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NEW DERIVATIONS AND MATRIX COMPARISONS OF THE BODWELL AND ODLE POSITION VECTOR SOLUTIONS FROM SIGHT LINE MEASUREMENTS

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CONTENTS

NEW DERIVATIONS AND MATRIX COMPARISONS OF THE BODWELL AND ODLE POSITION VECTOR SOLUTIONS FROM SIGHT LINE MEASUREMENTS

I. INTRODUCTION. A number of the test-ranges including WSMR use computer programs to process pointing angle data or unit sight-line (direction cosine) measurements. Tracking optical, and electronic instruments such as VTM's, tracking cines, AME, radars, etc., come in this category.

The mathware programmed into the software for these operating systems which are used as input to both real-time and post-flight recursive least squares filters is studied. The study in this report is limited to new state space derivations and comparisons between two well-known types of solutions; the Bodwell solution and the Odle solution references 1,2,3,4.

In most of these reports and studies, scalar equations that fill pages are presented, some of these single scalar equations may stretch out for one half to three quarters of a page and are very tedious. The modern analytical tools for the formulation and analysis of multivariable statistical estimation problems are state-vector and matrix methods. The dynamical propogation and cross coupling between the variance matrices for the interacting vectors are described by matrix Ricatti differential equations or their discrete recusive analogs. Solutions for these variance matrices at time points along the trajectories are used to weight the measurement vectors to produce state-estimates. Thus, it is very important to obtain the vector-matrix math models of the classical geometry associated with a sequence of discrete space points along a dynamical trajectory as observed by redundant instruments at each point. For it is exactly the mathematics of this non-linear geometry which relates the vector of measurements to the estimated state-vector namely position, velocity, and accelerations, which is a primary objective in most flight test.

For example, one of the ranges proposed an Odle solution on VTM data as input to an exponentially weighted on-line polynomial smoothing proceedure.

The math models for the propagation of these errors when fed through real-time or post-flight filters should be obtained and used as a simulation tool to compare filters and design improved processing schemes. The WSMR BET utilizes techniques from the Odle solution. A number of other WSMR programs utilize Bodwell solutions. Nothing conclusive can be said about the goodness of one technique over the other, since they are used as inputs to other recusive estimation subsystems. However, some ideas and comparisons between the two schemes from the stand point of computing efficiency can be obtained. How each of the solutions affect variance estimates for any particular type of polynomial estimator can be obtained on a particular case basis.

II. BODWELL AND ODLE EQUATIONS COMPARISONS. The mathematical complexity of the computations to obtain three coordinates of test vehicle position based on a system on n instrument stations measuring unit sight-line vectors (pointing angles or direction cosines) is presented in this section. The newly derived n station Bodwell solution as a 3x3 matrix inversion is compared with the n station Odle solution as a 3x3 matrix inversion. There are less mathematical operations necessary for the Odle solution.

The scalar derivations have been derived and utilized for a number of years in reference 1.

A matrix derivation of the two solutions was derived in reference 3, in which the n station Bodwell solution involved the inverse of an nxn matrix. The matrix solution was also obtained by R. Dale in reference 2 as an nxn matrix inversion.

In reference 3 the n station Bodwell solution was derived using orthogonal projections and the matrix generalized-inverse. The solution for the estimate of a point in 3-space involved the inversion of an nxn matrix (the number of stations). In the same report the n-station Odle solution was derived involving the inversion of a 3x3 matrix. One can also solve the n station Bodwell by taking two stations at a time and all combinations of two stations; for example with four stations, one has six combinations. The inversion of 2x2 matrices repeated six times can thus also be done. Of course in both methods the necessary number of computations, additions, multiplications, square roots, etc., can be done with no knowledge of matrix operations. These mathematical operations are of significance when one is considering computing efficiency and computing time as must be done for on-line or real-time applications.

In reference 2 Dale makes the statement that the drawback to the Bodwell solution is that an nxn matrix must be inverted; this is no longer necessary.

This study and report has been done for two basic reasons:

1. To obtain the n station Bodwell solution as a 3x3 matrix inversion and to compare the mathematical computational complexity with the corresponding n-station 3x3 matrix formulation of the Odle solution.

2. To show the simplicity of the algebraic approach and the vector-matrix structural relations when the problems are embedded in space of dimensions 3, n, 3n, and 3m, etc., where m is the number of combinations of difference vectors associated with n space points.

Future error vector analysis and variance matrix analysis studies should be made. Redundant measurements the degree of the approximating polynonial (usually 2 or 3) and the span of the filter are all parameters in the variance matrices.

This study was made to obtain the 3x3 matrix inversion solution to the n station Bodwell solution.

The n station Odle solution by Equation (93) is

$$\hat{y}\,(3\rangle_{op} = \left(\sum_{i=1}^{n} \underset{3\times3}{\tilde{P}_{ii}} \right)^{-1} \left[\mu\,(3\rangle_a - \sum \underset{3\times3}{P_{ii}}\, a\,(3\rangle_i \frac{1}{n} \right] \tag{1}$$

and the n-station Bodwell solution by Equation (74) is

$$\hat{y}\,(3\rangle_{ob} = -\left(\sum_{i=1}^{n} \underset{3\times3}{\tilde{P}_{ii}} \right)^{-1} \left[\left(\frac{\sum P_{ii}}{n} \right) \mu\,(3\rangle_a - \sum P_{ii}\, a\rangle_i \cdot \frac{1}{n} \right]$$
$$+ \left(\underset{3\times3}{I} + \sum_{i=1}^{n} \underset{3\times3}{P_{ii}} \right) \mu\,(3\rangle_a - \sum P_{ii}\, a\rangle_i \frac{1}{n} \tag{2}$$

where the 3x3 orthogonal projectors onto the measured unit sight line vectors $y\,(3\rangle_{ui}$ are

$$\underset{3\times3}{P_{ii}} = y\,(3\rangle_{ui} \langle 3)\, y_{ui} \tag{3}$$

and trace

$$t_r P_{ii} = \langle 3) y_{ui}\, y\, (3\rangle_{ui} = 1 \tag{4}$$

The orthogonal-complement projectors are

$$\underset{3x3}{\tilde{P}_{ii}} = \underset{3x3}{I} - \underset{3x3}{P_{ii}} \tag{5}$$

