SPECIAL REPORT NR 64

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# TO THE GIBBSION VECTORS
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# TO FLIGHT DYNAMICS

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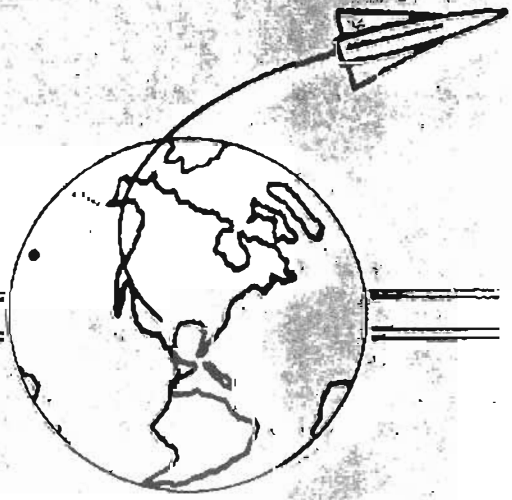

THE ARMY MISSILE TEST AND
EVALUATION DIRECTORATE

WHITE SANDS MISSILE RANGE
NEW MEXICO

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# MATRIX METHODS APPLIED TO THE GIBBSION VECTORS AND DYADICS WITH APPLICATIONS TO FLIGHT DYNAMICS

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# ABSTRACT

Matrix methods applied to the vector, dyadic, and polyadic analysis of Gibbs' simplifies the transformations encountered in deriving mathematical models of dynamical systems in moving bases. The matrix methods eliminate the need for mentally tracking hordes of tedious indices.

The techniques are applied to the derivation of equations of motion of flight systems in ortho-normal as well as non-ortho-normal moving bases.

# CONTENTS

# INTRODUCTION

Dynamical aero-space flight systems are characterized by moving bases, many of which occur naturally, or least expensively, as non-orthonormal bases. Theoretically one can always utilize a Schmidt orthogonalization process and obtain an O. N. basis. Thus most textbooks eliminate early in the pages a discussion of non-orthonormal moving bases. However, from a computer mechanization standpoint (whether flight borne or laboratory) this may not always be the best thing to do. Hence a re-examination of conventional transformation techniques should be made with computer mechanization in mind.

In the opinion of the author the classical representations of vectors and matrices and their applications to physical problems of three space leave something to be desired. This section points out some of the shortcomings. A study of the literature has revealed that one can neither find a standardization of representation techniques nor of notation. In fact the study of representation is, and should be, actively pursued to bridge the gap between classical and modern applied mathematics.

A number of pros and cons for the various representations are given below:

Kondo(19) alludes to the irritating slowness of the penetration of tensor concepts to practical circles. He claims that the tensor language is undoubtedly the most appropriate unifying apparatus for the systematical and transparent representation of physical and engineering sciences.

Halmos(13) states that the classical treatises of vector spaces as numerical n-tuples rather than abstract entities necessitates the introduction of some "cumbersome terminology." Halmos then proceeds to give a brief glossary of what he calls some of the more "baffling" terms and notations, such as covariant, contravariant, cogrediently, and contragrediently, which express the whole "tangle" of ideas of the classical terminology which arise in connection with dual spaces and adjoint transformations (see Figure 5 ).

Oravas states in the Applied Mechanics Reviews(1) that the success of Einstein's relativity is unfortunate for engineering in that it has brought on a fashion of the so-called tensor analysis, which is a cabalistic method in the exteme. He says it is now being applied to all sorts of engineering problems regardless of whether the nature of such problems warrants the application of this abstract and intricate tool in mathematics. Oravas states further that Gibbs' dyadic analysis on the other hand avoids this unnecessary abstraction and keeps its feet on the ground by never losing sight of the vector--provided that the vector constitutes the most fundamental and physically conceivable concept in mechanics--since it enters into a dyadic in a natural way. Oravas says that Gibbs' great contribution has been all but neglected by engineers in his own country; however, its great pedagogical power in engineering mechanics has been amply demonstrated by many engineers in Europe, such as Professor Müller of Germany and Professor Lure'e of Russia.

Drew( ) presents some of his techniques utilized for teaching engineering graduate students in fluid mechanics, heat transmission, and the theory of diffusional processes. He states that the use of tensor analysis in the form of Gibbsian polyadics clears away the fog of multiple scalar partial differential equations in which the student otherwise finds himself without requiring him to learn a secret code.

When one looks at the cumbersome representation of a dyadic in the <u>nonion</u>[+] form, and all of the other verbose terminology developed around the dyadics as presented by Gibbs and Wills, one appreciates why mathematicians have not popularized these methods.

The conventional representation of the dyadic in the nonion form[+] is

$$\begin{aligned}\bar{\bar{T}} = {} & t_{11}\,\bar{b}_1\bar{b}_1 + t_{12}\,\bar{b}_1\bar{b}_2 + t_{13}\bar{b}_1\bar{b}_3 \\ & + t_{21}\,\bar{b}_2\bar{b}_1 + t_{22}\,\bar{b}_2\bar{b}_2 + t_{23}\,\bar{b}_2\bar{b}_3 \qquad \text{i-(1)} \\ & + t_{31}\,\bar{b}_3\bar{b}_1 + t_{32}\,\bar{b}_3\bar{b}_2 + t_{33}\,\bar{b}_3\bar{b}_3,\end{aligned}$$

where $t_{ij}$ are scalars and $\bar{b}_i$ are basal elements (base vectors).

Quite obviously when one considers the operations of dot products or cross products on other dyadics or vectors things become quite "messy". Also time differentiation of the dyadic as represented by Eq. i-(1) as well as a change of basis becomes very tedious.[≠]

The conventional matrix representation of a scalar quadratic form is

$$Q = (x_1, x_2, x_3) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \text{i-(2)}$$

which when multiplied out yields terms quadratic in the scalar variables $x_i$.

---

[+] Goldstein

[≠] In the modern terminology these mappings to higher and to lower dimensional vector spaces are discussed under the headings of tensoral product spaces.

Recognizing that the nonion form of the dyadic of Eq. 1-(1) is quadratic in the base vectors rather than the field elements, by analogy one may write

$$\bar{\bar{T}} = (\bar{b}_1, \bar{b}_2, \bar{b}_3) \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{31} & t_{33} \end{pmatrix} \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{pmatrix} = < \bar{b}\ T\ \bar{b} >, \qquad 1\text{-}(3)$$

where the symbol[+] $<$ is a row matrix and $>$ is a column matrix.

Equation 1-(3) is a $(1 \times 1)$ -one row, one column -element generated as a matrix product[≠].

The vector analog of Eq. 1-(1) is the Gibbsian vector represented as a linear combination of the base vectors, that is:

$$\bar{R} = y_1\bar{b}_1 + y_2\bar{b}_2 + y_3\bar{b}_3 \ . \qquad 1\text{-}(4)$$

Expressing Eq. 1-(4) as a matrix product and still as a $1 \times 1$ element, one obtains

$$R = (y_1, y_2, y_3) \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{pmatrix} = < y\ \bar{b} > = < \bar{b}\ y > \ . \qquad 1\text{-}(5)$$

In Eq. 1-(3) and Eq. 1-(5) the field elements are separated from the base elements and transformations may operate on either the scalar elements, the base elements or both. For example, the familiar product rule of time differentiation may be applied to the dyadic of Eq. 1-(3) yielding

$$\frac{d\bar{\bar{T}}}{dt} = < \dot{\bar{b}}\ T\ \bar{b} > + < \bar{b}\ \dot{T}\ \bar{b} > + < \bar{b}\ T\ \dot{\bar{b}} > \ . \qquad 1\text{-}(6)$$

Since $\bar{b} >$ forms a basis for the space a matrix of field elements exist such that

$$\dot{\bar{b}} > = S\ \bar{b} >, \qquad 1\text{-}(7)$$

that is, the velocity vectors $\dot{\bar{b}} >$ may be expressed in the $\bar{b} >$ basis. The elements of the matrix of Eq. 1-(7) are derived in the paper.

---

[+] Freidman

[≠] Familiar matrix product row by column rules hold.

The vector cross product has always been a bothersome thing as a multiplicative operator and has generated such bizarre ideas as "pseudo vectors" and "axial vectors".

The physics and engineering texts have clung to the mnemonic representation device for a cross product in a basis as

$$\bar{R}_1 \times \bar{R}_2 = \begin{vmatrix} \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \qquad i\text{-}(8)$$

which is neither an array of vectors nor of field elements. Equation i-(8) has very little analytical utility and higher order vector products on vectors in different bases become extremely awkward.

The classical matrix texts strip the vector of its true identity by disregarding the bases and representing only the scalar coordinates of the vector+. Thus one finds the vector cross product designated either as

$$\bar{R}_1 \times \bar{R}_2 = \begin{pmatrix} y_2 z_3 - y_3 z_2 \\ y_3 z_1 - y_1 z_3 \\ y_1 z_2 - y_2 z_1 \end{pmatrix} \qquad i\text{-}(9)$$

or occasionally≠ as

$$\bar{R}_1 \times \bar{R}_2 = \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}. \qquad i\text{-}(10)$$

Equations i-(9) and i-(10) are only valid for ortho-normal bases and proper transformations (no inversion of axes, etc.).

---

+. Halmos

≠ Heading

It is shown in the paper that the vector-cross product is equivalent to a dyadic dot product, hence many of the transformations occurring in classical vector analysis may be expressed as a dyadic operator mapping one vector $\bar{R}_1$ to a new vector $\bar{R}_2$, i.e.,

$$\bar{R}_2 = \bar{\bar{T}} \cdot \bar{R}_1, \qquad 1\text{-}(11)$$

where $\bar{\bar{T}}$ itself may be a function of a vector. Geometrically Eq. 1-(11) may be considered as mapping the vector $\bar{R}_1$ to a new vector $\bar{R}_2$ through a rotation about an axis normal to the plane of the two vectors and a stretching parallel to the final vector.

Transformations due to a change of basis may be obtained from the special case of Eq. 1-(11) when the dyadic is the unitary dyadic, for example,

$$\bar{R}_1 = \bar{\bar{U}} \cdot \bar{R}_1. \qquad 1\text{-}(12)$$

The unitary dyadic, (discussed in the paper) does not rotate the vector, but can be considered to rotate the basis.

If the transformation dyadic merely stretches the vector, that is maps the vector $\bar{R}_1$ to a parallel vector $\lambda \bar{R}_1$ as shown in Eq. 1-(13),

$$\lambda \bar{R}_1 = \bar{\bar{T}} \cdot \bar{R}_1, \qquad 1\text{-}(13)$$

one is led to the eigen-value, eigen-vector problem.

Many of the conventional texts discuss a change of basis by the introduction of the mnemonic device of Fig. (1)-1.

| | $\bar{r}_1$ | $\bar{r}_2$ | $\bar{r}_3$ |
|---|---|---|---|
| $\bar{b}_1$ | $b_{11}$ | $b_{12}$ | $b_{13}$ |
| $\bar{b}_2$ | $b_{21}$ | $b_{22}$ | $b_{23}$ |
| $\bar{b}_3$ | $b_{31}$ | $b_{32}$ | $b_{33}$ |

Fig. (1)-1 Mnemonic Device for Change of Basis.

Most of the modern texts on linear vector spaces or dynamics as well as the aero-space industries technical reports go one step farther and express the transformation in matrix form, i.e.,

$$\begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} \qquad 1\text{-}(14)$$

Using Eq. 1-(14) one can achieve coordinate transformations due to a change of basis by algebraic substitution as discussed in the paper.

A literature search has revealed that very few authors do much more than coordinate transformations with the expression of Eq. 1-(14). In actuality Eq. 1-(14) exhibits very nice analytical properties as is to be shown in the paper. Equation 1-(15) is a special case of the representation of a 1 x 1 dyadic as an ordered - n - tuple of vectors, the analog of the representation of a 1 x 1 vector as an ordered n- tuple of scalars.

It is felt that the matrix representation techniques developed in this paper will make the classical vectors and dynamics of Gibbs more palatable to modern mathematicians. Above all it is hoped that the techniques presented in the paper will prove useful to the applied theoreticians performing transformations in moving non-orthogonal bases with computer mechanizations in mind. The representation techniques, the algebraic substitution - like transformation rules as presented here seem so simple and natural that surely other workers must have utilized them accordingly However the authors have not found a single paper or text that attempted to unify by showing the exact mappings which take vectors to n- tuples of scalars, dyadics to n- tuples of vectors or to square matrices of scalars Perhaps this is due in part to the fact that very few modern texts on linear spaces and modern algebra are interested in dynamical mathematical systems in which the base vectors are varying in time and space.

The direct mapping of the vector-dyadic equations of dynamical systems to matrix of scalar equations suggests the direct mechanization of the matrix equations on computers. Matrix analog computers should be designed to handle the vast number of flight system simulation studies required throughout the country. The Euler Angle Transformation Computer (E.A.T) designed for the Flight Simulation Laboratory at White Sands Missile Range is a first step in this direction.([illegible])

## VECTORS AS PRODUCTS OF A MATRIX OF SCALARS AND A MATRIX OF BASE VECTORS

A set of any abstract elements that satisfy the postulates of a vector space shown in Figure (1) are vectors. The "barred" vectors (e.g., $\bar{R}$) of physics and engineering are the most familiar to many technical people.

One of the most useful models of an abstract vector space has been the geometrical model of three space. The set of directed lines from a reference point as origin to every point in three space and the parallelogram rule of combination constitutes a vector space. The set of ordered n-tuples with the usual addition operator is another example of a vector space.

The term vector in the remainder of the report is reserved for the classical "barred" vector unless otherwise noted.

