ -s\phi_1 c\phi_2 s\phi_3 & c\phi_1 c\phi_2 s\phi_3 & s\phi_2 c\phi_3 \\ -c\phi_1 s\phi_3 & -s\phi_1 s\phi_3 & s\phi_2 c\phi_3 \\ -s\phi_1 c\phi_2 c\phi_3 & c\phi_1 c\phi_2 c\phi_3 & c\phi_2 \\ s\phi_1 s\phi_2 & -c\phi_1 s\phi_2 & c\phi_2 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} + \begin{bmatrix} \dot{\phi}_1 \bar{r}_1 \\ \dot{\phi}_2 \bar{r}_2 \\ \dot{\phi}_3 \bar{r}_3 \end{bmatrix}$$

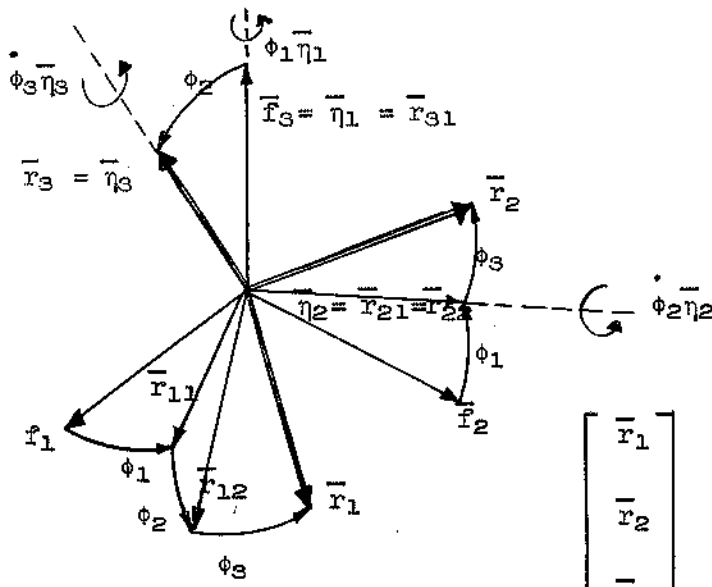
$$\begin{bmatrix} \dot{\phi}_1 \bar{r}_1 \\ \dot{\phi}_2 \bar{r}_2 \\ \dot{\phi}_3 \bar{r}_3 \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} s\phi_3 & c\phi_3 & 0 \\ s\phi_2 c\phi_3 & -s\phi_2 s\phi_3 & 0 \\ -c\phi_2 s\phi_3 & -c\phi_2 c\phi_3 & s\phi_2 \end{bmatrix} \begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} + \begin{bmatrix} -s\phi_1 c\phi_2 & c\phi_1 c\phi_2 & s\phi_2 \\ c\phi_1 s\phi_2 & s\phi_1 s\phi_2 & 0 \\ s\phi_1 & -c\phi_1 & 0 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix}$$

$$\begin{bmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \\ \bar{\eta}_3 \end{bmatrix} = \begin{bmatrix} s\phi_2 s\phi_3 & s\phi_2 c\phi_3 & c\phi_2 \\ c\phi_3 & -s\phi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ c\phi_1 & s\phi_1 & 0 \\ s\phi_1 s\phi_2 & -c\phi_1 s\phi_2 & c\phi_2 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix}$$

$$\begin{bmatrix} \bar{\eta}_1^* \\ \bar{\eta}_2^* \\ \bar{\eta}_3^* \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} s\phi_3 & c\phi_3 & 0 \\ s\phi_2 c\phi_3 & -s\phi_2 s\phi_3 & 0 \\ -c\phi_2 s\phi_3 & -c\phi_2 c\phi_3 & s\phi_2 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} -s\phi_1 c\phi_2 & c\phi_1 c\phi_2 & s\phi_2 \\ c\phi_1 s\phi_2 & s\phi_1 s\phi_2 & 0 \\ s\phi_1 & -c\phi_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix}$$

$$M_{rf} = M_3(\phi_3)M_1(\phi_2)M_3(\phi_1), \quad (\phi_1, \phi_2, \phi_3) = (\psi, \theta, \Psi) \quad \langle \bar{\eta} \rangle = \phi_1 \bar{r}_3 + \phi_2 \bar{r}_{11} + \phi_3 \bar{r}_3$$