The known surveyed position vector to each station i is

$$a\,(3\rangle_i = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix}_i \tag{6}$$

and the mean vector of the $a\,(3\rangle_i$ is

$$\mu\,(3\rangle_a = \left(a\,(3\rangle_1 + \ldots + a\,(3\rangle_n \right) \frac{1}{n} \tag{7}$$

The 3x3 matrix to be inverted in Equation (1) is

$$\left(\sum \underset{3x3}{\tilde{P}_{ii}} \right)^{-1} = \left(n \underset{3x3}{I} - \sum \underset{3x3}{P_{ii}} \right)^{-1} \tag{8}$$

and the 3 rectangular coordinates of the Odle solution position vector are

$$\hat{y}\left(3\right\rangle_{op} = \begin{pmatrix} \hat{y}^1 \\ \hat{y}^2 \\ \hat{y}^3 \end{pmatrix}_{op} \quad (9)$$

and the three rectangular coordinates of the n-station Bodwell solution vector is

$$\hat{y}\left(3\right\rangle_{ob} = \begin{pmatrix} \hat{y}^1 \\ \hat{y}^2 \\ \hat{y}^3 \end{pmatrix}_{ob} \quad (10)$$

Clearly Equation (2) can be rearranged in a number of ways but there are definitely more mathematical terms in the Bodwell solution.

The n-station Bodwell criterion is based on an error minimization function which considers all combinations of difference (or error) vectors between two points at a time, or $\frac{n\,(n-1)}{2}$ vectors.

Another criterion could be based on only n difference vectors instead of $\frac{n\,(n-1)}{2}$. Error analysis to show what is gained or lost should be done in future studies.

The next section is a detailed derivation of the two solutions.

III. <u>DERIVATION OF BODWELL AND ODLE SOLUTIONS.</u> The mathematical problems and solutions derived in this section are deterministic mathematical optimization problems and their statistical counterparts will be discussed in a later report.

Mathematical problems I and II consider the forms of the equations when n unit sight-line vectors from n different origins intersect in a point. Mathematical problem III considers the case when the rays do not intersect.

<u>Mathematical Problem I:</u> Given n constant vectors $a\,(3\rangle_i$ with respect to a common origin 0, and the n unit sight-line vectors $y\,(3\rangle_{ui}$ from the n different origins to a common point at m whose three coordinates are known with respect to the common origin $y\,(3\rangle_{om}$ to find the n ranges $(r, \ldots r_n)m$ to the point m.

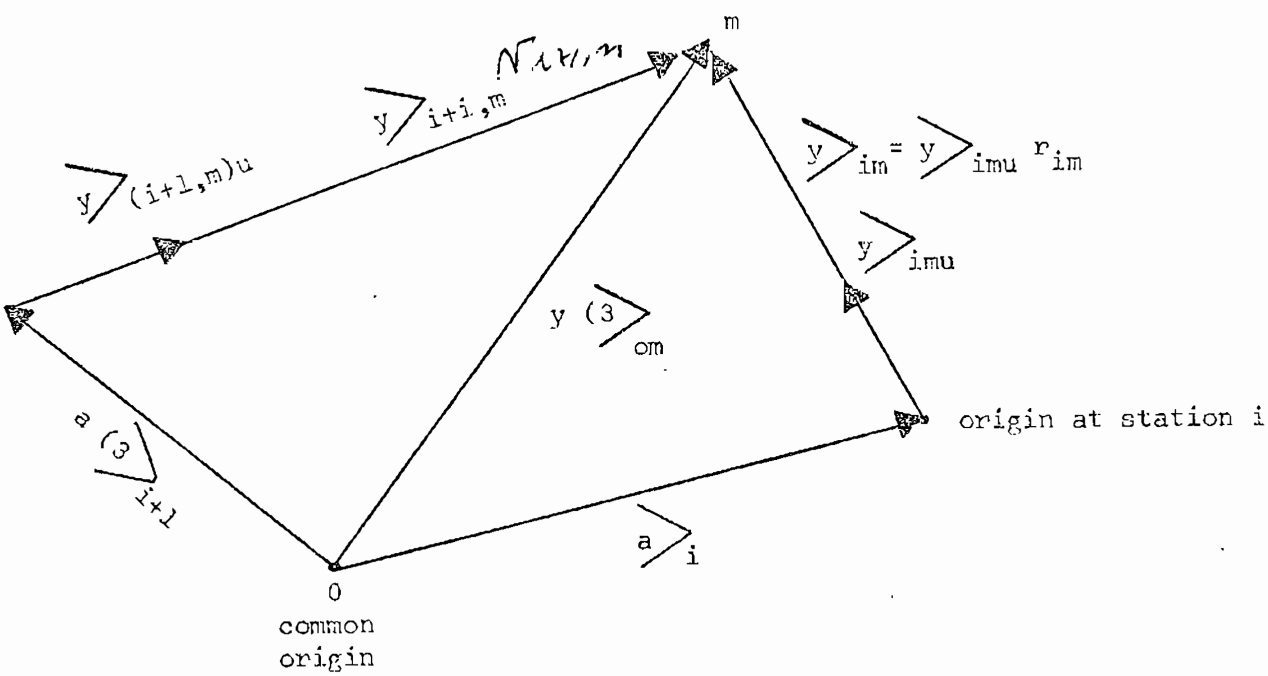

FIGURE (1)

N STATIONS LOOKING AT COMMON POINT

By Figure (1) we have n triangles

$$y\,(3\rangle_{om} = a\rangle_{i} + y\rangle_{imu}\, r_{im} \qquad (1)$$

$$i = 1, 2, \ldots n$$

It is assummed that all vectors have been transformed into a common base.