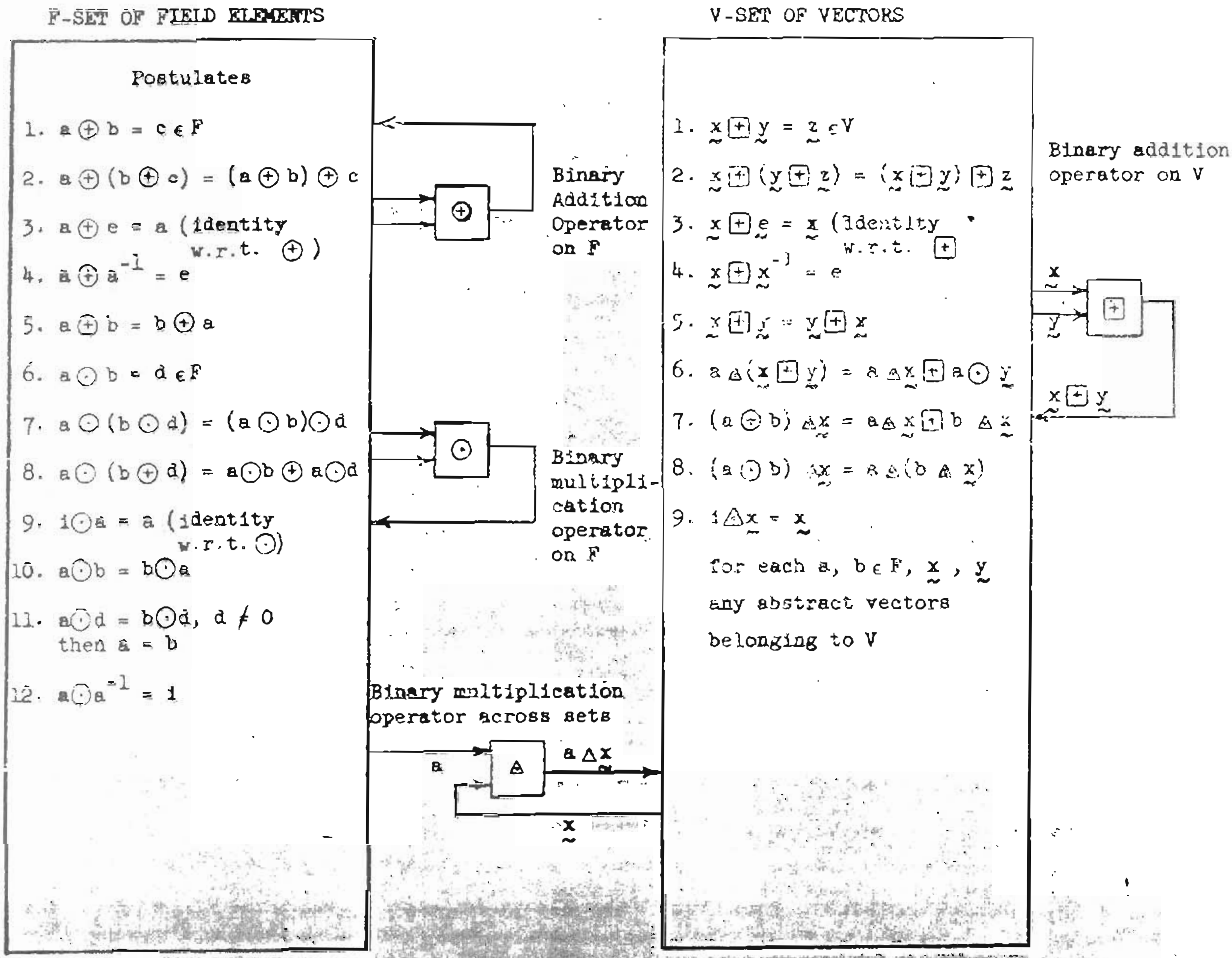
Let $( \bar{f}_1, \bar{f}_2, \bar{f}_3 )$ be a set of base vectors+ not necessarily orthonormal, (O.N.)≠ belonging to a linear vector space over the field of real numbers. Then geometrically the position vector $\bar{R}$ of every point with respect to an origin can be considered as

$$\bar{R} = y_1^f \bar{f}_1 + y_2^f \bar{f}_2 + y_3^f \bar{f}_3 . \qquad (1)$$

---

\+ If vectors $\bar{f}_1, \bar{f}_2, \ldots \bar{f}_n$ in a vector space V are linearly independent and span V, they are said to form a basis of V. (throughout this report the term "reference frame" is used synonymously with the term "basis".) A vector space V is said to be spanned by vectors $\bar{a}_1, \bar{a}_2, \ldots \bar{a}_t$ if (1) $(\bar{a}_1, \ldots \bar{a}_t)$ lie in V and (2) every vector in V is a linear combination of $\bar{a}_1, \bar{a}_2, \ldots \bar{a}_t$. A finite number of vectors $\bar{a}_1, \bar{a}_2, \ldots \bar{a}_n$ are said to be linearly dependent w.r.t. the field of real numbers if there exist real numbers $a_1, a_2, \ldots a_n$ not all zero such that $a_1\bar{a}_1 + a_2\bar{a}_2 + \ldots + a_n\bar{a}_n = \bar{0}$. The vectors $a_1 \ldots a_n$ are said to be linearly independent if they are not linearily dependent.

≠ Mutually orthogonal unit vectors.



Postulates

1. $a \oplus b = c \in F$
2. $a \oplus (b \oplus c) = (a \oplus b) \oplus c$
3. $a \oplus e = a$ (identity w.r.t. $\oplus$)
4. $a \oplus a^{-1} = e$
5. $a \oplus b = b \oplus a$
6. $a \odot b = d \in F$
7. $a \odot (b \odot d) = (a \odot b) \odot d$
8. $a \odot (b \oplus d) = a \odot b \oplus a \odot d$
9. $1 \odot a = a$ (identity w.r.t. $\odot$)
10. $a \odot b = b \odot a$
11. $a \odot d = b \odot d,\ d \neq 0$ then $a = b$
12. $a \odot a^{-1} = 1$

1. $\underset{\sim}{x} \boxplus \underset{\sim}{y} = \underset{\sim}{z} \in V$
2. $\underset{\sim}{x} \boxplus (\underset{\sim}{y} \boxplus \underset{\sim}{z}) = (\underset{\sim}{x} \boxplus \underset{\sim}{y}) \boxplus \underset{\sim}{z}$
3. $\underset{\sim}{x} \boxplus \underset{\sim}{e} = \underset{\sim}{x}$ (identity w.r.t. $\boxplus$
4. $\underset{\sim}{x} \boxplus \underset{\sim}{x}^{-1} = e$
5. $\underset{\sim}{x} \boxplus \underset{\sim}{y} = \underset{\sim}{y} \boxplus \underset{\sim}{x}$
6. $a \triangle (\underset{\sim}{x} \boxplus \underset{\sim}{y}) = a \triangle \underset{\sim}{x} \boxplus a \odot \underset{\sim}{y}$
7. $(a \oplus b) \triangle \underset{\sim}{x} = a \triangle \underset{\sim}{x} \boxplus b \triangle \underset{\sim}{x}$
8. $(a \odot b) \triangle \underset{\sim}{x} = a \triangle (b \triangle \underset{\sim}{x})$
9. $1 \triangle \underset{\sim}{x} = \underset{\sim}{x}$

for each a, $b \in F$, $\underset{\sim}{x}$, $\underset{\sim}{y}$ any abstract vectors belonging to V

Figure 1
Block Diagram of a Vector Space or Linear Space

As shown in Figure (2), the superscript f on the coordinates $y_i^f$ i = 1, 2, 3, indicate the basis to which the coordinates are referenced.

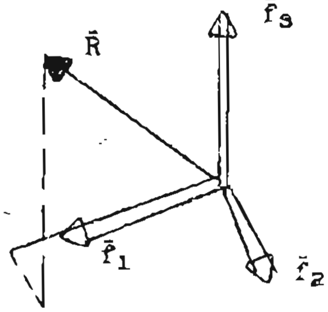


Figure 2

Components of Position Vector $\bar{R}$ in a Non-Orthogonal Reference Frame

Equation (1) may be rewritten as

$$\bar{R} = \begin{pmatrix} y_1^f, & y_2^f, & y_3^f \end{pmatrix} \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{pmatrix} = \langle y^f \ \bar{f} \rangle = \langle \bar{f} \ y^f \rangle \quad (2)$$

since scalar multiplication of a vector commutes.

The right hand side of equation (2) will be referred to as the vector-matrix representation of the vector $\bar{R}$. The elements of a matrix (defined as an array of abstract elements) are by no means restricted to be scalars. In fact allowing the elements to be scalars, vectors, dyadics, n-adics, and differential operators leads to very interesting results.

As indicated by Eq. (2) the symbol $>$ means a column (3x1)--read three row, one column--matrix. A column matrix of scalars and a column matrix of vectors are designated respectively as

$$\begin{pmatrix} y_1^f \\ y_2^f \\ y_3^f \end{pmatrix} = y^f> \tag{3}$$

and

$$\begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{pmatrix} = \bar{f}> \tag{4}$$

The conventional rules of row by column matrix multiplication are employed.

Consider an abstract binary operator on vectors, $\vartheta$, then in the $\bar{f}>$ basis

$$\bar{R}_1 \;\vartheta\; \bar{R}_2 = <y\,\bar{f}> \vartheta <\bar{f}\,z> \tag{5}$$

where $<y$ and $z>$ are the scalar coordinates of the two vectors. Multiplying the two matrices of Eq. (5) one obtains

$$\bar{f}> \vartheta <\bar{f} = \begin{pmatrix} \bar{f}_1 \vartheta \bar{f}_1 & \bar{f}_1 \vartheta \bar{f}_2 & \bar{f}_1 \vartheta \bar{f}_3 \\ \bar{f}_2 \vartheta \bar{f}_2 & \bar{f}_2 \vartheta \bar{f}_2 & \bar{f}_2 \vartheta \bar{f}_3 \\ \bar{f}_3 \vartheta \bar{f}_1 & \bar{f}_3 \vartheta \bar{f}_2 & \bar{f}_3 \vartheta \bar{f}_3 \end{pmatrix} \tag{6}$$

Multiplication of a row matrix by a column matrix must also make sense, in fact,

$$\text{trace } \bar{f}>\vartheta<\bar{f} = <\bar{f} \vartheta \bar{f}> = \bar{f}_1 \vartheta \bar{f}_1 + \bar{f}_2 \vartheta \bar{f}_2 + \bar{f}_3 \vartheta \bar{f}_3 \tag{7}$$

The three operations considered in the paper are:

(1) The dyadic product of two vectors

(2) The scalar product of two vectors

(3) The vector cross product of two vectors.[+]

These three operations are also applied to matrices of vectors as indicated by Eq. (6) and Eq. (7).

The scalar dot product of the physicist is an example of an inner-product that is a binary operator which maps two n-dimensional vectors to a one dimensional vector - a scalar. Figure (3) is a block diagram of an abstract mathematical system called an inner-product space.

The three operations enumerated above map the vectors into dyadics, scalars and vectors respectively as shown in Figure (4). The three operations applied to n-tuples of vectors as shown in Eq. (6) map to square matrices of dyadics, square matrices of vectors and square matrices of field elements respectively.

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[+] The vector cross product has meaning only in three space, however the analog can be obtained for n-space.

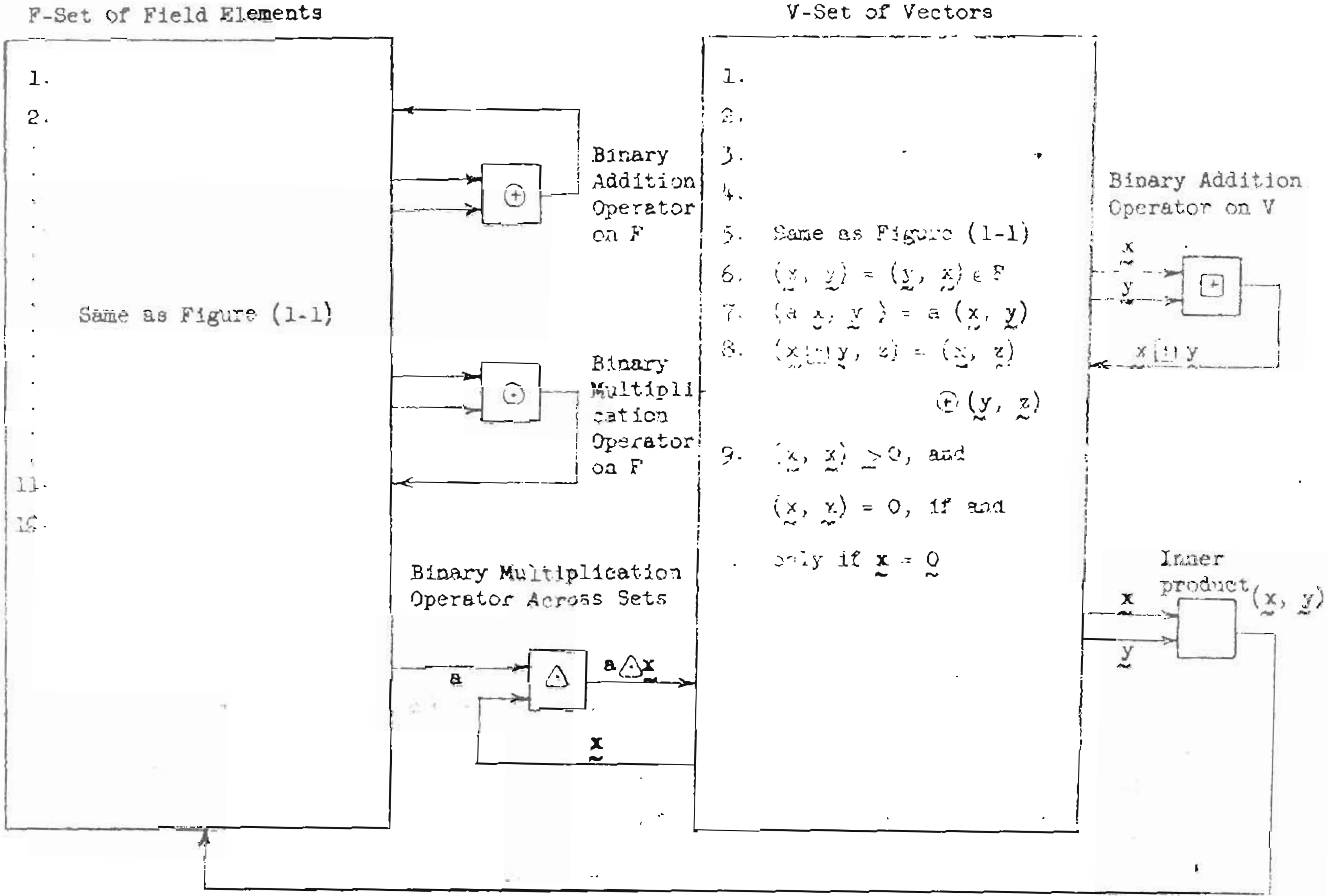

Figure 3
Block Diagram of Inner Product Space

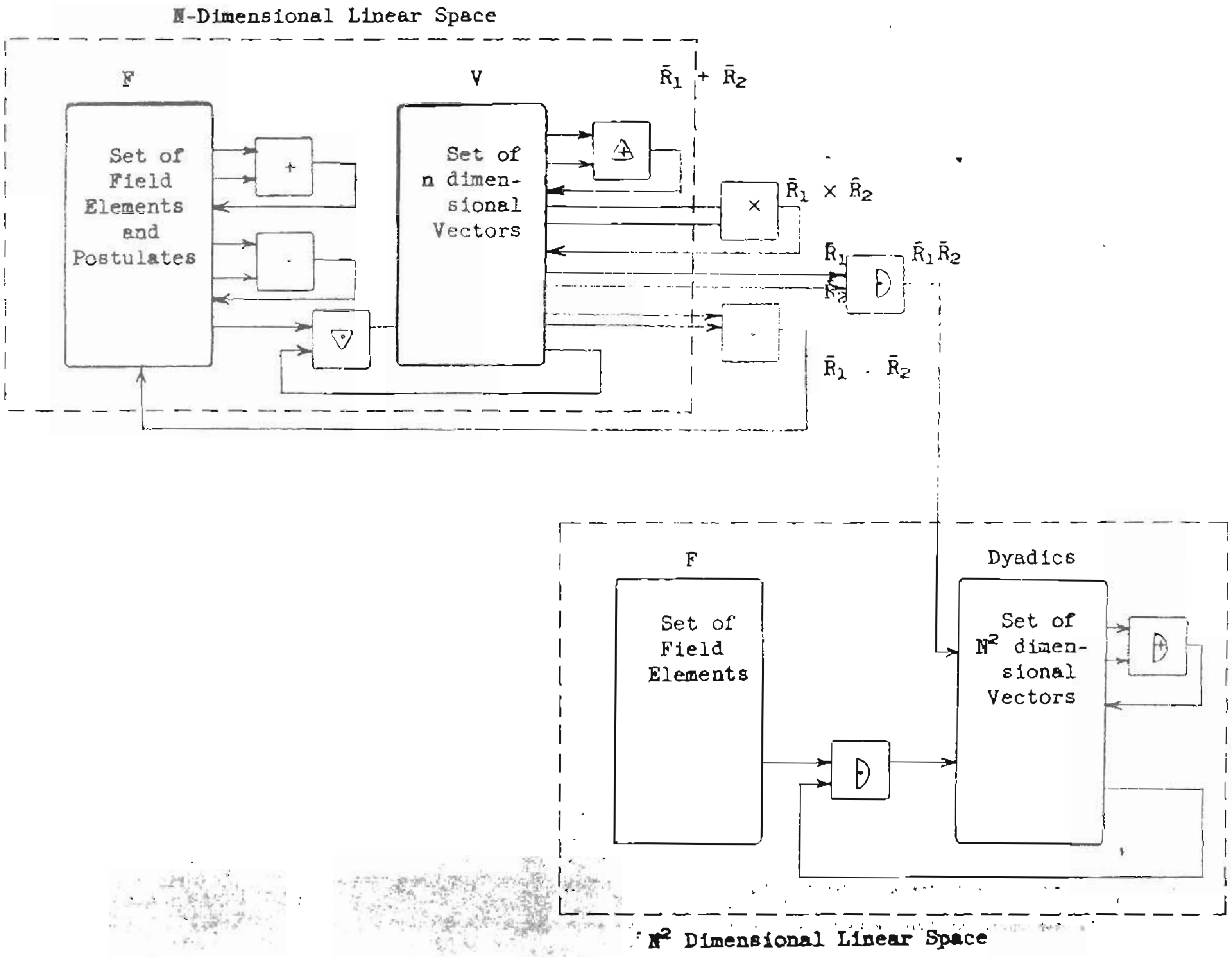

Figure 4
Mappings to one Dimensional, N Dimensional, and $N^2$ Dimensional Linear Spaces