Figure 15. Repetitive Rotation Sequence (3,1,3)



$$\begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix} = \begin{bmatrix} -s\phi_1 s\phi_3 & c\phi_1 s\phi_3 & -s\phi_2 c\phi_2 \\ +c\phi_1 c\phi_2 c\phi_3 & +s\phi_1 c\phi_2 c\phi_3 & \\ -s\phi_1 c\phi_3 & c\phi_1 c\phi_3 & s\phi_2 s\phi_3 \\ -c\phi_1 c\phi_2 s\phi_3 & -s\phi_1 c\phi_2 s\phi_3 & \\ c\phi_1 s\phi_2 & s\phi_1 s\phi_2 & c\phi_2 \end{bmatrix} \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{bmatrix}$$

$$\begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} = \begin{bmatrix} -s\phi_2 c\phi_3 & s\phi_3 & 0 \\ s\phi_2 s\phi_3 & c\phi_3 & 0 \\ c\phi_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi}_{r1} \\ \dot{\phi}_{r2} \\ \dot{\phi}_{r3} \end{bmatrix} + \begin{bmatrix} -s\phi_1 s\phi_3 & c\phi_1 s\phi_3 & -s\phi_2 c\phi_3 \\ +c\phi_1 c\phi_2 c\phi_3 & +s\phi_1 c\phi_2 c\phi_3 & \\ -s\phi_1 c\phi_3 & c\phi_1 c\phi_3 & s\phi_2 s\phi_3 \\ -c\phi_1 c\phi_2 s\phi_3 & -s\phi_1 c\phi_2 s\phi_3 & \\ c\phi_1 s\phi_2 & s\phi_1 s\phi_2 & c\phi_2 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\phi}_{rf1} \\ \dot{\phi}_{rf2} \\ \dot{\phi}_{rf3} \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} -c\phi_3 & s\phi_3 & 0 \\ s\phi_2 s\phi_3 & s\phi_2 c\phi_3 & 0 \\ c\phi_2 c\phi_3 & -c\phi_2 s\phi_3 & s\phi_2 \end{bmatrix} \begin{bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{bmatrix} - \frac{1}{s\phi_2} \begin{bmatrix} -c\phi_1 c\phi_2 & -s\phi_1 c\phi_2 & s\phi_2 \\ s\phi_1 s\phi_2 & c\phi_1 s\phi_2 & 0 \\ c\phi_1 & s\phi_1 & 0 \end{bmatrix} \begin{bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{bmatrix}$$

$$\begin{bmatrix} \bar{n}_1 \\ \bar{n}_2 \\ \bar{n}_3 \end{bmatrix} = \begin{bmatrix} -s\phi_2 c\phi_3 & s\phi_2 s\phi_3 & c\phi_2 \\ s\phi_3 & c\phi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -s\phi_1 & c\phi_1 & 0 \\ c\phi_1 s\phi_2 & s\phi_1 s\phi_2 & c\phi_2 \end{bmatrix} \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{bmatrix}$$

$$\begin{bmatrix} \bar{n}_1^* \\ \bar{n}_2^* \\ \bar{n}_3^* \end{bmatrix} = \frac{1}{s\phi_2} \begin{bmatrix} -c\phi_3 & s\phi_3 & 0 \\ s\phi_2 s\phi_3 & s\phi_2 c\phi_3 & 0 \\ c\phi_2 c\phi_3 & -c\phi_2 s\phi_3 & s\phi_2 \end{bmatrix} \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{bmatrix} - \frac{1}{s\phi_2} \begin{bmatrix} -c\phi_1 c\phi_2 & -s\phi_1 c\phi_2 & s\phi_2 \\ -s\phi_1 s\phi_2 & c\phi_1 s\phi_2 & 0 \\ c\phi_1 & s\phi_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{bmatrix}$$

$$M_{rf} = M_3(\phi_3)M_2(\phi_2)M_3(\phi_1), \quad (\phi_1, \phi_2, \phi_3) = (\psi, \theta, \Psi), \quad \langle \dot{\phi}\bar{n} \rangle = \dot{\phi}_1 \bar{f}_3 + \dot{\phi}_2 \bar{r}_{21} + \dot{\phi}_3 \bar{r}_3$$

Figure 16. Repetitive Rotation Sequence (3,2,3)

A P P E N D I X A

The vector-cross product of two vectors in non-orthonormal bases may be written as

$$\bar{R}_1 \times \bar{R}_2 = \langle y \bar{f} \rangle \times \langle \bar{f} z \rangle \quad (a-1)$$

where

$$\bar{R}_1 = \langle \bar{f} y \rangle = \langle y \bar{f} \rangle \quad (a-2)$$

$$\bar{R}_2 = \langle \bar{f} z \rangle = \langle z \bar{f} \rangle \quad (a-3)$$