The solution is very simple since the unit vector has magnitude of unity, multiply Equation (1) as shown in Equation (2)

$$\langle 3)_{i}\, y_{u} \left[y\,(3\rangle_{om} - a\,(3\rangle_{i} \right] = {}_{i}\langle y_{u}\, y\rangle_{ui}\, r_{im} = r_{im} \qquad (2)$$

where

$$^{i}\langle y_{u}\, y\rangle^{i}_{u} = 1 \qquad (3)$$

If we package the n ranges as a column vector

$$\begin{bmatrix} r^{1} \\ r^{2} \\ \cdot \\ \cdot \\ \cdot \\ r^{n} \end{bmatrix}_{m} = \begin{bmatrix} {}^{1}\langle 3)\, y_{u} \\ {}^{2}\langle 3)\, y_{u} \\ \cdot \\ \cdot \\ \cdot \\ {}^{n}\langle 3)\, y_{u} \end{bmatrix} y\,(3\rangle_{om} - \begin{bmatrix} g_{11} \\ g_{22} \\ \cdot \\ \cdot \\ \cdot \\ g_{nn} \end{bmatrix}_{y_{ua}} \qquad (4)$$

$$r\,(n\rangle = \underset{n\times 3}{Y_u^T}\; y\,(3\rangle_{om} - g\,(n\rangle_{yua}$$

$$r\,(n\rangle = \left[\underset{n\times 3}{Y_u^T},\; g\,(n\rangle_{yua}\right]\begin{pmatrix} y\,(3\rangle_{om} \\ -1 \end{pmatrix} \tag{5}$$

and

$$g_{iiya} = \langle 3)\; y_{ui}\; a\,(3\rangle_i \tag{6}$$

Thus the triangle constraints in $L^{(3)}$ with a common vertex point m, constrains the n dimensional vector of ranges to the four demensional subspace of $L^{(n)}$ spanned by the columns of M

$$\left[Y_u^T,\; g\,(n\rangle\right] = M = \left[m\rangle_1, m\rangle_2, m\rangle_3, m\rangle_4\right]. \tag{7}$$

The following derivations will show the matrix structural aspects of the constraints when the problem is considered in $L^{(n)}$ and in $L^{(3n)}$ as discussed in appendix A for a sequence of n vectors in 3-space.

If the n vectors in $L^{(3)}$ to the common point **m** are expressed as a row of column vectors, then the 3xn matrix has rank one and by Equation (A-9) is

$$y\,(3\rangle_{om}\langle n)\,1 = \underset{3\times n}{A} + \underset{3\times n}{Y_{um}}\;\underset{n\times n}{D\,(r)_m} \tag{8}$$

where the unit magnitude vectors to point m are

$$\underset{3\times n}{Y_{um}} = \left[y\rangle_{ui} \; \cdots \; y\rangle_{un} \right]_m \tag{9}$$

and

$$\underset{n\times n}{D(r)} = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & r_n \end{pmatrix}_m \tag{10}$$

The following are given

$$y\,(3\rangle_{om}, \; \underset{3\times n}{A} \quad \text{and} \quad \underset{3\times n}{Y_{um}},$$

find D(r).

Multiply Equation (8) by $1^{*}\rangle$ of Equation (A-12)

$$\left[y\,(3\rangle_{om}\langle n)\, 1 - \underset{3\times n}{A} \right] \underline{1\,(n\rangle}_{n} = Y_{um}\, D(r)\, \underline{1\rangle}_{n} \tag{11}$$

or

$$n\left(y\,(3\rangle_{om} - \mu\,(3\rangle_{a} \right) = \underset{3\times n}{Y_{um}}\; r\,(n\rangle \qquad (12)$$

Multiply Equation (12) by Y^*_{um}

$$\underset{n\times 3}{Y^*_{um}} \left(y\,(3\rangle_{om} - \mu\,(3\rangle_{a} \right) n = \underset{n\,(3)\,n}{Y^*_{um}\, Y_{um}}\; r\,(n\rangle \qquad (13)$$

The nxn projector has rank three when Y_{um} has rank 3 or

$$\underset{n\times n}{P_{yuyu}} = \underset{n\,(3)\,n}{Y^*_{um}\, Y_{um}} \; . \qquad (14)$$

For the special case of $n \leq 3$, for example n=2, and full rank Equation (14) becomes

$$\underset{2\times 2}{P} = \underset{2\times 3}{Y^*_{u}}\; \underset{3\times 2}{Y} = \underset{2\times 2}{I} \qquad (15)$$

and by Equation (15) in Equation (13)

$$r\,(2\rangle = \underset{2\times 3}{Y^*_{u}} \left[y\,(3\rangle_{om} - \mu\,(3\rangle_{a} \right] n \qquad (16)$$

where

$$\underset{2\times 3}{Y^*_{u}} = \left(\underset{2\times 2}{Y^T_{u}\, Y_{u}} \right)^{-1} \underset{2\times 3}{Y^{-T}_{u}} \qquad (17)$$

$$= \begin{pmatrix} 1 & g_{12} \\ g_{12} & 1 \end{pmatrix}^{-1} \begin{bmatrix} {}^{1}\langle 3)\ y_u \\ {}^{2}\langle 3)\ y_u \end{bmatrix} \tag{18}$$

and

$$g_{12} = {}^{2}\langle y_u \quad y\rangle^{1}_{u} \tag{19}$$

For the case of n > 3 the non-full rank projector condition of Equation (13) can be avoided if the vectors are considered in $L^{(3n)}$ or by Equation (1), a column of column vectors

$$\begin{bmatrix} y\ (3\rangle \\ \cdot \\ \cdot \\ \cdot \\ y\ (3\rangle \end{bmatrix}_{om} = \underset{3nx3}{I_c}\ y\ (3\rangle_{om} = a\ (3n\rangle + \underset{3nxn}{D(y\rangle_{um}}\Big)\ r\ (n\rangle \tag{20}$$