Since any matrix can be expressed as a symmetric plus a skew symmetric matrix Eq. (6) under the dyadic product can be written as

$$\bar{f} > < \bar{f} = \frac{1}{2} \begin{bmatrix} 2\bar{f}_1\bar{f}_1 & \bar{f}_1\bar{f}_2 + \bar{f}_2\bar{f}_1 & \bar{f}_1\bar{f}_3 + \bar{f}_3\bar{f}_1 \\ \bar{f}_2\bar{f}_1 + \bar{f}_1\bar{f}_2 & 2\bar{f}_2\bar{f}_2 & \bar{f}_2\bar{f}_3 + \bar{f}_3\bar{f}_2 \\ \bar{f}_3\bar{f}_1 + \bar{f}_1\bar{f}_3 & \bar{f}_3\bar{f}_2 + \bar{f}_2\bar{f}_3 & 2\bar{f}_3\bar{f}_3 \end{bmatrix}$$

$$+ \frac{1}{2} \begin{bmatrix} \bar{\bar{0}} & \bar{f}_1\bar{f}_2 - \bar{f}_2\bar{f}_1 & \bar{f}_1\bar{f}_3 - \bar{f}_3\bar{f}_1 \\ \bar{f}_2\bar{f}_1 - \bar{f}_1\bar{f}_2 & \bar{\bar{0}} & \bar{f}_3\bar{f}_1 - \bar{f}_1\bar{f}_3 \\ \bar{f}_3\bar{f}_1 - \bar{f}_1\bar{f}_3 & \bar{f}_3\bar{f}_2 - \bar{f}_2\bar{f}_3 & \bar{\bar{0}} \end{bmatrix} \quad (8)$$

Under the dot product (a commutative operator) the symmetrix matrix remains and the skew-symmetric matrix vanish. Under the cross product the skew symmetric matrix remains and the symmetric matrix vanishes, i.e.,

$$\bar{f} >\cdot< \bar{f} = \begin{bmatrix} \bar{f}_1 \cdot \bar{f}_1 & \bar{f}_1 \cdot \bar{f} & \bar{f}_1 \cdot \bar{f}_3 \\ f_2 \cdot \bar{f}_1 & \bar{f}_2 \cdot \bar{f}_2 & \bar{f}_2 \cdot \bar{f}_3 \\ \bar{f}_3 \cdot \bar{f}_1 & \bar{f}_3 \cdot \bar{f}_2 & \bar{f}_3 \cdot \bar{f}_3 \end{bmatrix} \quad (9)$$

and

$$\bar{f} > \times < \bar{f} = \begin{bmatrix} \bar{0} & \bar{f}_1 \times \bar{f}_2 & -\bar{f}_3 \times \bar{f}_1 \\ -\bar{f}_1 \times \bar{f}_2 & \bar{0} & \bar{f}_2 \times \bar{f}_3 \\ \bar{f}_3 \times \bar{f}_1 & -\bar{f}_2 \times \bar{f}_3 & \bar{0} \end{bmatrix} . \quad (10)$$

If the basis is O.N., then

$$\bar{f} > \times < \bar{f} = \begin{pmatrix} \bar{0} & \bar{f}_3 & -\bar{f}_2 \\ -\bar{f}_3 & \bar{0} & \bar{f}_1 \\ \bar{f}_2 & -\bar{f}_1 & \bar{0} \end{pmatrix} = S_{\bar{f}} , \quad (11)$$

where $S_{\bar{f}}$ is the skew-symmetric matrix of base vectors.

Utilizing Eq. (8), Eq. (9), and Eq. (10), in Eq. (5) one obtains for the three multiplications

$$\bar{R}_1\bar{R}_2 = <\bar{f}y><z\bar{f}> = <y\bar{f}><\bar{f}z> \tag{12}$$

$$\bar{R}_1\cdot\bar{R}_2 = <y\bar{f}>\cdot<\bar{f}z> \tag{13}$$

$$\bar{R}_1\times\bar{R}_2 = <y\bar{f}>\times<\bar{f}z> . \tag{14}$$

Equation (12) is a dyadic form bilinear in the field elements. Equation (13) is a scalar form bilinear in the field elements. Equation (14) is a vector form bilinear in the field elements. If $\bar{R}_1$ and $\bar{R}_2$ are the same vector then $\bar{R}$ has the familiar quadratic forms.

Mapping the vector to a scalar. - The scalar product of the vector $\bar{R}$ with any of the three base vectors $\bar{b}_j$ is the scalar magnitude of the projection of $\bar{R}$ onto that axis. The conventional form is

$$\bar{R}\cdot\bar{b}_j = |\bar{R}|\,|\bar{b}_j|\cos(\bar{R},\bar{b}_j) \tag{15}$$

$$= \sum_{i=1}^{3} y_i\bar{b}_i\cdot\bar{b}_j = y_1\bar{b}_1\cdot\bar{b}_j + y_2\bar{b}_2\cdot\bar{b}_j + y_3\bar{b}_3\cdot\bar{b}_j \tag{16}$$

where, in general, for a non-orthonormal basis

$$\bar{b}_i\cdot\bar{b}_j \neq 0 \tag{17}$$

The reciprocal base vectors $\bar{b}^* >$ have the property

$$\bar{b}_i\cdot\bar{b}_j^* = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \tag{18}$$

Operating on $\bar{R}$ with $\bar{b}_j^*$ one obtains

$$\bar{R}\cdot\bar{b}_j^* = y_j \tag{19}$$

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thus the projection of $\bar{R}$ onto the reciprocal base vectors yields the scalar coordinates of $\bar{R}$ in the $\bar{b}>$ basis.

Mapping the vector to a row or column matrix of scalars. - The vector of Eq. (7) may be mapped directly to a matrix of scalars by operating with $\bar{b}>$, hence

$$\bar{b}> \cdot \bar{R} = \begin{pmatrix} \bar{b}_1 \cdot \bar{R} \\ \bar{b}_2 \cdot \bar{R} \\ \bar{b}_3 \cdot \bar{R} \end{pmatrix} = \bar{b}> \cdot < \bar{b}\ y> = M_{bb}y>, \quad (20)$$

where

$$M_{bb} = \bar{b}> \cdot <\bar{b} = \begin{pmatrix} \bar{b}_1 \cdot \bar{b}_1 & \bar{b}_1 \cdot \bar{b}_2 & \bar{b}_1 \cdot \bar{b}_3 \\ \bar{b}_2 \cdot \bar{b}_1 & \bar{b}_2 \cdot \bar{b}_2 & \bar{b}_2 \cdot \bar{b}_3 \\ \bar{b}_3 \cdot \bar{b}_1 & \bar{b}_3 \cdot \bar{b}_2 & \bar{b}_3 \cdot \bar{b}_3 \end{pmatrix}. \quad (21)$$

The matrix $M_{bb}$ is the metric-matrix of the $\bar{b}>$ space and its elements are defined in terms of the scaler dot product, hence it is a real symmetric matrix.

If $\bar{R}$ is projected onto the three reciprocal base vectors, one obtains

$$\bar{b}^*> \cdot \bar{R} = \begin{pmatrix} \bar{b}_1^* \cdot \bar{R} \\ \bar{b}_2^* \cdot \bar{R} \\ \bar{b}_3^* \cdot \bar{R} \end{pmatrix} = \bar{b}^*> \cdot < \bar{b}\ y> = y> \quad (22)$$

since

$$\bar{b}^*> \cdot < \bar{b} = \left[\ \bar{b}_i^* \cdot \bar{b}_j\ \right] = I. \quad (23)$$

If $\bar{b}>$ is an O.N. basis, it is self-reciprocal, that is

$$\bar{b}^*> = \bar{b}>. \quad (24)$$

Scalar product between two vectors. - The scalar product between two vectors $\bar{R}_1$ and $\bar{R}_2$ where

$$\bar{R}_2 = < \bar{b} Z > \tag{25}$$

is

$$\bar{R}_1 \cdot \bar{R}_2 = < y \bar{b} > \quad . \quad < \bar{b} Z > = < y M_{bb} Z > . \tag{26}$$

The bilinear scalar form of Eq. (26) contains nine terms, conventionally written as

$$\bar{R}_1 \cdot \bar{R}_2 = \sum_{i=1}^{3} \sum_{j=1}^{3} \bar{b}_i \cdot \bar{b}_j y_i Z_j \tag{27}$$

If the vector $\bar{R}_2$ is expressed in the reciprocal basis, then

$$\bar{R}_1 \cdot \bar{R}_2 = < y \bar{b} > \quad . \quad < \bar{b}^* Z^* > \quad = < y Z^* > , \tag{28}$$

which is a summation of only three terms, i.e.,

$$\bar{R}_1 \cdot \bar{R}_2 = y_1 Z_1^* + y_2 Z_2^* + y_3 Z_3^* . \tag{29}$$

Mapping a dyadic to a square matrix of scalars. - An arbitrary dyadic $\bar{\bar{D}}$ in a given basis may be written as

$$\bar{\bar{D}} = < \bar{b} D \bar{b} > \tag{30}$$

where D is a square matrix of scalars (variables or constants). The dyadic of Eq. (30) may be mapped to a matrix of scalars by operating with $\bar{b} > .$ on the left and $. < \bar{b}$ on the right, hence

$$\bar{b} > . \bar{\bar{D}} . < \bar{b} = \bar{b} > . < \bar{b} D \bar{b} > . < \bar{b} = M_{bb} D M_{bb} . \tag{31}$$

The matrix D may be recovered analogous to Eq. (22) by operating with the reciprocal base vectors, i.e.,

$$\bar{b}^* > . \bar{\bar{D}} . < \bar{b}^* = \bar{b}^* > . < \bar{b} D \bar{b} > . < \bar{b}^* = D . \tag{32}$$

As an example consider one of the nine base dyads from the matrix of dyads

$$\bar{b}> <\bar{b} = \left[ \bar{b}_i \bar{b}_j \right] = \left[ \bar{b}_{ij} \right] \tag{33}$$

say $\bar{b}_{12}$. In the vector matrix form

$$\bar{b}_1 = (\bar{b}_1, \ \bar{b}_2, \ \bar{b}_3) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{34}$$

and

$$\bar{b}_2 = (\bar{b}_1, \ \bar{b}_2, \ \bar{b}_3) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{35}$$

hence

$$\bar{b}_1 \bar{b}_2 = <\bar{b} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (0, 1, 0)\ \bar{b} \quad = <\bar{b} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bar{b}> . \tag{36}$$

Mapping the base dyad $\bar{b}_{12}$ to the matrix of scalars, one obtains

$$\bar{b}^* > . \ \bar{b}_{12} \ . < \bar{b}^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv E_{12} \tag{37}$$

which is one of the nine basal elements for the set of (3x3) matrices whose elements are real scalars.

It is well known in the theory of linear vector spaces that the set of all nxn matrices form a vector space. The nine matrices of which equation (37) is an example form a basis for such a nine dimensional space, since any (3x3) matrix may be expressed as

$$A = \left[ a_{ij} \right] = (a_{11}, \ a_{12}, \ a_{13} \ . \ . \ . \ a_{33}) \left( \begin{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix} \right) \tag{38}$$

or in matrix form

$$A = < a \; E > \tag{39}$$

where $E>$ is the column matrix of base matrices having matrix elements $E_{ij}$.

*The unitary dyadic*. - The trace of the square matrix of dyads is the sum of the main diagonal terms, i.e.,

$$\text{trace} \; \bar{b}>< \bar{b} = < \bar{b} \; \bar{b}> = \bar{b}_1\bar{b}_1 + \bar{b}_2\bar{b}_2 + \bar{b}_3\bar{b}_3 . \tag{40}$$

The unitary dyadic $\bar{\bar{U}}$ is

$$\text{trace} \; \bar{b}>< \bar{b}* = < \bar{b} \; \bar{b}*> = < \bar{b}* \; \bar{b}> \tag{41}$$

or

$$\bar{\bar{U}} = \bar{b}_1\bar{b}_1* + \bar{b}_2\bar{b}_2* + \bar{b}_3\bar{b}_3* \tag{42}$$

and similarly in the $\bar{r}>$ basis

$$\bar{\bar{U}} = < \bar{r} \; \bar{r}*> = < \bar{b} \; \bar{b}*> . \tag{43}$$

*The scalar product between the unitary dyadic and a vector*. - The scalar product between the unitary dyadic and a vector $\bar{R}$ is

$$\bar{R} \cdot \bar{\bar{U}} = < y \; \bar{b} > . < \bar{b}*\bar{b} > = < y \; \bar{b} > = \bar{R} = \bar{\bar{U}} . \bar{R} \tag{44}$$

*The scalar product between the unitary dyadic and a dyadic*. - The scalar product between the unitary dyadic and a dyadic $\bar{\bar{D}}$ is

$$\bar{\bar{U}} . \bar{\bar{D}} = < \bar{b} \; \bar{b}* > . < \bar{b} \; D \; \bar{b}> = < \bar{b} \; D \; \bar{b} > = \bar{\bar{D}} . \tag{45}$$

It is easily shown that commutativity holds with respect to the scalar product of a dyadic or a vector with the unitary dyadic.

Transformations on the base vectors. - Familiarity with the reciprocal base vectors of the previous sections was assumed. By the definition of a basis a set of scalars exist such that

$$\bar{b}^{*}> \; = D_{b^{*}b}\,\bar{b}> . \tag{46}$$

The scalar elements of the transformation matrix $D_{b^{*}b}$ may be evaluated through the concept of the scalar product, hence

$$\bar{b}^{*}> \cdot <\bar{b}^{*} = D_{b^{*}b}\,\bar{b}> \cdot <\bar{b}^{*} = D_{b^{*}b}, \tag{47}$$

if one can find a basis $\bar{b}^{*}>$ oriented w.r.t. $\bar{b}>$ such that

$$\bar{b}> \cdot <\bar{b}^{*} = I. \tag{48}$$

The basis having the property of Eq. (48) is called the reciprocal basis.