Consider the square matrix $\bar{f} \rangle \times \langle \bar{f}$ of Equation (a-1), i.e.,

$$\bar{f} \rangle \times \langle \bar{f} \begin{pmatrix} \bar{0} & \bar{f}_1 \times \bar{f}_2 & \bar{f}_1 \times \bar{f}_3 \\ \bar{f}_2 \times \bar{f}_1 & \bar{0} & \bar{f}_2 \times \bar{f}_3 \\ \bar{f}_3 \times \bar{f}_1 & \bar{f}_3 \times \bar{f}_2 & \bar{0} \end{pmatrix} \quad (a-4)$$

where

$$\bar{f}_i \times \bar{f}_j = -\bar{f}_j \times \bar{f}_i \quad (a-5)$$

Then Equation (a-1) may be written as

$$\bar{R}_1 \times \bar{R}_2 = \begin{pmatrix} \bar{f}_2 \times \bar{f}_3 \\ \bar{f}_3 \times \bar{f}_1 \\ \bar{f}_1 \times \bar{f}_2 \end{pmatrix} \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \quad (a-6)$$

It is shown in reference () that the reciprocal base vectors $\bar{b}^* \rangle$ are related to the base vectors $\bar{b} \rangle$ under the cross product operation by the following

$$\begin{pmatrix} \bar{f}_1^* \\ \bar{f}_2^* \\ \bar{f}_3^* \end{pmatrix} = K_V^f \begin{pmatrix} \bar{f}_2 \times \bar{f}_3 \\ \bar{f}_3 \times \bar{f}_1 \\ \bar{f}_1 \times \bar{f}_2 \end{pmatrix} \quad (a-7)$$

where K_V^f is the volume of the base vectors, i.e.,

$$K_V^f = \bar{f}_1 \cdot (\bar{f}_2 \times \bar{f}_3) \quad (a-8)$$

Appendix A

Using Equation (a-7) in Equation (a-6)

$$\bar{R}_1 \times \bar{R}_2 = K_V^f \langle \bar{f}^* \ S_y' \ Z \rangle \quad (a-9)$$

The scalar product of the vector $\bar{R}_1 \times \bar{R}_2$ of Equation (a-8) with a third vector $\langle x \ \bar{f} \rangle$ is

$$\bar{R}_3 \cdot (\bar{R}_1 \times \bar{R}_2) = \langle x \ \bar{f} \rangle \cdot \langle \bar{f}^* \ S_y' \ Z \rangle K_V^f \quad (a-10)$$

The reciprocal base vectors have the property

$$\bar{f} \rangle \cdot \langle \bar{f}^* = I \quad (a-11)$$

hence

$$\bar{R}_3 \cdot (\bar{R}_1 \times \bar{R}_2) = \langle x \ S_y' \ Z \rangle K_V^f \quad (a-12)$$

Consider the mapping between two arbitrary basis

$$\begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{pmatrix} = D_{rf} \bar{f} \quad (a-13)$$

by Equation (a-12),

$$K_V^r = \bar{r}_1 \cdot (\bar{r}_2 \times \bar{r}_3) = K_V^f (r_{11}, r_{12}, r_{13}) \begin{pmatrix} 0 & r_{23} - r_{22} \\ -r_{23} & 0 & r_{21} \\ r_{22} & -r_{21} & 0 \end{pmatrix} \begin{pmatrix} r_{31} \\ r_{32} \\ r_{33} \end{pmatrix} \quad (a-14)$$

or

$$K_V^r = K_V^f (r_{11}, r_{12}, r_{13}) \begin{pmatrix} r_{22} r_{33} - r_{32} r_{23} \\ r_{23} r_{31} - r_{21} r_{33} \\ r_{21} r_{32} - r_{31} r_{22} \end{pmatrix} \quad (a-15)$$

The scalar equation resulting from the above matrix product is seen to be the well known determinant of D_{rf} expanded about the first row, hence

$$K_V^r = |D_{rf}| K_V^f \quad (a-16)$$

The conventional inverse of a matrix is

$$D_{rf}^{-1} = \frac{\text{adj } D_{rf}}{|D_{rf}|} \quad . \quad (\text{a-17})$$

Operating on Equation (a-13) with $\langle \bar{r}^* |$

$$I = D_{rf} \bar{f} \rangle . \langle \bar{r}^* \quad (\text{a-18})$$

hence

$$D_{rf}^{-1} = \bar{f} \rangle . \langle \bar{r}^* \quad . \quad (\text{a-19})$$

By Equation (a-17), Equation (a-16) and Equation (a-19)

$$D_{rf}^{-1} = \frac{K_V^f}{K_V^r} \text{adj } D_{rf} \quad . \quad (\text{a-20})$$

The adjoint matrix by Equation (a-20) and Equation (a-19) is

$$\text{adj } D_{rf} = \bar{f} \rangle . \langle \bar{r}^* | D_{rf} | \quad . \quad (\text{a-21})$$

The adjoint matrix is also expressed as the transpose of the matrix of cofactors.

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