where the 3nx3 matrix I_c is given by Equation (A-25), and where the 3nxn matrix of unit vectors times the nx1 matrix of ranges is by Equation (B-54)

$$\underset{3n\times n}{D}\left(\left(y\right\rangle_{nm}\right)\underset{n\times 1}{r\rangle} = \begin{bmatrix} y\,(3\rangle_{u1} & o\,(3\rangle & \cdot & 0\rangle \\ 0\,(3\rangle & y\,(3\rangle_{u2} & & \vdots \\ \vdots & \vdots & & \vdots \\ 0\,(3\rangle & 0\rangle & & y\,(3\rangle_{un} \end{bmatrix}_m \qquad (21)$$

$$\times \begin{bmatrix} r^1 \\ r^2 \\ \vdots \\ r^n \end{bmatrix}_m$$

Multiply Equation (23) by the psuedo inverse of D and solve for the vector of ranges

$$\underset{n\times 3n}{D^*}\left(\underset{3n\times 3}{I_c}\; y\,(3\rangle_{om} - a\,(3n\rangle\right) = r\,(n\rangle \qquad (22)$$

where by Equation (B-120)

$$\underset{n\times 3n}{D^*} = \underset{n\times 3n}{D^T}$$

23

Using Equation (23) in Equation (22) and noting that

$$D_y^{*} I_c = D_y^{T} I_c = \underset{n\times 3}{Y_u^{T}} \tag{24}$$

we have

$$r\,(n\rangle = \underset{n\times 3}{Y_u^{T}}\; y\,(3\rangle_{om} - g\,(n\rangle_{ya} \tag{25}$$

which is Equation (5) derived using psuedo inverses.

The previous analysis has been done to show the vector space structure of the problem when cast into the right vector space setting. To be sure Equation (1) can be solved n different times; but one looses the state-space techniques and when the equations become coupled get quickly snarled in the scalar tedium.

MATHEMATICAL PROBLEM II. This problem is the same as the previous with the exception that the three coordinates of the common point are not known but are also to be solved for. Stating the problem in terms of the matrices of Equation (8), that is

$$y\,(3\rangle_{mo}\langle n)\;1 = \underset{3\times n}{A} + Y_{um}\;\underset{n\times n}{D(r)} \tag{26}$$

we have given:

$$\underset{3xn}{A} \text{ and } \underset{3xn}{Y_{um}}$$

to solve for

$$\underset{nxn}{D(r)} \text{ and } y \left(3\right\rangle_{mo}$$

If we take the representation of Equation (23) and look at the previous solution for $r \left(n\right\rangle$ Equation (25) we do not know $y \left(3\right\rangle_{om}$.

Consider again Equation (20)

$$I_c \, y \left(3\right\rangle_{om} - a \left(3n\right\rangle = D(yu) \, r \left(n\right\rangle \tag{27}$$

and multiply (27) by the difference operator of Equation (B-48)

$$\underset{3mx3n}{\delta F} \left[I_c \, y \left(3\right\rangle_{om} - a \left(3n\right\rangle \right] = \delta F \, D(yu) \, r \left(n\right\rangle \tag{28}$$

and observe

$$\underset{3m\times 3n}{\mathcal{F}} \quad \underset{3n\times 3}{I_c} = \underset{3m\times 3}{[0]} \tag{29}$$

By (B-48) we see Equation (29) is

$$\underset{3m \times 3n}{\begin{bmatrix} -I, & I, & 0 & . & . & 0 \\ -I & 0 & I & 0 & . & . \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & & & & -I & I \end{bmatrix}} \underset{3n\times 3}{\begin{pmatrix} I \\ I \\ \\ \\ \\ I \end{pmatrix}} = \underset{3m\times 3}{\begin{pmatrix} 0_{3\times 3} \\ . \\ . \\ . \\ . \\ 0_{3\times 3} \end{pmatrix}} \tag{30}$$

Using Equation (29) in Equation (28)

$$-\mathcal{F}\, a \,(3n\rangle = S_{yu}\, r\,(n\rangle \tag{31}$$

or

$$-\Delta\, a\,(3m\rangle = S_{yu}\, r\,(n\rangle = -\mathcal{F}\, a\,(3n\rangle \tag{32}$$

where the 3m vector $\Delta\, a\, (3m\rangle$ is the column vector of all combinations of column vector differences of the known three dimensional station vectors $a\,(3\rangle_i$, and by Equation (B-57)

$$\underset{3m\times n}{\mathcal{F}\, D(yu)} = S_{yu} = \begin{bmatrix} -y\,(3\rangle_{u1} & +y\rangle_{u2} & \cdots & 0\rangle \\ -y\rangle_{u1} & 0\rangle & y\rangle_{u3} & \cdots \\ 0\rangle & y\rangle_{u3} & & \\ \cdot & 0\rangle & & \\ \cdot & \cdot & & -y\rangle_{u\,n-1} \\ 0\,(3\rangle & 0\rangle & & y\rangle_{un} \end{bmatrix}_m \qquad (33)$$

The vector $r\,(n\rangle$ is obtained from Equation (32) as

$$r\,(n\rangle = -S^{*}_{yu}\ \Delta\, a\,(3m\rangle = -S^{*}_{yu}\,\mathcal{F}\, a\,(3n\rangle \qquad (34)$$