Operating on Eq. (46) with $\cdot <\bar{b}$ one obtains

$$I = D_{b^{*}b}\,M_{bb}, \tag{49}$$

or by Eq. (47)

$$I = M_{b^{*}b^{*}}\,M_{bb}, \tag{50}$$

hence the metric matrix of the $\bar{b}>$ basis and its reciprocal basis are inversely related, i.e.,

$$M_{b^{*}b^{*}} = M_{bb}^{-1} = D_{b^{*}b}. \tag{51}$$

Thus, by Eq. (46) and (51) the inverse of the metric matrix of the $\bar{b}>$ space maps the base vectors $b>$ to the reciprocal base vectors $\bar{b}^{*}>$ which are the basis for the dual space, i.e.,

$$\bar{b}^{*}> \; = M_{bb}^{-1}\,\bar{b}> . \tag{52}$$

Equation (50) may be derived directly from the unitary dyadic, hence

$$\bar{\bar{U}} \cdot \bar{\bar{U}} = \bar{\bar{U}}, \tag{53}$$

and in the $\bar{b} >$ basis[+]

$$< \bar{b}* \bar{b} > \cdot < \bar{b} \bar{b}* > = < \bar{b}* M_{bb} \bar{b}* > = < \bar{b} \bar{b}* > . \tag{54}$$

Operating on Eq. (54) with $\bar{b}* >$ · on the left and · $< \bar{b}$ on the right,

$$M_{b*b*} M_{bb} = I. \tag{55}$$

The scalar elements of a transformation matrix which maps the base vectors $\bar{b}>$ to an arbitary basis $\bar{r}>$ may be found in a similar manner. If $\bar{r} >$ is a basis, then a matrix of scalars exist such that

$$\bar{r} > = D_{rb} \bar{b} > , \tag{56}$$

hence

$$\bar{r} > \cdot < \bar{b}* = M_{rb*} = D_{rb}. \tag{57}$$

Expressing the unitary dyadic in the $\bar{b} >$ and the $\bar{r} >$ basis

$$\bar{\bar{U}} \cdot \bar{\bar{U}} = < \bar{b} \bar{b}* > \cdot < \bar{r} \bar{r}* > = < \bar{b} \bar{b}* > , \tag{58}$$

and operating on Eq. (58) with $\bar{b}* >$ · on the left and · $< \bar{b}$ on the right

$$\bar{b}* > \cdot \bar{\bar{U}} \cdot \bar{\bar{U}} < \bar{b} = M_{b*r} M_{r*b} = I. \tag{59}$$

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[+] In general the transpose of a dyadic say $< \bar{b} \bar{r} >$ is $< \bar{r} \bar{b} >$ and the expressions are not equal, however the unitary dyadic has this special property.

Transposing Eq. (59) one obtains

$$M_{br*} M_{rb*} = 1, \tag{60}$$

since

$$M'_{r*b} = (\bar{r}* \cdot \cdot < \bar{b})' = \bar{b} > \cdot < \bar{r}* . \tag{61}$$

By similar manipulations it can be shown that

$$M_{br} M_{r*b*} = 1 = M_{rr} M_{r*r*} . \tag{62}$$

The reciprocity relationships between the four sets of base vectors which naturally arise when considering two arbitrary bases $\bar{r} >$ and $\bar{b} >$ are given by Eqs (57), (59), (60), and (62) and shown in Figure (5).

Change of basis on a vector by substitution. - Let the set of vectors $(\bar{r}_1, \bar{r}_2, \bar{r}_3)$ be a basis different from $\bar{b} >$ and $\bar{b}* >$, then by the definition of a basis there exists scalars such that

$$\bar{r} > = D_{rb}\, \bar{b} > = D_{rb*}\, \bar{b}* > , \tag{63}$$

also

$$\bar{b} > = D_{br}\, \bar{r} > = D_{br*}\, \bar{r}* > . \tag{64}$$

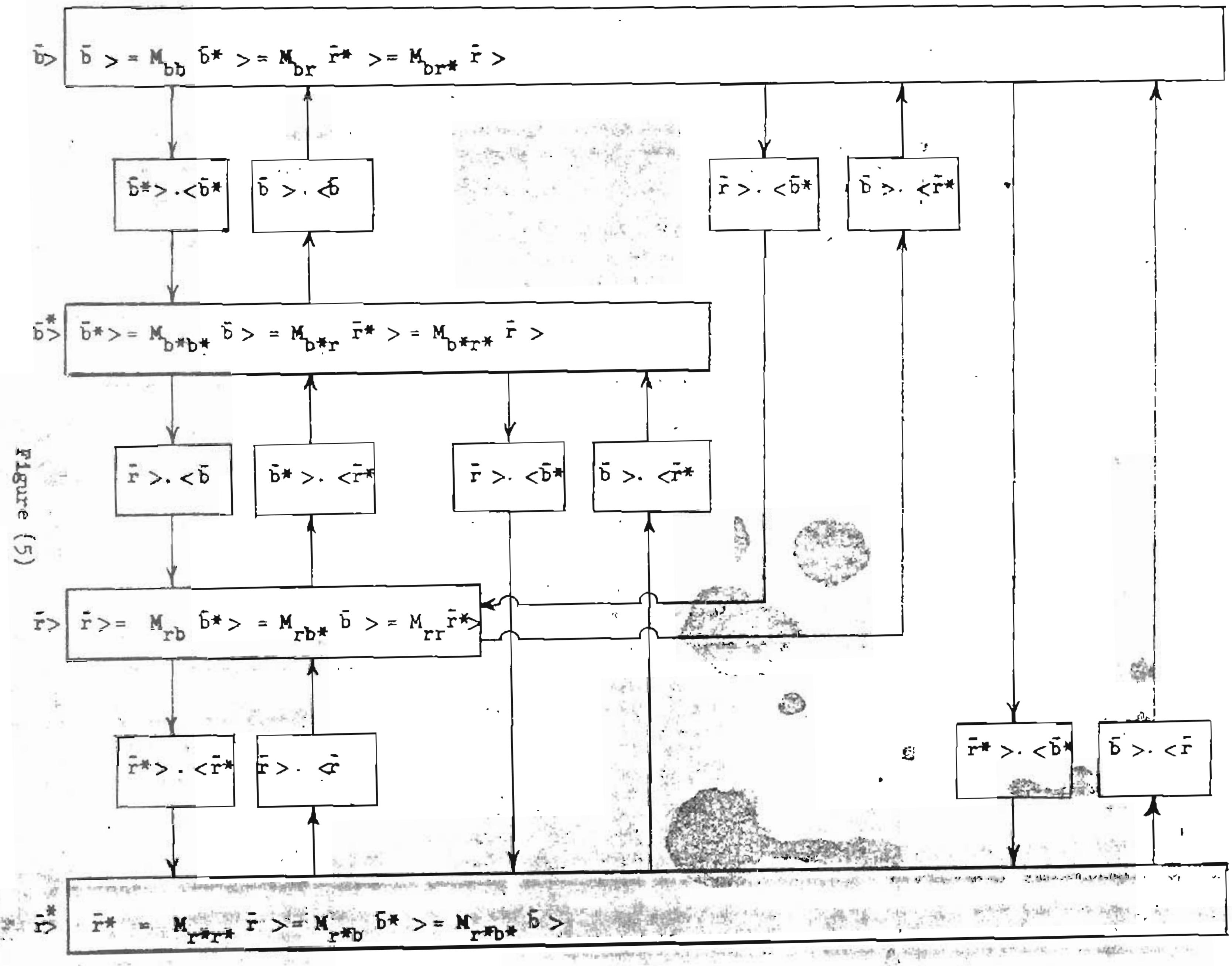
Utilizing the first equality of Eq. (64) in Eq. (2),

$$\bar{R} = < y\, \bar{b} > = < y\, D_{br}\, \bar{r} > = < y^{r}\, \bar{r} > \tag{65}$$

where $< y^{r}$ are the scalar coordinates of the vector $\bar{R}$ in the $\bar{r} >$ basis.

Operating on Eq (65) with $\cdot < r*$

$$< y^{r} = < y D_{br} . \tag{66}$$



Transformations on two arbitrary sets of base vectors and their reciprocal bases

Figure (5)

Equation (66) may be expressed as a column rather than a row by transposing, hence,

$$y^{r} > = D'_{br}\ y > , \tag{67}$$

which gives rise to the often discussed phenomena of "turning around of indices".(1)+

Change of basis on a vector by unitary dyadic operation. - The transformation achieved by algebraic substitution of Eq. (65) may be achieved by taking the scalar product of the unitary dyadic in the r > space and the vector in the $\bar{b} >$ space, hence

$$\bar{R} \cdot \bar{\bar{U}} = < y\ \bar{b} > . < \bar{r}^{*}\bar{r} > = < y\ M_{br*}\ \bar{r} > . \tag{68}$$

Since the vector $\bar{R}$ in the $\bar{r}$ basis is by Eq. (67)

$$\bar{R} = \bar{R} \ . \ \bar{\bar{U}} = < y^{r}\ \bar{r} > = < y\ M_{br*}\ \bar{r} > \tag{69}$$

the matrix of scalars may be obtained by operating with . $<\bar{r}'$, hence

$$\bar{R} \ . \ < \bar{r}^{*} = < y^{r} \quad = < y\ M_{br*} \tag{70}$$

Change of basis on a dyadic. - A change of basis on a dyadic may be achieved by direct algebraic substitution or by the unitary dyadic operation. If the transformation on the basis is

$$\bar{b} > = D_{br}\ \bar{r} > \tag{71}$$

the transpose is

$$< \bar{b} = < \bar{r}\ D'_{br} , \tag{72}$$

hence by substitution

$$\bar{\bar{D}} = < \bar{b}\ D\ \bar{b} > = < \bar{r}\ D'_{br}\ D\ D_{br}\ \bar{r} > , = < \bar{r}\ D^{r}\ \bar{r} > \tag{73}$$

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+(1) Halmos, P. 66.

and

$$\bar{r}^{*} > \cdot \bar{\bar{D}} \cdot < \bar{r}^{*} = D'_{br} \, D \, D_{br} = D^{r}. \qquad (74)$$

Transformation by the unitary dyadic operation yields

$$\bar{\bar{U}} \cdot \bar{\bar{D}} \cdot \bar{\bar{U}} = < \bar{r} \, \bar{r}^{*} > \cdot < \bar{b} \, \bar{D} \, \bar{b} > \cdot < \bar{r}^{*} \, \bar{r} > =$$

$$< \bar{r} \, M'_{br*} \, D \, M_{br*} \, \bar{r} > \qquad (75)$$

A matrix transformation of the form of Eq. (74) is called a congruent-transformation. If the dyadic $\bar{\bar{D}}$ is expressed in a particular mixed basis, that is a basis $\bar{b}>$ and its reciprocal basis and the change of basis is to $\bar{r} >$ and $\bar{r}^{*} >$, then a similarity transformation results.

<u>Transformations on the column or row matrix of scalars resulting from a change of basis.</u> - The vector $\bar{R}$ may be mapped directly to a matrix of scalars in any one of the four bases under discussion, i.e.,

$$\bar{R} = < \bar{b} \, y > = < \bar{b}^{*} \, y^{*} > = < \bar{r} \, y^{r} > = < \bar{r}^{*} \, y^{r*} > \qquad (76)$$

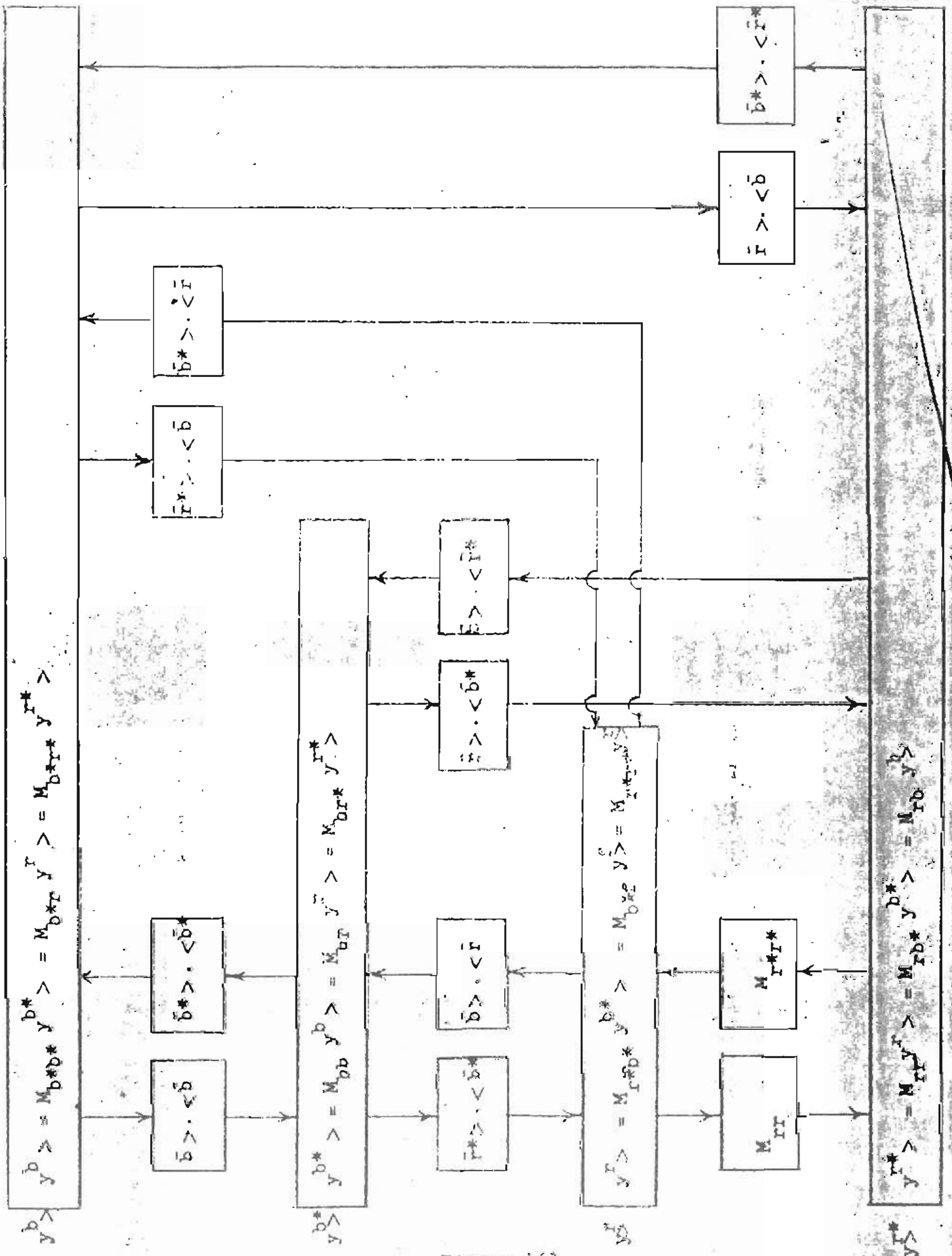
Operating on $\bar{R}$ with $\bar{b} > \cdot$ maps to a matrix of scalar equations, the projections of $\bar{R}$ onto $\bar{b}_i$ axes,

$$\bar{b} > \cdot \bar{R} = M_{bb} \, y > = y^{*} > = M_{br} \, y^{r} > = M_{br*} \, y^{r*} > . \qquad (77)$$

Operating on Eq. (77) with $\bar{b}^{*} > \cdot$ yields

$$\bar{b}^{*} > \cdot \bar{R} = y > = M_{b*b*} \, y^{*} > = M_{b*r} \, y^{r} > = M_{b*r*} \, y^{r*} > . \qquad (78)$$

Two other relations are obtained with $\bar{r} > \cdot$ and $\bar{r}^{*} > \cdot$ on all terms of Eq. (76), as done in deriving Eq. (77) and Eq. (78). The relationships between the coordinates are shown in Figure (6).



Transformations on the scalar coordinates of a Vector in two arbitrary bases and their reciprocal bases.

Figure (6)

## DYADIC REPRESENTATION OF SYSTEMS OF LINEAR EQUATIONS

Consider the system of equations linear in the variables $y_i$ and $z_i$,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \qquad (79)$$

or

$$y> = T z>. \qquad (80)$$

The transformation matrix T of real scalars may be considered as an operator which acting on z> (the scalar coordinates of one vector) transform it into a different vector having coordinates y >. Without changing the mathematics one may consider y > and z > of equation (1) as the scalar coordinates of the same vector but measured in different bases. Examples of the former are abundant, angular momentum matrix equals the inertia matrix times the angular velocity matrix, i.e.,

$$h> = J\omega> \qquad (81)$$

One may establish a geometrical model for the system of equation (79), hence

$$\bar{R}_1 = <\bar{b}\, y> \qquad (82)$$

$$\bar{R}_2 = <\bar{b}\, z> \qquad (83)$$

that is y> and z> represent vectors in the $\bar{b}>$ basis.

The transformation T as a dyadic operator in the $\bar{b}>$ and $\bar{b}^*>$ bases has the form

$$\bar{T} = <\bar{b}\, T\, \bar{b}^*> \qquad (84)$$

Thus the geometrical analog of equation (79) is

$$\bar{R}_1 = \bar{T} \cdot \bar{R}_2 \qquad (85)$$

since

$$< \bar{b}\ y > = < \bar{b}\ T\ \bar{b}^* > \quad < \bar{b}\ z > \tag{86}$$

or

$$< \bar{b}\ y > = < \bar{b}\ T\ z > . \tag{87}$$

By Equation (52)

$$\bar{b}^* > = M_{bb}^{-1}\ \bar{b} > \tag{88}$$

hence the dyadic operator $\bar{\bar{T}}$ of equation (84) may be written as

$$\bar{\bar{T}} = < \bar{b}\ T\ M_{bb}^{-1}\ \bar{b} > = < \bar{b}\ D\ \bar{b} > \tag{89}$$

Thus one may consider T as an operator solely in the $\bar{b} >$ space if it is multiplied by the inverse of the metric matrix and the matrix of base vector terms.

## DYADIC AS AN ORDERED TRIPLE OF VECTORS

Just as a vector was expressed as an ordered triple of scalars so a dyadic may be expressed as an ordered triple of vectors. For example the unitary dyadic was shown to be the sum of three dyads (dyadic products of base vectors).

Consider the dyadic representation of a square matrix of scalars in a given basis $\bar{b} >$, i.e.,

$$\bar{\bar{D}} = < \bar{b}\ D\ \bar{b} > \tag{90}$$

and the result of partitioning D into three rows,

$$\bar{\bar{D}} = \ < \bar{f} \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} \bar{f} > \tag{91}$$

or

$$\bar{\bar{D}} = \ < \bar{f} \begin{pmatrix} < d_1 & \bar{f} > \\ < d_2 & \bar{f} > \\ < d_3 & \bar{f} > \end{pmatrix} \tag{92}$$

The three terms in the column matrix of equation (92) are vectors, i.e.,

$$\bar{R}_1 = \langle d_1 \; \bar{f} \rangle \qquad (93)$$

$$\bar{R}_2 = \langle d_2 \; \bar{f} \rangle \qquad (94)$$

$$\bar{R}_3 = \langle d_3 \; \bar{f} \rangle \qquad (95)$$

hence

$$\bar{D} = \langle \bar{f} \begin{pmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \bar{R}_3 \end{pmatrix} \qquad (96)$$

The vector analog of equation (96) is

$$\bar{R} = \langle \bar{b} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \qquad (97)$$

Thus one may consider a dyadic as an ordered triple of vectors. A special case quite often used in space vehicle work is the ordered triple of base vectors,

$$\begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} = D_{rb} \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{pmatrix} \qquad (98)$$

representing a change of basis.

The three vectors of equation (98) are linearly independent since it is assumed that they are a basis. It is interesting to observe the form of the matrix of vectors representation of the dyadic form as a juxtaposition of two vectors. By equation (97)

$$\bar{D}_{12} = \bar{R}_1\bar{R}_2 = \langle \bar{b} \; y \rangle \bar{R}_2 \qquad (99)$$

or

$$\bar{R}_1\bar{R}_2 = \langle \bar{b} \begin{pmatrix} y_1\bar{R}_2 \\ y_2\bar{R}_2 \\ y_3\bar{R}_2 \end{pmatrix} \qquad (100)$$

Thus equation (100) is representable as an ordered triple of parallel vectors, whereas the general dyadic (over three space) is representable as an ordered triple of linearly independent (non-coplanar) vectors.

Vector cross product between two vectors. - The vector cross product between two vectors in an arbitrary basis in 3 space is

$$\bar{R}_1 \times \bar{R}_2 = \langle y\, \bar{b}\rangle \times \langle \bar{b}\, z\rangle \tag{101}$$

or

$$\bar{R}_1 \times \bar{R}_2 = \left\langle y \begin{pmatrix} \bar{0} & \bar{b}_1 \times \bar{b}_2 & \bar{b}_1 \times \bar{b}_3 \\ \bar{b}_2 \times \bar{b}_1 & \bar{0} & \bar{b}_2 \times \bar{b}_3 \\ \bar{b}_3 \times \bar{b}_1 & \bar{b}_3 \times \bar{b}_2 & \bar{0} \end{pmatrix} z \right\rangle . \tag{102}$$

It may be seen by the definition of the reciprocal base vectors that in a three space the vectors $\bar{b}_1^*$, $\bar{b}_2^*$, $\bar{b}_3^*$ are perpendicular to the planes determined by $\bar{b}_2$, $\bar{b}_3$; $\bar{b}_3$, $\bar{b}_1$; $\bar{b}_1$, $\bar{b}_2$; respectively. Hence,

$$\bar{b}_1{}^* = k\,(\bar{b}_2 \times \bar{b}_3) \tag{103}$$

where k is an undetermined scalar. Operating on Eq. (103) with $\bar{b}_1 \cdot$ and utilizing Eq. (18)

$$\bar{b}_1 \cdot \bar{b}_1{}^* = 1 = K\, \bar{b}_1 \cdot (\bar{b}_2 \times \bar{b}_3) \tag{104}$$

or

$$\bar{b}_1 \cdot (\bar{b}_2 \times \bar{b}_3) = \frac{1}{K} = K_v{}^b , \tag{105}$$

where $K_v{}^b$ is seen to be the well known scalar triple product or the volume generated by the base vectors $\bar{b}\rangle$. It is well known that the volume generated by a basis $\bar{f}\rangle$ and its reciprocal basis are inversely related, i.e.,

$$K_v{}^b\, K_v{}^{b*} = 1, \tag{106}$$

which is the scalar analog of the reciprocity relation between the metric matrix of the $b\rangle$ space and the dual space $b^*\rangle$ given by Eq. (55).

Writing the well-known+ cross product relationship between the base vectors and the reciprocal base vectors in matrix form one obtains

$$K_v^{\,b}\begin{pmatrix}\bar{b}_1{*}\\ \bar{b}_2{*}\\ \bar{b}_3{*}\end{pmatrix}=\begin{pmatrix}\bar{b}_2\times\bar{b}_3\\ \bar{b}_3\times\bar{b}_1\\ \bar{b}_1\times\bar{b}_2\end{pmatrix}. \qquad (107)$$

By use of Eq. (107) one may write the useful relation

$$\bar{b}>\times<\bar{b}=K_v^{\,b}\begin{pmatrix}\bar{0} & \bar{b}_3{*} & -\bar{b}_2{*}\\ -\bar{b}_3{*} & \bar{0} & \bar{b}_1{*}\\ \bar{b}_2{*} & -\bar{b}_1{*} & \bar{0}\end{pmatrix}=K_v^{\,b}S_{\bar{b}*} \qquad (108)$$

Utilizing Eq. (108) in Eq. (102)

$$\bar{R}_1\times\bar{R}_2=\ <y\begin{pmatrix}\bar{0} & \bar{b}_3{*} & -\bar{b}_2{*}\\ -\bar{b}_3{*} & \bar{0} & \bar{b}_1{*}\\ \bar{b}_2{*} & \bar{b}_1^{*} & \bar{0}\end{pmatrix}z>K_v^{\,b}. \qquad (109)$$

It is easily demonstrated by performing the multiplication that

$$(y_1,\ y_2,\ y_3)\begin{pmatrix}\bar{0} & \bar{b}_3{*} & -\bar{b}_2{*}\\ -\bar{b}_3{*} & \bar{0} & \bar{b}_1{*}\\ \bar{b}_2{*} & \bar{b}_1{*} & \bar{0}\end{pmatrix}=-<\bar{b}{*}\begin{pmatrix}0 & y_3 & -y_2\\ -y_3 & 0 & y_1\\ y_2 & -y_1 & 0\end{pmatrix} \qquad (110)$$

hence

$$\bar{R}_1\times\bar{R}_2=K_v^{\,b}<\bar{b}{*}\begin{pmatrix}0 & y_3 & -y_2\\ -y_3 & 0 & y_1\\ y_2 & -y_1 & 0\end{pmatrix}\begin{pmatrix}z_1\\ z_2\\ z_3\end{pmatrix}. \qquad (111)$$

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+ Taylor, p. (50)

By Eq. (52) one may write Eq. (107) as

$$\bar{R}_1 \times \bar{R}_2 = - < \bar{b}\, K_v{}^b\, M_{bb}{}^{-1} \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} . \qquad (112)$$

If $\bar{b}>$ is an O.N. basis, the volume factor $K_v{}^b$ equals unity and the metric matrix $M_{bb}$ is the identity matrix, hence for an O.N. basis

$$\bar{R}_1 \times \bar{R}_2 = - < \bar{b} \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} . \qquad (113)$$

Dyadic representation of the vector cross operation. - The scalar product of a vector with the unitary dyadic is the vector, hence by Eq (44)

$$\bar{R}_1 \times \bar{R}_2 = \bar{R}_1 \cdot \bar{U} \times \bar{U} \cdot \bar{R}_2 \,. \qquad (114)$$

Expressing all elements of Eq. (114) in the $\bar{b}>$ basis except $\bar{R}_2$,

$$\bar{R}_1 \times \bar{R}_2 = <y\, \bar{b} > \cdot < \bar{b}^*\bar{b} > \times < \bar{b}\, \bar{b}^* > \cdot \bar{R}_2 , \qquad (115)$$

and by Eq. (48), Eq. (108) and Eq. (110)

$$\bar{R}_1 \times \bar{R}_2 = k_v{}^b < \bar{b}^*\, S'_{R_1}\, \bar{b}^* > \cdot \bar{R}_2 \,. \qquad (116)$$

By Eq. (116) one may write

$$\bar{R}_1 \times = \bar{\bar{S}}_{R_1} \,. \qquad (117)$$

where the skew dyadic is

$$\bar{S}_{R_1} \cdot \equiv <\bar{b}^*\, S'_{R_1}\, \bar{b}^* > k_v{}^b \cdot \equiv < \bar{b}\, M_{bb}^{-1}\, S'_{R_1}\, \bar{b}^* > . \qquad (118)$$

Eq. (116) expresses the vector product operation as a scalar product. One may also write by Eq. (114) and Eq. (117)

$$\bar{R}_1 \times = \bar{R}_1 \cdot \bar{\bar{U}} \times \bar{\bar{U}} \cdot = \bar{\bar{S}}_{R_1} \cdot \qquad (119)$$

The vector cross product expressed in the $\bar{b} >$ basis by Eq. (112) Eq. (118) and Eq. (119) is

$$\bar{R}_1 \times \bar{R}_2 = k_v{}^b \; < \bar{b} \, M_{bb}{}^{-1} \; S'_{R_1} \; Z > = \bar{\bar{S}}_{R_1} \cdot \bar{R}_2 \quad . \qquad (120)$$

<u>The vector triple product</u>. - The vector triple product is conventionally shown to be

$$\bar{R}_3 \times (\bar{R}_1 \times \bar{R}_2) = \bar{R}_3 \cdot \bar{R}_2 \, \bar{R}_1 - \bar{R}_3 \cdot \bar{R}_1 \, \bar{R}_2 \; . \qquad (121)$$

In conventional[+] dyadic form, Eq. (121) is written as

$$\bar{R}_3 \times (\bar{R}_1 \times \bar{R}_2) = (\bar{R}_1 \, \bar{R}_3 - \bar{R}_1 \cdot \bar{R}_3 \, \bar{\bar{U}}) \cdot \bar{R}_2 \qquad (123)$$

By use of the skew dyadic of Eq. (117), one may write directly

$$\bar{R}_3 \times (\bar{R}_1 \times \bar{R}_2) = \bar{\bar{S}}_{R_3} \cdot (\bar{\bar{S}}_{R_1} \cdot \bar{R}_2) \qquad (123)$$

or in the $\bar{b}^* >$ basis

$$\bar{R}_3 \times (\bar{R}_1 \times \bar{R}_2) = (k_v{}^b)^2 \; < \bar{b}^* S_{R_3} b^* > . < \bar{b}^* S_{R_1} \bar{b}^* > \; . \; < b \; Z > \qquad (124)$$

or

$$\bar{R}_3 \times (\bar{R}_1 \times \bar{R}_2) = (k_v{}^b)^2 \; < \bar{b}^* S_{R_3} M_{b^* b^*} S_{R_1} \; Z > . \qquad (125)$$

Transposing the triple product vector of Eq. (125) into the $\bar{b}>$ basis and using Eq. (51) and Eq. (52), one obtains

$$\bar{R}_3 \times (\bar{R}_1 \times \bar{R}_2) = (k_v{}^b)^2 \; < \bar{b} \, M_{BB}{}^{-1} \, S'_{R_3} \, M_{bb}{}^{-1} \, S'_{R_1} \; Z > . \qquad (126)$$

---

[+] Goldstein, p. (149)

The vector triple product occurs quite often as an acceleration term $\bar{\omega} \times (\bar{\omega} \times \bar{R})$ in translational equations of motion and in the rigid body rotational equations of motion as angular momentum, i.e.,

$$\bar{H} = \sum_{i=1}^{\infty} \bar{R}_i \times \dot{\bar{R}}_i m_i = \sum_{i=1}^{\infty} \bar{R}_i \times (\bar{\omega} \times \bar{R}_i)\, m_i \qquad (127)$$

Scalar triple product. - The scalar triple product in a given basis may be written by use of Eq. (26) as

$$\bar{R}_3 \cdot (\bar{R}_1 \times \bar{R}_2) = < W\, b > \cdot < b^* S_{R_1}\, Z > k_v^{\,b} \qquad (128)$$

where

$$\bar{R}_3 = < W\, b > , \qquad (129)$$

or

$$\bar{R}_3 \cdot (\bar{R}_1 \times \bar{R}_2) = (W_1, W_2, W_3) \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} k_v^{\,b} \qquad (130)$$

As an example of the utility of Eq. (130) let the three vectors be the base vectors $\bar{r} >$ expressed in the $\bar{b} >$ basis, i.e.,

$$\bar{r}_1 = (d_{11}, d_{12}, d_{13})\, \bar{b} > \qquad (131)$$

$$\bar{r}_2 = (d_{21}, d_{22}, d_{23})\, \bar{b} > \qquad (132)$$

$$\bar{r}_3 = (d_{31}, d_{32}, d_{33})\, \bar{b} > \qquad (133)$$

where the transformation on the base vectors is

$$\bar{r} > = D_{rb}\, \bar{b} > = \left[ d_{ij} \right] \bar{b} > \qquad (134)$$

Utilizing Eqs. (131), (132), and (133), in Eq. (130

$$k_v^{\,r} = \bar{r}_1 \cdot (\bar{r}_2 \times \bar{r}_3) = k_v^{\,b}(d_{11},\ d_{12},\ d_{13}) \begin{pmatrix} 0 & -d_{23} & d_{22} \\ d_{23} & 0 & -d_{21} \\ -d_{22} & d_{21} & 0 \end{pmatrix} \begin{pmatrix} d_{31} \\ d_{32} \\ d_{33} \end{pmatrix} \quad (135)$$

and multiplying the last two matrices

$$k_v^{\,r} = (d_{11},\ d_{12},\ d_{13}) \begin{pmatrix} d_{22}\ d_{33} - d_{32}\ d_{23} \\ d_{23}\ d_{31} - d_{21}\ d_{33} \\ d_{21}\ d_{32} - d_{31}\ d_{22} \end{pmatrix} k_v^{\,b} \ . \quad (136)$$

The scalar equation resulting from the above matrix product is seen to be the well-known determinant of $D_{rb}$ expanded about the first row, hence

$$k_v^{\,r} = |\, D_{rb} \,|\ k_v^{\,b} \ . \quad (137)$$

Equation (137) relates the determinant of the transformation on the two bases to the volumes generated by the base vectors.

## OTHER OPERATIONS BETWEEN VECTORS AND DYADICS

Any number of important relations between dyadics and vectors under the two operations x and . are easily derivable in an arbitrary basis $\bar{b} >$ or any mixed bases.