Equation (34) by Equation (B-121),(B-117) is

$$r\,(n\rangle_m = \underset{n\times n}{-D^{T}_{yu}}\,\tilde{a}\,(3n\rangle - \underset{n\times 3}{Y^{T}_{u}} \left(\sum_{i=1}^{n} \underset{3\times 3}{\tilde{P}_{ii}} \right)^{-1} \left(\underset{3\times 3}{P_{11}}\ ,\ \ldots\ \underset{3\times 3}{P_{nn}} \right) \tilde{a}\,(3n\rangle_I \qquad (35)$$

where

$$\tilde{a}\,(3n\rangle_I = \begin{pmatrix} a\,(3\rangle_1 - \mu\,(3\rangle_a \\ \cdot \\ \cdot \\ \cdot \\ a\,(3\rangle_n^1 - \mu\,(3\rangle_a \end{pmatrix} \tag{36}$$

and the 3x3 projectors are

$$\underset{3\times3}{P_{ii}} = y\,(3\rangle_{ui}\langle 3)\,y_{ui} \tag{37}$$

and the mean of the $a\,(3\rangle_i$ vectors is

$$\mu\,(3\rangle_a = \sum_{i=1}^{n} a\,(3\rangle_i \,\frac{1}{n} \tag{38}$$

The three dimensional position vector $y\,(3\rangle_{om}$ can be obtained from Equation (26) by multiplying on the right by $1\,^*(n\rangle$ or

$$y\,(3\rangle_{om} = A\,1\,^*(n\rangle + Y_u\,r\,(n\rangle\left(\frac{1}{n}\right) \tag{39}$$

and by Equation (A-38) for the mean

$$y\,(3\rangle_{om} = \mu\,(3\rangle_{a} + \underset{3\times n}{Y_u}\,\underset{n\times 1}{r\rangle}\left(\frac{1}{n}\right) \tag{40}$$

Equation (40) can also be obtained from Equation (27) by multiplying on the left by I_c^* or

$$y\,(3\rangle_{om} = I_c^*\,a\,(3n\rangle + I_c^*\,D(yu)\,r\,(n\rangle \tag{41}$$

where

$$I_c^*\,a\,(3n\rangle = I_c^T\,a\,(3n\rangle\,\frac{1}{n} = \mu\,(3\rangle_{a} \tag{42}$$

and

$$I_c^*\,D(yu) = I_c^T\,D(yu)\,\frac{1}{n} = Y_u\left(\frac{1}{n}\right) \tag{43}$$

Using Equations (42) and (43) in Equation (41)

$$y\,(3\rangle_{om} = \mu\,(3\rangle_{a} + Y_u\,r\,(n\rangle\left(\frac{1}{n}\right) \tag{44}$$

Y_M

Using r (n$\rangle$ of Equation (35) in Equation (44)

$= (P_{11}, P_{22} \cdot \underset{3\times3}{P_{nn}})$

$$y\,(3\rangle_{om} = \mu\,(3\rangle_a - \underset{3\times n}{Y_M}{}^{T} D(yu)\,\tilde{a}\,(3n\rangle_I \frac{1}{n}$$

$$- \Big(\underset{3\times3}{\sum \tilde{P}_{ii}}\Big)^{-1}\Big(P_{11}, \dots P_{nn}\Big)\,\tilde{a}\,(3n\rangle_I \left(\frac{1}{n}\right) \qquad (45)$$

By Equation (B-120) Equation (45) can be written as

$$y\,(3\rangle_{om} = \mu\,(3\rangle_a - \left[\underset{3\times3}{I} - \Big(\underset{3\times3}{\sum \tilde{P}_{ii}}\Big)^{-1}\right]\left[\underset{3\times3}{P_{11}}, \dots \underset{3\times3}{P_{nn}}\right]\tilde{a}\,(3n\rangle \left(\frac{1}{n}\right) \qquad (46)$$

By Equation (B-114)

$$\left(\frac{I}{n}\right)\underset{3\times3n}{\left[\underset{3\times3}{P_{11}}, P_{22} \dots \underset{3\times3}{P_{nn}}\right]}\,\tilde{a}\,(3n\rangle_I = \underset{3\times3}{\sum P_{ii}}\, a\,(3\rangle_i \left(\frac{1}{n}\right) \qquad (47)$$

$$- \Big(\underset{3\times3}{\sum P_{ii}}\Big)\,\mu\,(3\rangle_a \left(\frac{1}{n}\right)$$

Using Equation (47) in Equation (46)

$$y \left(3\right\rangle_{om} = \mu \left(3\right\rangle_a - \left[\underset{3x3}{I} - \left(\sum \underset{3x3}{\tilde{P}_{ii}} \right)^{-1} \right] \left[\sum P_{ii}\, a \left(3\right\rangle_i \frac{1}{n} - \sum P_{ii} \quad \mu \left(3\right\rangle_a \frac{1}{n} \right] \quad (48)$$

$$y \left(3\right\rangle_{om} = \mu \left(3\right\rangle_a - \left\{ \sum P_{ii}\, a \left(3\right\rangle_i \frac{1}{n} - \sum P_{ii}\, \mu \left(3\right\rangle_a \left(\frac{1}{n}\right) \right\} \quad (49)$$

$$+ \left(\sum \tilde{P}_{ii} \right)^{-1} \sum P_{ii}\, a \left(3\right\rangle_i \left(\frac{1}{n}\right) - \left(\sum \tilde{P}_{ii} \right)^{-1} \left(\sum P_{ii} \right) \mu \left(3\right\rangle_a \left(\frac{1}{n}\right)$$

$$= \left[\underset{3x3}{nI} + \underset{3x3}{\sum P_{ii}} - \left(\sum \tilde{P}_{ii} \right)^{-1} \left(\sum P_{ii} \right) \right] \mu \left(3\right\rangle_a \left(\frac{1}{n}\right)$$

$$- \left(\frac{1}{n}\right) \sum P_{ii}\, a \left(3\right\rangle_i + \left(\sum \tilde{P}_{ii} \right)^{-1} \left(\sum P_{ii} \right) a \left(3\right\rangle_i \left(\frac{1}{n}\right) \quad (50)$$

$$= \left[\underset{3x3}{In} + \sum P_{ii} - \left(\sum \tilde{P}_{ii} \right)^{-1} \sum P_{ii} \right] \mu \left(3\right\rangle_a \left(\frac{1}{n}\right)$$