A dyadic and a vector map under the dot product to a vector, hence

$$\bar{\bar{D}} \cdot \bar{R} = <\bar{b}\ D\ \bar{b}> \cdot <\bar{b}\ y> = <\bar{b}\ D\ M_{bb} y> \quad (138)$$

The resulting vector of the above operation maps to a scalar under the dot product operation, hence

$$\bar{R}_1 \cdot \bar{\bar{D}} \cdot \bar{R}_2 = <y\ \bar{b}> \cdot <\bar{b}\ D\ \bar{b}> \cdot <\bar{b}\ Z> \quad (139)$$

$$= <y\ M_{bb} D\ M_{bb}\ Z> \ . \quad (140)$$

A vector crossed with a dyadic yields a dyadic

$$\bar{R} \text{ x } \bar{\bar{D}} = < y \; \bar{b} > \text{ x } < \bar{b} \; D \; \bar{b} > \; = \; < y \; S_{\bar{b}*} \; D \; \bar{b} > k_{v}{}^{b} \tag{141}$$

$$= \; < \bar{b}* \; S'_{y} \; D \; \bar{b} > k_{v}{}^{b} \tag{142}$$

or in the $\bar{b} >$ basis

$$\bar{R} \text{ x } \bar{\bar{D}} = < \bar{b} \; M_{bb}{}^{-1} \; S'_{y} \; D \; \bar{b} > k_{v}{}^{b} \; . \tag{143}$$

If the dyadic of Eq. (154) is dotted with a vector, one obtains

$$(\bar{R}_1 \text{ x } \bar{\bar{D}}) \;\; \bar{R}_2 = < \bar{b} \; M_{bb}{}^{-1} \; S'_{y} \; D \; \bar{b} > \cdot < \bar{b} \; Z \;\; k_{v}{}^{b} > \tag{144}$$

$$= \; < \bar{b} \; M_{bb}{}^{-1} \; S'_{y} \; D \; M_{bb} \; Z > k_{v}{}^{b} \; , \tag{145}$$

which is a vector.

The matrix orienting one O.N. basis w.r.t. a second O.N. basis may be expressed as the product of three orthogonal matrices, i.e.,

$$\bar{r}> = M_{rf} \; \bar{f}> \; = M_{k} \; (\phi_3) \; M_{j} \; (\phi_2) \; M_{i} \; (\phi_1) \; \bar{f} >$$

where

$$M_1 \; (\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\,\phi & s\,\phi \\ 0 & -s\,\phi & c\,\phi \end{pmatrix} \tag{146}$$

$$M_2 \; (\phi) = \begin{pmatrix} c\,\phi & 0 & -s\,\phi \\ 0 & 1 & 0 \\ s\,\phi & 0 & c\,\phi \end{pmatrix} \tag{147}$$

$$M_3 \; (\phi) = \begin{pmatrix} c\,\phi & s\,\phi & 0 \\ -s\,\phi & c\,\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{148}$$

If $i \neq j \neq k$, that is if one uses successive rotation sequence and small angle assumptions then

$$M_{rf} = I + \begin{pmatrix} 0 & \Delta\phi_3 & -\Delta\phi_2 \\ -\Delta\phi_3 & 0 & \Delta\phi_1 \\ \Delta\phi_2 & -\Delta\phi_1 & 0 \end{pmatrix} \tag{149}$$

Utilizing the small angle matrix of Eq. (149) one may write

$$\bar{b}\,(t + \Delta t)> \; = (I + S_{\Delta\phi})\,\bar{b}(t)> , \tag{150}$$

or by subtracting $I\,\bar{b}(t)>$

$$\bar{b}\,(t + \Delta t) - \bar{b}(t) > \; = S_{\Delta\phi}\,\bar{b}(t) > \quad . \tag{151}$$

Dividing by $\Delta t$ and passing to the limit, one obtains

$$\dot{\bar{b}} > \; = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \bar{b} > \tag{152}$$

where

$$\omega_i = \lim_{\Delta t \to 0} \frac{\Delta\phi_i}{\Delta t}$$

If $\bar{b}>$ is not an O.N. basis, it may be shown that $\dot{\bar{b}}>$ may be written as

$$\dot{\bar{b}} > \; = k_v^{\,b}\, S_{\omega_b}\, \bar{b}* > \; = K_v^{\,b}\, S_{\omega_b}\, M_{bb}^{-1}\, \bar{b} > \quad . \tag{153}$$

Time derivative of a vector in a moving basis. - The time derivative of a vector in a moving basis is conventionally written as

$$\frac{d\bar{R}}{dt} = \left.\frac{\partial \bar{R}}{\partial t}\right)_b + \bar{\omega}_b \times \bar{R}, \tag{154}$$

where $\bar{\omega}_b$ is the inertial angular velocity of the $\bar{b}>$ basis. If $\bar{b}>$ is an orthonormal basis, the second term of Eq. (157) is quite often written as

$$\bar{\omega}_b \times \bar{R} = \left|\begin{matrix} \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ y_1 & y_2 & y_3 \end{matrix}\right| \tag{155}$$

which is neither a matrix nor a determinant but a notation which by convention means

$$\bar{\omega}_b \times \bar{R} = (\omega_2 y_3 - \omega_3 y_2)\bar{b}_1 + (\omega_3 y_1 - \omega_1 y_3)\bar{b}_2$$

$$+ (\omega_1 y_2 - \omega_2 y_1)\bar{b}_3 \; . \tag{156}$$

A straight forward derivation of the time derivative of $\bar{R}$ in a moving basis may be obtained.

By Eq (31)

$$\bar{R} = <y\,\bar{b}> = <\bar{b}\,y> \tag{157}$$

and

$$\dot{\bar{R}} = <\dot{y}\,\bar{b}> + <y\,\dot{\bar{b}}> = <\dot{\bar{b}}\,y> + <\bar{b}\,\dot{y}> . \tag{158}$$

Utilizing Eq. (153) in Eq. (158) one obtains for an O.N. basis

$$\dot{\bar{R}} = \left(<\dot{y} + <y\,S_{\omega}\right)\bar{b}> = \left.\frac{\partial \bar{R}}{\partial t}\right)_b - \bar{R} \cdot \bar{\bar{S}}_{\omega_b} \tag{159}$$

or transposing Eq. (159)

$$\dot{\bar{R}} = \langle \bar{b} \left( \dot{y} \rangle + S'_{\omega} y \rangle \right) = \left. \frac{\partial \bar{R}}{\partial t} \right)_{b} + \bar{\bar{S}}_{\omega b} \cdot \bar{R} \qquad (160)$$

By Eq. (154) and Eq. (160)

$$\left. \frac{\partial \bar{R}}{\partial t} \right)_{\bar{b}} = \langle \bar{b} \, \dot{y} \rangle \qquad (161)$$

and

$$\bar{\omega}_{b} \times \bar{R} = \langle \bar{b} \, S' \, y \rangle \, . \qquad (162)$$

The velocity vector $\dot{\bar{R}}$ may be mapped to a matrix of scalars as

$$\bar{b} \rangle \cdot \dot{\bar{R}} = \begin{pmatrix} \bar{b}_1 \cdot \dot{\bar{R}} \\ \bar{b}_2 \cdot \dot{\bar{R}} \\ \bar{b}_3 \cdot \dot{\bar{R}} \end{pmatrix} = \dot{y} \rangle + S'_{\omega} \, y \rangle \, . \qquad (163)$$

If $\bar{b} \rangle$ is not an O.N. basis then the expression of Eq. (153) must be used.

<u>Time derivative of a dyadic in a moving basis.</u> - The time derivative of a dyadic in a moving basis is

$$\frac{d}{dt} \bar{\bar{D}} = \frac{d}{dt} \left( \langle \bar{b} \, D \, \bar{b} \rangle \right) \qquad (164)$$

or

$$\dot{\bar{\bar{D}}} = \langle \dot{\bar{b}} \, D \, \bar{b} \rangle + \langle \bar{b} \, \dot{D} \, \bar{b} \rangle + \langle \bar{b} \, D \, \dot{\bar{b}} \rangle \qquad (165)$$

Utilizing Eq. (153) for an ortho-normal basis in Eq. (165)

$$\dot{\bar{\bar{D}}} = \langle \bar{b} \, \dot{D} \, \bar{b} \rangle + \langle \bar{b} \, S'_{\omega} \, D \, \bar{b} \rangle + \langle \bar{b} \, D \, S_{\omega} \, \bar{b} \rangle \, , \qquad (166)$$

or in an analogous form to Eq. (154)

$$\dot{\bar{\bar{D}}} = \frac{\partial \bar{\bar{D}}}{\partial t}\Bigg)_b + \bar{\omega}_b \times \bar{\bar{D}} - \bar{\bar{D}} \times \bar{\omega}_b = \frac{\partial \bar{\bar{D}}}{\partial t}\Bigg)_b$$

$$+ \bar{\bar{S}}_{\omega_b} \cdot \bar{\bar{D}} - \bar{\bar{D}} \cdot \bar{\bar{S}}_{\omega_b} \tag{167}$$

The derivative of the unitary dyadic in a moving ortho-normal basis may be shown to be zero, for by Eq. (43)

$$\bar{\bar{U}} = < \bar{b} \; I \; \bar{b} > \tag{168}$$

and

$$\dot{\bar{\bar{U}}} = < \dot{\bar{b}} \; I \; \bar{b} > + < \bar{b} \; \dot{I} \; \bar{b} > + < \bar{b} \; I \; \dot{\bar{b}} > \tag{169}$$

or

$$\dot{\bar{\bar{U}}} = - < \bar{b} \; S_{\omega} \; \bar{b} > + < \bar{b} \; S_{\omega} \; \bar{b} > = \bar{\bar{0}} \tag{170}$$

A similar proof may be given for a non-ortho-normal basis.

## PART II. APPLICATIONS

The mathematical methods of the previous section are applied to translational and rotational dynamics as occur in flight mechanics.

### Translational Kinematics

The velocity and acceleration vectors of a point as observed by an inertial observer are derived in a moving basis to demonstrate the flexibility of transition from vectors and dyadics to matrix equations.

Expressing the position vector $\bar{R}$ of figure (a-1) in a moving reference frame say $\bar{b}>$ one obtains

$$\bar{R} = < y \, \bar{b} > = < \bar{b} \, y > . \qquad \text{(a-1)}$$

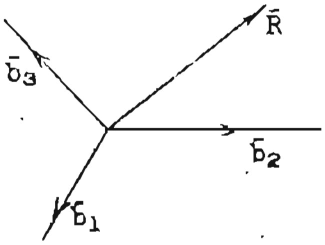

Figure (a-1) - Position Vector $\bar{R}$ in a Moving Basis

When the position vector $\bar{R}$ is expressed as a column matrix of scalars (right hand side of equation (a-1)), then by equation (160)

$$\dot{\bar{R}} = \left.\frac{\partial \bar{R}}{\partial t}\right)_{\bar{b}} + \bar{\bar{S}}_{\omega_b} \cdot \bar{R} = \left.\frac{\partial \bar{R}}{\partial t}\right)_{f} + \bar{\omega}_b \times \bar{R} . \qquad \text{(a-2)}$$

The corresponding vector-matrix expression by equation (153) is

$$\dot{\bar{R}} = <\bar{b} \, \dot{y}> - k_v^{\,b} < \bar{b}^* S_{\omega_b} \, y > = <\bar{b} \left[ \dot{y} > - k_v \, {}^{b}M_{bb}^{-1} \, S_{\omega_b} y> \right] \qquad \text{(a-3)}$$

The matrix block diagrams of equation (a-3) is given in figure (a-2).

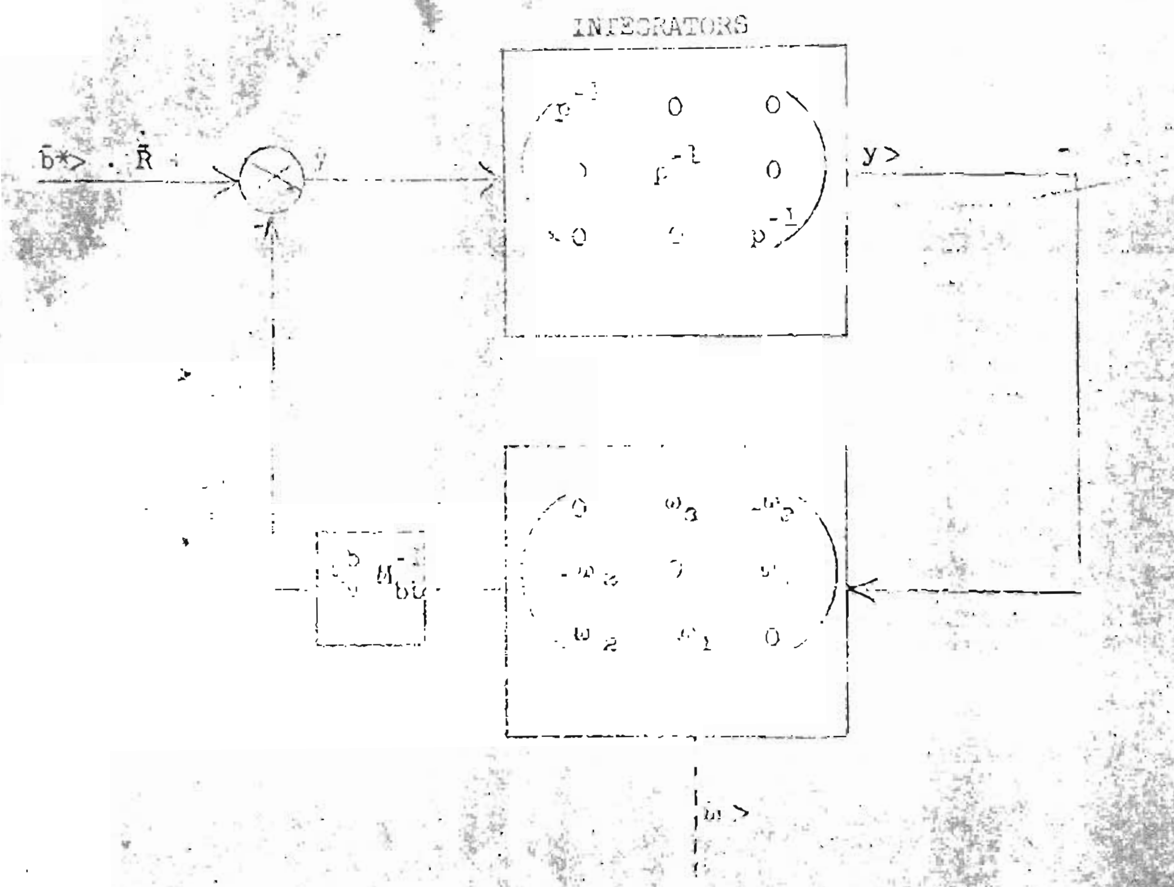

Figure (a-2) - Matrix Block Diagrams

The inertially observed velocities as a column matrix of scalars is obtained by operating on equation (a-3) with $\bar{b}^* >$. hence

$$\bar{b}^* > \cdot \dot{\bar{R}} = \begin{pmatrix} \bar{b}_1^* \cdot \dot{\bar{R}} \\ \bar{b}_2^* \cdot \dot{\bar{R}} \\ \bar{b}_3^* \cdot \dot{\bar{R}} \end{pmatrix} = \dot{y} > - K_v^b M_{bb}^{-1} S_{\omega_b} y > \qquad \text{(a-4)}$$

If the velocities are projected onto the $\bar{b} >$ basis, equation (a-3) becomes

$$\bar{b} > \cdot \dot{\bar{R}} = \begin{pmatrix} \bar{b}_1 \cdot \dot{\bar{R}}_1 \\ \bar{b}_2 \cdot \dot{\bar{R}}_1 \\ \bar{b}_3 \cdot \dot{\bar{R}}_1 \end{pmatrix} = M_{bb} \dot{y} > - K_v^b S_{\omega_b} y >. \qquad \text{(a-5)}$$

Clearly, if $\bar{b}>$ is O.N.

$$M_{bb} = \bar{I}, \qquad \text{(a-6)}$$

and Equation (a-5) becomes the familiar expression,

$$\begin{pmatrix} \bar{b}_1 \cdot \dot{\bar{R}} \\ \bar{b}_2 \cdot \dot{\bar{R}} \\ \bar{b}_3 \cdot \dot{\bar{R}} \end{pmatrix} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} + \begin{pmatrix} \omega_{b_2} & y_3 & -\omega_{b_3} & y_2 \\ \omega_{b_3} & y_1 & -\omega_{b_1} & y_3 \\ \omega_{b_1} & y_2 & -\omega_{b_2} & y_1 \end{pmatrix} \qquad \text{(a-7)}$$

The time derivative of the velocity vector of equation (a-2) w.r.t. an inertial observer is the acceleration vector usually written in the familiar form,

$$\ddot{\bar{R}} = \frac{\partial^2 \bar{R}}{\partial t^2}\Big)_b + 2\bar{\omega}_b \times \frac{\partial \bar{R}}{\partial t}\Big)_b + \bar{\omega}_b \times \left(\bar{\omega}_b \times \bar{R}\right) + \dot{\bar{\omega}}_b \times \bar{R}$$

where

$$\dot{\bar{\omega}}_b = \frac{\partial \bar{\omega}_b}{\partial t}\Big)_b + \bar{\omega}_b \times \bar{\omega}_b = \frac{\partial \bar{\omega}_b}{\partial t}\Big)_b . \qquad \text{(a-9)}$$

It follows from equation (2) by replacing $\bar{R}$ by $\dot{\bar{R}}$, that

$$\ddot{\bar{R}} = \frac{\partial^2 \dot{\bar{R}}}{\partial t^2}\Big)_b + \bar{\bar{S}}_{\omega_b} \cdot \dot{\bar{R}} \qquad \text{(a-10)}$$

and substituting for $\dot{\bar{R}}$ from equation (a-2) yields

$$\ddot{\bar{R}} = \frac{\partial^2 \bar{R}}{\partial t^2}\Big)_b + 2\bar{\bar{S}}_{\omega_b} \cdot \frac{\partial \bar{R}}{\partial t}\Big)_b + \frac{\partial \bar{\bar{S}}_{\omega_b}}{\partial t}\Big)_b \cdot \bar{R} + \bar{\bar{S}}_{\omega_b} \cdot \left(\bar{\bar{S}}_{\omega_b} \cdot \bar{R}\right) \qquad \text{(a-11)}$$

If $M_{bb} = I$, i.e., $\bar{b}>$ is an O.N. basis, then equation (a-11) in open form is

$$\begin{pmatrix} \ddot{\bar{R}} \cdot \bar{b}_1 \\ \ddot{\bar{R}} \cdot \bar{b}_2 \\ \ddot{\bar{R}} \cdot \bar{b}_3 \end{pmatrix} = \begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{pmatrix} + 2 \begin{pmatrix} 0 & \omega_3 & \omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}^2 \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \dot{\omega}_3 & -\dot{\omega}_2 \\ -\dot{\omega}_3 & 0 & \dot{\omega}_1 \\ \omega_2 & -\omega_1 & \omega_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} . \qquad \text{(a-12)}$$

## TRANSLATIONAL KINETIC ENERGY IN MOVING BASES

A scalar proportional to the kinetic energy $\dot{\bar{R}} \cdot \dot{\bar{R}}$ in a moving non-orthonormal basis may be obtained as matrix products. The transpose of Eq. (a-3) is

$$\dot{\bar{R}} = \left[ <\dot{y} + K_v^{\,b} < y\, S_{\omega_b} M_{bb}^{-1} \right] \bar{b}> \qquad \text{(a-13)}$$

and by Eq. (a-3) and Eq. (a-13)

$$\dot{\bar{R}} \cdot \dot{\bar{R}} = \left[ <\dot{y} + K_v^{\,b} <y\, S_{\omega_b} M_{bb}^{-1} \right] M_{bb} \left[ \dot{y}> - K_v^{\,b} M_{bb}^{-1} S_{\omega_b}\, y> \right] \qquad \text{(a-14)}$$

$$\dot{\bar{R}} \cdot \dot{\bar{R}} = <\dot{y}\, M_{bb}\, \dot{y}> - 2\, K_v^{\,b} <\dot{y}\, S_{\omega_b}\, y>$$

$$-\left(K_v^{\,b}\right)^2 <y\, S_{\omega_b} M_{bb}^{-1} S_{\omega_b}\, y> . \qquad \text{(a-15)}$$

If $\bar{b}>$ is O.N. then Eq. (a-15) becomes

$$\dot{\bar{R}} \cdot \dot{\bar{R}} = \langle \dot{y}\, \dot{y} \rangle - 2 \langle \dot{y}\, S_{\omega_b}\, y \rangle - \langle y\, S_{\omega_b} S_{\omega_b}\, y \rangle . \qquad (a\text{-}16)$$

## Rotational Dynamics of two Rigid Bodies

Consider the conventional angular momentum expression as the first moment of the mass-weighted velocity vector, i.e.,

$$\bar{H}_s = \sum_{K=1}^{\infty} \bar{R}_K \times \dot{\bar{R}}_K\, m_K \qquad (a\text{-}17)$$

where $\bar{R}_K$ is the position vector of the kth particle with respect to the center of mass. The velocity vector $\dot{\bar{R}}_K$ for a rigid body in a moving body-fixed reference frame is

$$\dot{\bar{R}}_K = \left.\frac{\partial \bar{R}_K}{\partial t}\right)_b + \bar{\omega}_b \times \bar{R}_K = - \bar{R}_K \times \bar{\omega}_b . \qquad (a\text{-}18)$$

Utilizing Eq. (120) and Eq. (a-18) in Eq. (a-17)

$$\bar{H}_s = \left( -\sum_{K=1}^{\infty} \bar{\bar{S}}_{R_K} \cdot \bar{\bar{S}}_{R_K}\, m_K \right) \cdot \bar{\omega}_b . \qquad (a\text{-}19)$$

The term in parenthesis of Eq. (a-19) in an arbitrary basis is

$$- \sum_{K=1}^{\infty} \bar{\bar{S}}_{R_K} \cdot \bar{\bar{S}}_{R_K}\, m_K = - \sum_{K=1}^{\infty} \langle \bar{b}^*\, S_{R_K} M_{bb}^{-1} S_{R_K}\, \bar{b}^* \rangle\, m_K, \qquad (a\text{-}20)$$

or in an orthonormal basis

$$- \sum_{K=1}^{\infty} \bar{\bar{S}}_{R_K} \cdot \bar{\bar{S}}_{R_K}\, m_K = - \sum_{K=1}^{\infty} \langle \bar{b}\, S_{R_K}^{2}\, \bar{b} \rangle\, m_K \qquad (a\text{-}21)$$

The square of the skew-symmetric matrix of Eq. (a-21) is a symmetric matrix and is

$$\sum_{K=1}^{\infty} <\bar{b}\, S_{R_K}^{\,2}\, \bar{b}>\, m_K =$$

$$<\bar{b} \begin{pmatrix} \sum (y_{2_k}^{2}+y_{3_k}^{2})m_k & -\sum y_{1_k} y_{2_k} m_k & -\sum y_{1_k} y_{3_k} m_k \\ -\sum y_{2_k} y_{1_k} m_k & \sum (y_{1_k}^{2}+y_{3_k}^{2})m_k & -\sum y_{2_k} y_{3_k} m_k \\ -\sum y_{3_k} y_{1_k} m_k & \sum y_{3_k} y_{2_k} m_k & \sum (y_{1_k}^{2}+y_{2_k}^{2})m_k \end{pmatrix} \bar{b}> \qquad (a\text{-}22)$$

The scalar elements of the above matrix are the familiar moment and product of inertia terms, hence the inertia dyadic $\bar{\bar{I}}$ may be defined as

$$\bar{\bar{I}}_s = -\sum_{K=1}^{\infty} \bar{\bar{S}}_{R_K} \cdot \bar{\bar{S}}_{R_K}\, m_K = <\bar{b}\, I_s\, \bar{b}> . \qquad (a\text{-}23)$$

The conventional form$^{+}$ for writing the inertia dyadic is

$$\bar{\bar{I}}_s = -\sum_{K=1}^{\infty} (\bar{R}_K \bar{R}_K - \bar{R}_K \cdot \bar{R}_K\, \bar{\bar{U}})\, m_K \qquad (a\text{-}24)$$

The angular momentum may be written in the well-known form as

$$\bar{H} = \bar{\bar{I}}_s \cdot \bar{\omega} . \qquad (a\text{-}25)$$

Consider a translating flight system composed of two rigid bodies, a main body B and a second body R having one rotational degree of freedom with respect to body B, Figure (a-3). This condition occurs in stable platforms, solar paddles in satellites, or inertia wheel control systems, and other gimbaled body problems with the exception that one needs to account for each additional body.

---

+ Goldstein, p. 149

Assume further that an ortho-normal reference frame $\bar{r}>$ fixed in the rotating body R has one degree of freedom with respect to a case frame $\bar{c}>$, for example

$$\bar{r}> = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\,\phi_{rc} & s\,\phi_{rc} \\ 0 & -s\,\phi_{rc} & c\,\phi_{rc} \end{pmatrix} \bar{c}> = M_{rc}\,\bar{c}> \qquad \text{(a-26)}$$

as shown in Figure (a-3).

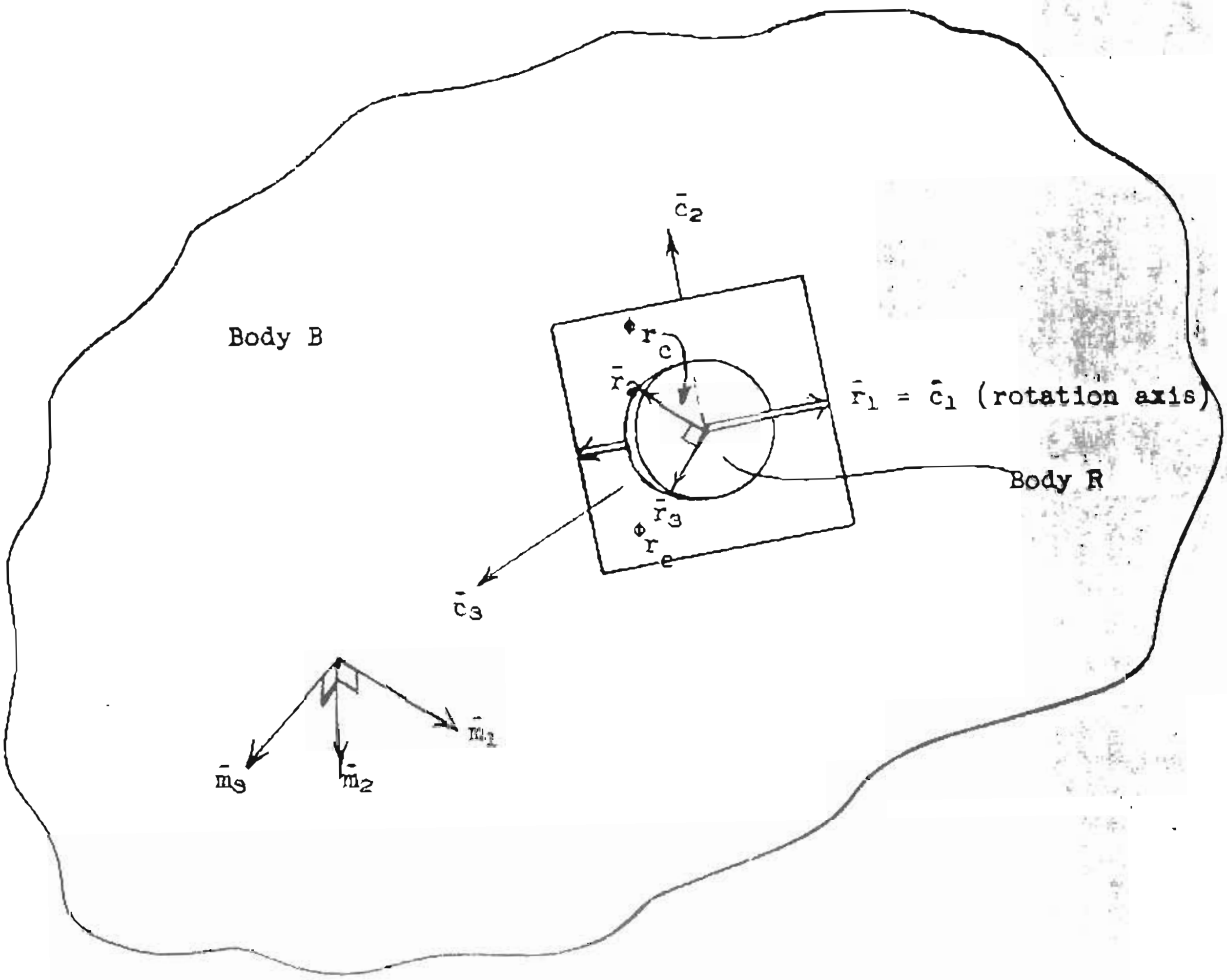

Figure (a-3) Orientation of Reference Frames

The introduction of the idea of a case frame $\bar{c}>$ allows for an arbitrary orientation of the case frame (that is the zero gimbal angle position for the rotating component) with respect to the main body frame $\bar{m}>$ , thus

$$\bar{c}> = M_{cm} \; \bar{m}> = \left[ \; c_{ij} \; \right] \bar{m}> . \qquad \text{(a-27)}$$

By Eq. (a-26) and Eq. (a-27)

$$\bar{r}> = M_{rc} \, M_{cm} \, \bar{m}> = M_{rm} \, \bar{m}> . \qquad \text{(a-28)}$$

The rotational dynamics of the system may be written as

$$\dot{\bar{H}}_s = \dot{\bar{H}}_v + \dot{\bar{H}}_r = \bar{T}_s \qquad \text{(a-29)}$$

where

$$\bar{H}_v = \bar{\bar{I}}_v \cdot \bar{\omega}_m . \qquad \text{(a-30)}$$

$\bar{\bar{I}}_v$ is the inertia dyadic of the composite vehicle when the angle between the two bodies is zero; and

$$\bar{H}_r = \bar{\bar{I}}_r \cdot \bar{\omega}_r . \qquad \text{(a-31)}$$

The angular velocity vectors $\bar{\omega}_m$ and $\bar{\omega}_r$ are inertial angular velocities and

$$\bar{\omega}_r = \bar{\omega}_{rm} + \bar{\omega}_m , \qquad \text{(a-32)}$$

hence Eq. (a-29) may be written as

$$\dot{\bar{H}}_s = \frac{d}{dt} \left[ \left( \bar{\bar{I}}_v + \bar{\bar{I}}_r \right) \cdot \bar{\omega}_m + \bar{\bar{I}}_r \cdot \bar{\omega}_{rm} \right] = \bar{T}_s \qquad \text{(a-33)}$$

Taking the time derivative with respect to inertial space of Eq. (a-33) and utilizing the relation

$$\frac{d}{dt}\left(\bar{\bar{D}} \cdot \bar{R}\right) = \frac{d\bar{\bar{D}}}{dt} \cdot \bar{R} + \bar{\bar{D}} \cdot \frac{d\bar{R}}{dt} \qquad \text{(a-34)}$$

one obtains

$$\dot{\bar{H}}_s = (\bar{\bar{I}}_v + \bar{\bar{I}}_r) \cdot \left.\frac{\partial \bar{\omega}_m}{\partial t}\right)_m + \bar{\bar{I}}_r \cdot \left.\frac{\partial \bar{\omega}_{rc}}{\partial t}\right)_r$$

$$+ \bar{\omega}_m \times (\bar{\bar{I}}_v + \bar{\bar{I}}_r) \cdot \bar{\omega}_m + \left(\bar{\omega}_{rc} \times \bar{\bar{I}}_r\right) \cdot \bar{\omega}_{rc} + \left(\bar{\omega}_m \times \bar{\bar{I}}_r\right) \cdot \bar{\omega}_{rc}$$

$$+ \left(\bar{\omega}_{rc} \times \bar{\bar{I}}_r\right) \cdot \bar{\omega}_m - \bar{\bar{I}}_r \cdot \left(\bar{\omega}_{rc} \times \bar{\omega}_m\right) = \bar{\bar{T}}_s \qquad \text{(a-35)}$$