$$- \left[\sum P_{ii}\, a \left(3\right\rangle_i \left(\frac{1}{n}\right) - \left(\sum \tilde{P}_{ii} \right)^{-1} \sum P_{ii}\, a \left(3\right\rangle_i \left(\frac{1}{n}\right) \right] \quad (51)$$

$$= \left\{ n \underset{3\times3}{I} + \left[\underset{3\times3}{I} - \left(\sum \tilde{P}_{ii} \right)^{-1} \right] \sum P_{ii} \right\} \mu\, (3\rangle_a \left(\frac{1}{n} \right)$$

$$- \left[\underset{3\times3}{I} - \left(\sum \tilde{P}_{ii} \right)^{-1} \right] \sum P_{ii}\, a\, (3\rangle_i \left(\frac{1}{n} \right) \qquad (52)$$

$$y\, (3\rangle_{om} = \mu\, (3\rangle_a + \left[\underset{3\times3}{I} - \left(\underset{3\times3}{\sum \tilde{P}_{ii}} \right)^{-1} \right] \underset{3\times3}{\sum P_{ii}} \left(\frac{1}{n} \right) \mu\, (3\rangle_a$$

$$- \left[\underset{3\times3}{I} - \left(\underset{3\times3}{\sum \tilde{P}_{ii}} \right)^{-1} \right] \underset{3\times3}{\sum P_{ii}}\, a\, (3\rangle_i \left(\frac{1}{n} \right) \qquad (53)$$

$$y\, (3\rangle_{om} = \mu\, (3\rangle_a + \left[\underset{3\times3}{I} - \left(\underset{3\times3}{\sum \tilde{P}_{ii}} \right)^{-1} \right] \left[\left(\underset{3\times3}{\sum P_{ii}} \right) \mu\, (3\rangle_a - \underset{3\times3}{\sum P_{ii}}\, a\, (3\rangle_i \right] \frac{1}{n} \qquad (54)$$

MATHEMATICAL PROBLEM (III-a) MINIMIZATION PROBLEM): Given n constant vectors $a\,(3\rangle_i$ with respect to a common origin 0 and n unit sight-line vectors $y\,(3\rangle_{ui}$ from the n different origins (the rays in general do not intersect) to find n different vectors $y\,(3\rangle_i$ from the n different origins each vector of which lies along the unit sight-line vectors (which is to say, find n ranges r_i) that is $y\,(3\rangle_i = y\,(3\rangle_{ui}$ such that the sums of the squares of the magnitudes of all combinations of difference vectors of the n position vectors with respect to the common origin is minimized.

(b). Given the n vectors $y\,(3\rangle_i$ obtained above, find a single vector with **respect** to the common origin which is the mean or the weighted mean of the n vectors $y\,(3\rangle_i$.

The vectors for this case are shown in Figure (2)

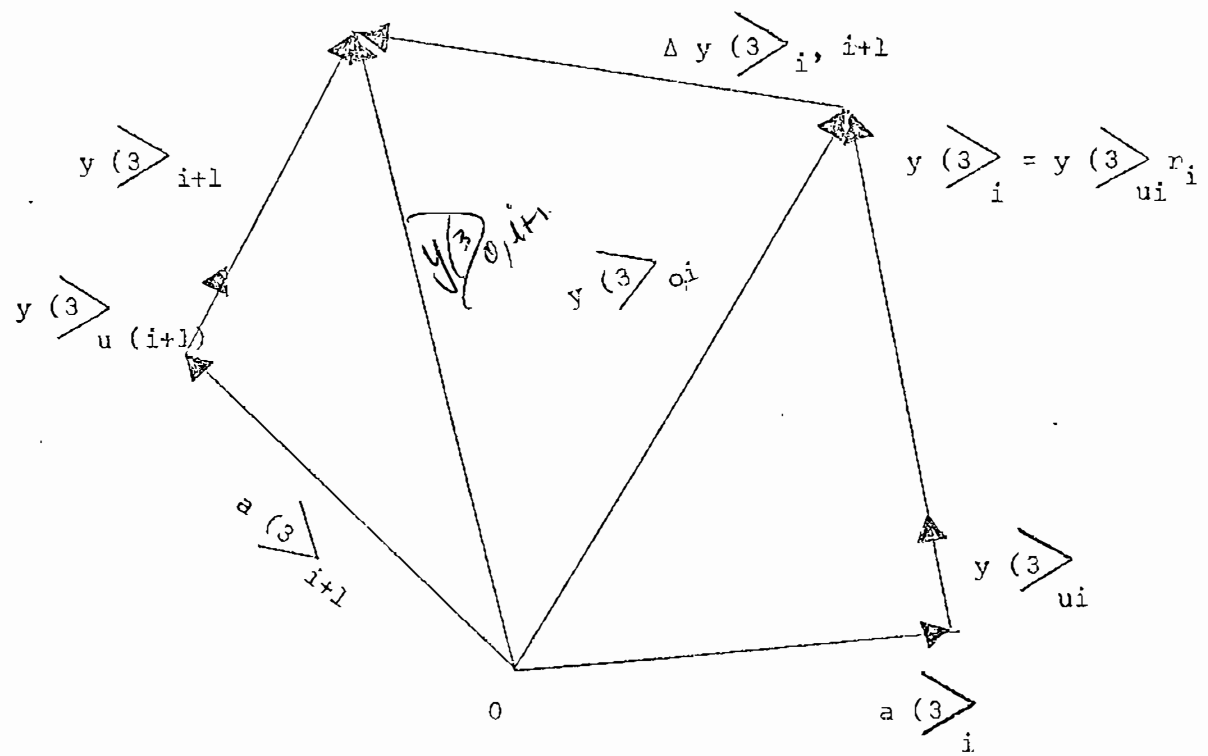

FIGURE (2)