There are four rotational degrees of freedom for the system, three independent equations are given by Eq. (a-35) and the fourth equation may be obtained from

$$\bar{r}_1 \cdot \dot{\bar{H}}_r = \bar{r}_1 \cdot \bar{T}_r \qquad \text{(a-36)}$$

where

$$\dot{\bar{H}}_r = \bar{\bar{I}}_r \cdot \left[ \left.\frac{\partial \bar{\omega}_{rc}}{\partial t}\right)_r + \left.\frac{\partial \bar{\omega}_m}{\partial t}\right)_m + \bar{\omega}_m \times \bar{\omega}_{rc}\right]$$

$$+ \left(\bar{\omega}_{rc} \times \bar{\bar{I}}_r\right) \cdot \bar{\omega}_{rc} + \left(\bar{\omega}_m \times \bar{\bar{I}}_r\right) \cdot \bar{\omega}_{rc}$$

$$+ \left(\bar{\omega}_{rc} \times \bar{\bar{I}}_r\right) \cdot \bar{\omega}_m + \left(\bar{\omega}_m \times I_r\right) \cdot \bar{\omega}_m \, . \qquad \text{(a-37)}$$

Equation (a-35) may be mapped to a matrix of scalar equations by operating with $\bar{m} > \cdot$ , hence

$$\bar{m}> \cdot \dot{\bar{H}}_s = \begin{pmatrix} \bar{m}_1 \cdot \dot{\bar{H}}_s \\ \bar{m}_2 \cdot \dot{\bar{H}}_s \\ \bar{m}_3 \cdot \dot{\bar{H}}_s \end{pmatrix} = (I_v + M'_{rm} I_r M_{rm}) \, \dot{\omega}_m >$$

$$+ M_{mr} I_r \dot{\omega}_{rc} > + S'_{\omega_m} \left[ I_v + M'_{rm} I_r M_{rm} \right] \omega_m >$$

$$+ M_{mr} \Big[ S'_{\omega_{rc}} I_r \omega_{rc} > + S'_{\omega^r_m} I_r \omega_{rc} >$$

$$+ \left( S'_{\omega_{rc}} I_r - I_r S'_{\omega_{rc}} \right) \omega^r_m > \Big] \qquad \text{(a-38)}$$

and

$$r_1 \cdot \dot{\bar{H}}_r = (1,0,0) \Big\{ I_r \left( \omega_{rc} > + \dot{\omega}^r_m > \right) + S'_{\omega^r_m} I_r \omega^r_m >$$

$$+ S'_{\omega_{rc}} I_r \omega_{rc} > + S'_{\omega^r_m} I_r \omega_{rc} > + \left( S'_{\omega_{rc}} I_r - I_r S'_{\omega_{rc}} \right)$$

$$\omega^r_m > \Big\} \qquad \text{(a-39)}$$

If the $\bar{r}>$ frame is the principal axis frame for body R, then

$$\bar{\omega}_{rc} \times \bar{\bar{I}}_r \cdot \bar{\omega}_{rc} = <\bar{r} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\phi}_{rc} \\ 0 & \dot{\phi}_{rc} & 0 \end{pmatrix} \begin{pmatrix} I_{r11} \dot{\phi}_{rc} \\ 0 \\ 0 \end{pmatrix} = \vec{0} \qquad \text{(a-40)}$$

Utilizing the principal axis assumptions of Eq. (a-40) the system equations may be written in open matrix form as shown in Fig (a-4). The matrix system block diagram is shown in Fig. (a-5).

$$
\left\{ \begin{pmatrix} I_{v_{11}} & I_{v_{12}} & I_{v_{13}} \\ I_{v_{21}} & I_{v_{22}} & I_{v_{23}} \\ I_{v_{31}} & I_{v_{32}} & I_{v_{33}} \end{pmatrix} + \begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi_{rc} & -s\phi_{rc} \\ 0 & s\phi_{rc} & c\phi_{rc} \end{pmatrix} \begin{pmatrix} I_{r_{11}} & 0 & 0 \\ 0 & I_{r_{22}} & 0 \\ 0 & 0 & I_{r_{33}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi_{rc} & s\phi_{rc} \\ 0 & -s\phi_{rc} & c\phi_{rc} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \right\} \begin{pmatrix} \dot{\omega}_{m_3} \\ \dot{\omega}_{m_2} \\ \dot{\omega}_{m_3} \end{pmatrix}
$$

$$
\begin{pmatrix} 0 & \omega_{m_3} & -\omega_{m_2} \\ \omega_{m_3} & 0 & \omega_{m_1} \\ \omega_{m_2} & \omega_{m_1} & 0 \end{pmatrix} \left\{ \begin{pmatrix} I_{v_{11}} & I_{v_{12}} & I_{v_{13}} \\ I_{v_{21}} & I_{v_{22}} & I_{v_{23}} \\ I_{v_{31}} & I_{v_{32}} & I_{v_{33}} \end{pmatrix} + \begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi_{rc} & -s\phi_{rc} \\ 0 & s\phi_{rc} & c\phi_{rc} \end{pmatrix} \begin{pmatrix} I_{r_{11}} & 0 & 0 \\ 0 & I_{r_{22}} & 0 \\ 0 & 0 & I_{r_{33}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi_{rc} & -s\phi_{rc} \\ 0 & s\phi_{rc} & c\phi_{rc} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} \omega_{m_1} \\ \omega_{m_2} \\ \omega_{m_3} \end{pmatrix}
$$

$$
+ \begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{rc} & s_{rc} \\ 0 & -s_{rc} & c_{rc} \end{pmatrix} \left\{ \begin{pmatrix} I_{r_{11}} & 0 & 0 \\ 0 & I_{r_{22}} & 0 \\ 0 & 0 & I_{r_{33}} \end{pmatrix} \begin{pmatrix} \dot{\phi}_{rc} \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & \omega^{r}_{m_3} & -\omega^{r}_{m_2} \\ -\omega^{r}_{m_3} & 0 & \omega^{r}_{m_1} \\ \omega^{r}_{m_2} & -\omega^{r}_{m_1} & 0 \end{pmatrix} \begin{pmatrix} I_{r_{11}}\dot{\phi}_{rc} \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (I_{r_{22}}-I_{r_{33}})\phi_{rc} \\ 0 & (I_{r_{22}}-I_{r_{33}})\phi_{rc} & 0 \end{pmatrix} \begin{pmatrix} \omega^{r}_{m_1} \\ \omega^{r}_{m_2} \\ \omega^{r}_{m_3} \end{pmatrix} \right\}
$$

$$
= \begin{pmatrix} T_{s_1} \\ T_{s_2} \\ T_{s_3} \end{pmatrix}
$$

$$
I_{r_{11}} \left( \ddot{\phi}_{rc} + \dot{\omega}^{r}_{m_1} \right) - \omega^{r}_{m_3} \quad \omega^{r}_{m_2} \left( I_{r_{22}} - I_{r_{33}} \right) = T_{r_1}
$$

Figure (a-4) Nonlinear Four Degree of Freedom System Equations

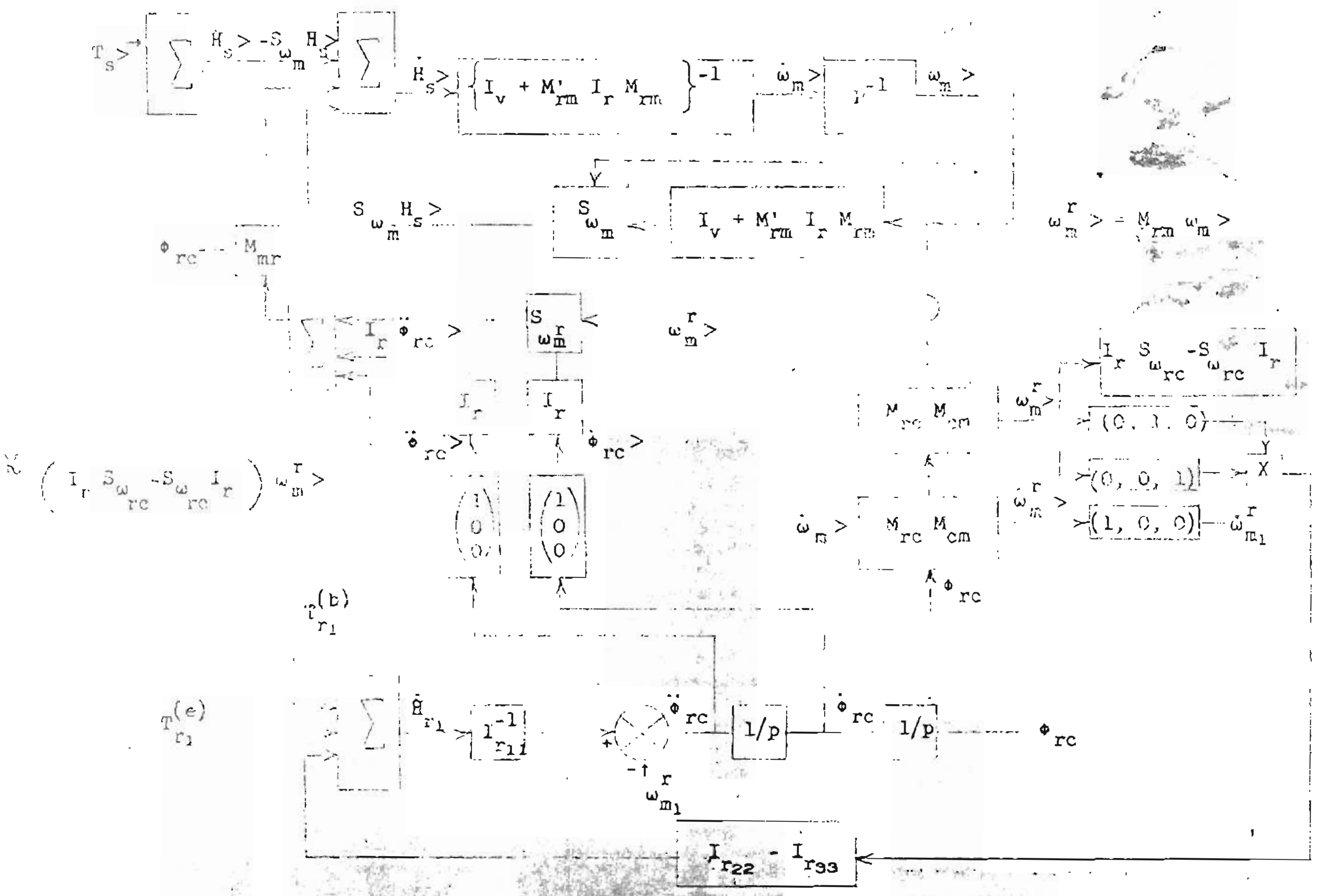

Figure (a-5) Matrix Block Diagram of Non-Linear Four Degree of Freedom System Equations

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## SYMBOLS AND NOMENCLATURE

Reference frame as used in this report refers to a triple of base vectors, whereas a coordinate system refers to a triple of base vectors and an origin.

### Symbols

$< = ( \,,\, ,\, .\, .\, .\, )$ a row array or matrix, ordered n-tuples.

$> = \left( \quad \right)$ a column array or matrix.

' Superscript, transposition, or interchange of rows and columns of an array or matrix, e.g., $>' = <$, or the transpose of a matrix A by A'.

w.r.t.: with respect to

O.N.: Ortho-normal--mutually perpendicular unit magnitude base vectors.

$\sum_{i=1}^{n}$ : summation

$\left[ \quad \right]$ square matrix

$\left.\begin{matrix} |A| \\ \det A \end{matrix}\right\}$ determinant of matrix A.

$|\bar{R}|$ Magnitude of a vector.

p: $\frac{d}{dt}$

$p^{-1}$: $\int dt$

$\left.\begin{matrix} s \\ c \end{matrix}\right\}$ sine function, cosine function

$$P = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} p^{-1} & 0 & 0 \\ 0 & p^{-1} & 0 \\ 0 & 0 & p^{-1} \end{pmatrix}$$

Scalars - No distinction is made between coordinates, real numbers, and scalar invariants as done by some authors. Scalars are generally designated by small letters, e.g., y, [illegible], etc.

Matrices - A matrix is used as an array of elements, e.g., (1x3) a row matrix, (3x1) column matrix, (3x3) a three row, three column array. Capital letters are used to designate matrices. The elements of the arrays may be vectors, scalars, dyadics, etc.

Vectors - Vectors are used in the Gibbsian sense as a barred entity, e.g., $\bar{R}$. The barred representation of the vector may be thought of as a 1x1 (1 row, 1 column) matrix. The representation of a vector as an ordered n-tuple of scalars is referred to as a column or row matrix of scalars.

Polyadic - Used in the Gibbsian sense. The majority of applications in this report are restricted to Dyadics.

Dyadic - A 1x1 entity designated with two bars as $\bar{\bar{D}}$. The juxtaposition (dyadic product) of two vectors yields a dyadic. It is shown in the report that an ordered triple of vectors (row matrix or column matrix of vectors) can be mapped to the 1x1 representation of the dyadic.

Transformation - The terms correspondence, transformations, map, operator, and functions are synonymous.

Metric Matrix - (Restricted meaning) Square matrices whose elements are scalar products of the base vectors spanning the space $\bar{b}_i \cdot \bar{b}_j$, Examples:

$$M_{bb} = \bar{b} > \cdot < \bar{b},\ M_{rr} = \bar{r} > \cdot < \bar{r} = \left[ \bar{r}_i \cdot \bar{r}_j \right]$$

Base Vectors:

The most basic entities of guidance and control problems are the base vectors. They are designated as

$$\bar{r}_i,\ \bar{r}_i,\ \bar{b}_i,\ \text{etc.}\ (i = 1, 2, 3).$$

$< \bar{r} = (\bar{r}_1, \bar{r}_2, \bar{r}_3)$: a row matrix or an ordered triple of base vectors.

$\bar{r}> = \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix}$: A column matrix of base vectors.

$M_{bb} = \bar{b} > . < \bar{b} = [\bar{b}_i \cdot \bar{b}_j]$ - Matric of inner products, metric matric of the $\bar{b} >$ space.

$M_{br} = \bar{b} > . < \bar{r} = [\bar{b}_i \cdot \bar{r}_j]$ - Transformation matrix between two O.N. bases, i.e., $\bar{b} > = M_{br} \bar{r} >$

$K_v^b = \bar{b}_1 \cdot (\bar{b}_2 \times \bar{b}_3)$ - Volume factor generated by $\bar{b}>$ basis, equals unity if $\bar{b} >$ is O.N.

$$S_{\omega_b} = \begin{pmatrix} 0 & \omega_{3b} & -\omega_{2b} \\ -\omega_{3b} & 0 & \omega_{1b} \\ \omega_{2b} & \omega_{1b} & 0 \end{pmatrix}$$

$\omega_m> = \begin{pmatrix} \omega_{m_1} \\ \omega_{m_2} \\ \omega_{m_3} \end{pmatrix}$ inertial angular velocity of $\bar{m}$ frame components for the $\bar{m} >$ space.

$\omega_m^r > = M_{rm} \, \omega_m>$ - Inertial angular velocity of $\bar{m} >$ frame transformed to the $\bar{r} >$ space.

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