DIFFERENCE VECTORS BETWEEN RAYS

The position vectors with respect to the common origin is

$$y\,(3\rangle_{oi} = a\,(3\rangle_{i} + y\,(3\rangle_{ui}\; r_i \qquad (55)$$

or packaged in $L^{(3)}$ for n vectors

$$\underset{3xn}{Y_o} = \underset{3xn}{A} + \underset{3xn}{Y_u}\, D(r) \qquad (56)$$

where Y_o has rank 3. In $L^{(3n)}$

$$y\,(3n\rangle_{o} = a\,(3n\rangle + \underset{3nxn}{D(yu)}\; r\,(n\rangle \qquad (57)$$

Two approaches will be shown. Multiply Equation (56) on the right by the differencing operator of Equation (B-34)

$$\underset{3xn}{Y_o}\;\underset{nxm}{\mathcal{F}} = \underset{3xm}{\Delta Y_o} = \underset{3xm}{\Delta A} + Y_u\, D(r)\, \mathcal{F} \qquad (58)$$

Transpose Equation (58)

$$\underset{mx3}{\Delta Y_o^T} = \underset{mx3}{\Delta A^T} + \mathcal{F}^T\, D(r)\, Y_u^T \qquad (59)$$

Take the product of Equation (58) and Equation (59)

$$\Delta Y_o \, \Delta Y_o^T = \left(\Delta A + Y_u D(r) \mathcal{F} \right) \left(\Delta A^T + \mathcal{F}^T D(r) \, Y_u^T \right)$$

$$= \Delta A \Delta A^T + \Delta A \mathcal{F}^T D(r) \, Y_u^T$$

$$+ \; Y_u D(r) \mathcal{F} \, \Delta A^T + Y_u D(r) \mathcal{F} \mathcal{F}^T D(r) \, Y_u^T \qquad (60)$$

Take the trace of Equation (60) and the partial derivative with respect to the r_i. The steps will not be shown here; they are derived in reference 3.

The algebraic orthogonal projection approach will be derived. Multiply the $L^{(3n)}$ expression of Equation (57) by the differencing operator of Equation (b-48)

$$\underset{3m\times 3n}{\mathcal{F}} \; y \, (3n\rangle_o = \Delta y \, (3m\rangle_o = \Delta a \, (3m\rangle + \underset{3m\times n}{S_{(yu)}} \, r \, (n\rangle \qquad (61)$$

where

$$\underset{3m\times n}{S_{(yu)}} = \underset{3m\times 3n \; 3n\times n}{\mathcal{F} \, D(yu)} \qquad (62)$$

and is given by Equation (B-58).

Note in Equation (61) $\Delta a\ (3m\rangle$ and S(yu) are given, we want to find $r\ (n\rangle$ and $y\ (3m\rangle_o$ to satisfy definable mathematical criterion, namely the least squares criterion shown in Figure (3) in $L^{(3m)}$.

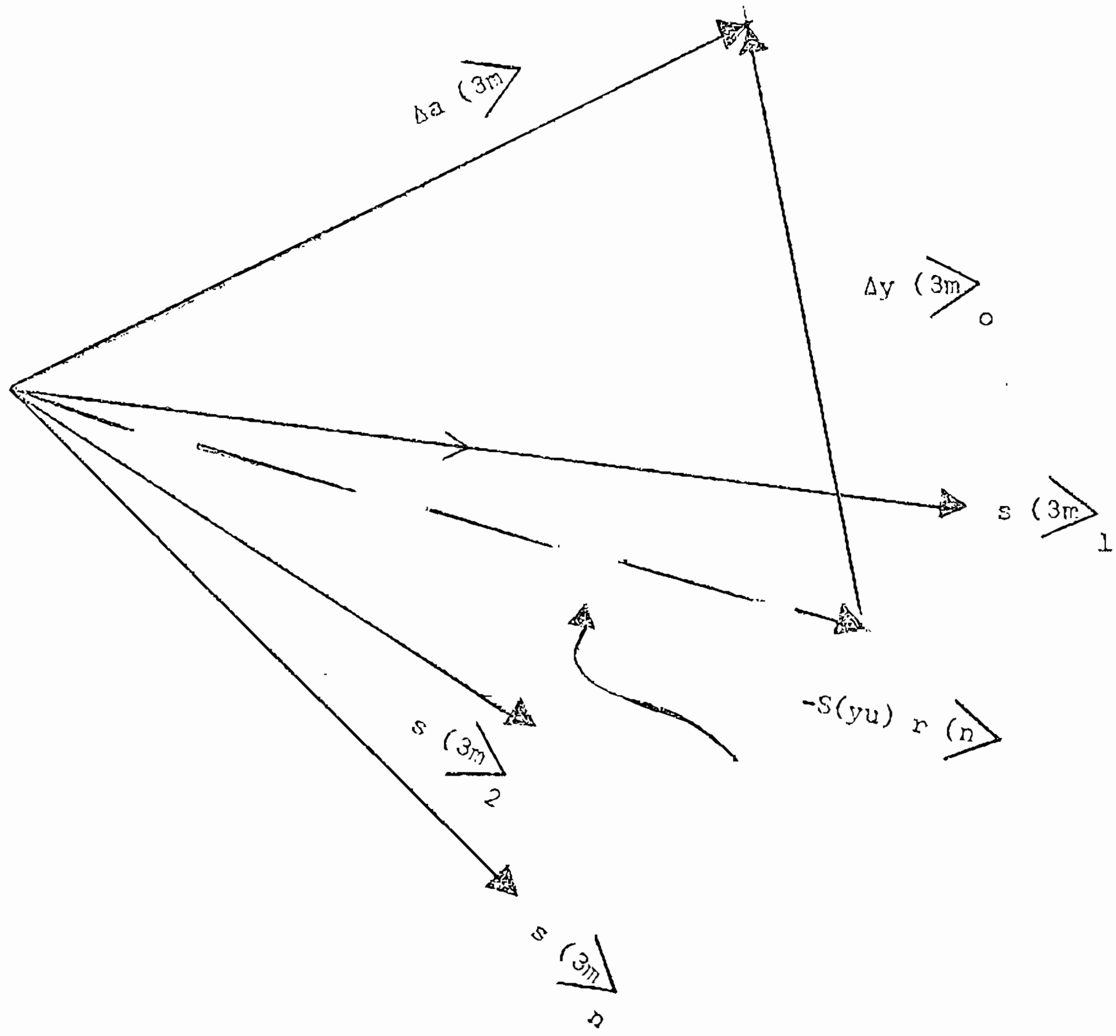

FIGURE (3)
N-DIMENSIONAL SUBSPACE OF $L^{(3m)}$ FOR ARBITRARY $r\ (n\rangle$

The n-dimensional subspace spanned by the n column vectors of S(yu) is shown in Figure (3). Equation (62) can be written as

$$\Delta\ (3m\rangle = -S_{(yu)}\ r\ (n\rangle + \Delta y\ (3m\rangle_{o} \qquad (63)$$

for any n coordinates $r\ (n\rangle$, where $\Delta y\ (3m\rangle$ is not perpendicular to the subspace.

The set of n coordinates $r\ (n\rangle$ which make $\Delta y\ (3n\rangle_{o}$ orthogonal to the subspace spanned by $S_{(yu)}$ is obtained by multiplying Equation (63) by the psuedo inverse

$$S^{*}_{(yu)}\ \Delta a\ (3m\rangle = -\underbrace{S^{*}_{(yu)}\ S_{(yu)}}_{nxn}\ \hat{r}\ (n\rangle = -\hat{r}\ (n\rangle \qquad (64)$$

where

$$S^{*}_{yu}\ \Delta y\ (3m\rangle_{o} = 0\ (n\rangle \qquad (65)$$

as shown in Figure (4)

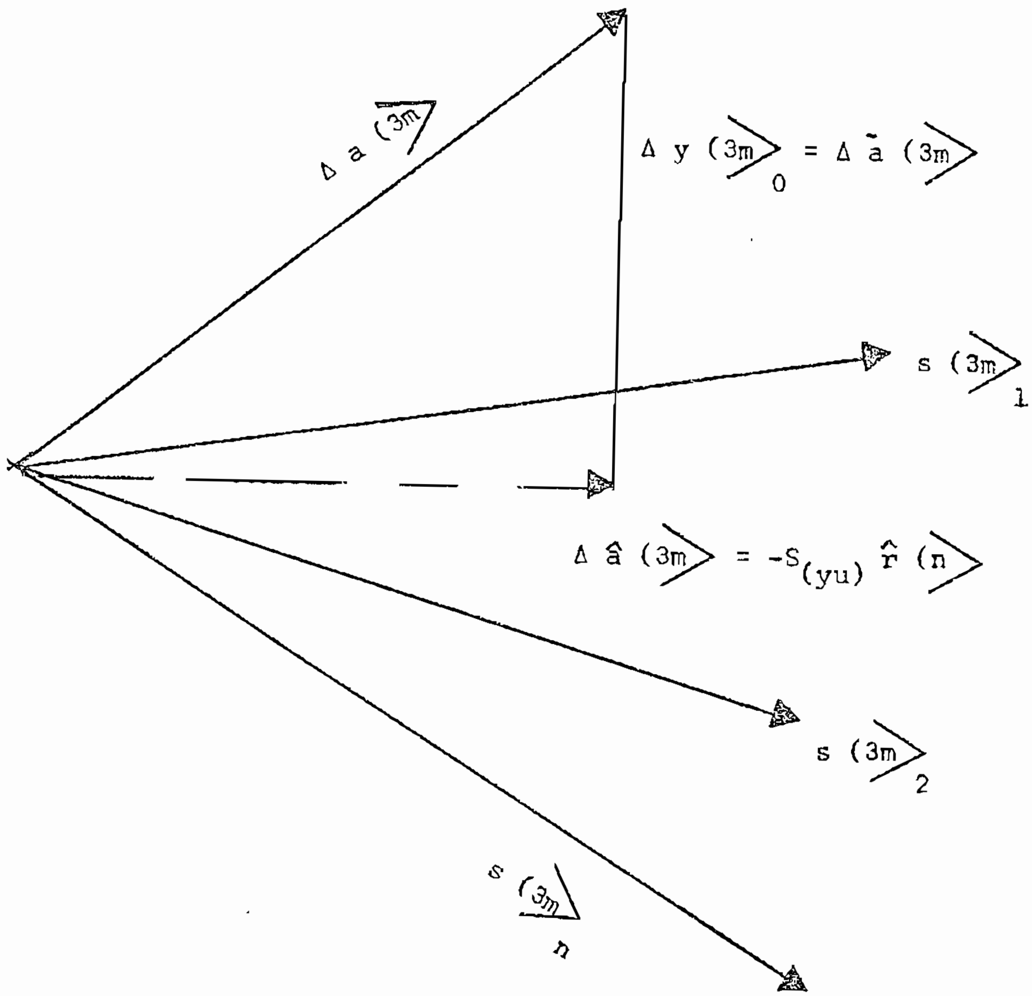

FIGURE (4)
ORTHOGONAL PROJECTION IN $L^{(3m)}$

The vector of n ranges satisfying the geometry Figure (4) is by Equation (64)

$$r\,(n\rangle = -\underset{nx3m}{S^{*}_{(yu)}}\;\underset{3mx3n}{F}\;a\,(3n\rangle \qquad (66)$$

The expression for $r\ (n\rangle$ is given by Equation (35) where it is to be understood that in Equation (35) the unit sight-line vectors are all directed to a common point m and the exact notation is $y\ (3\rangle_{uim}$ where as the unit sight-line vectors in Equation (66) do not intersect. If we use Equation (66) in Equation (67)

$$y\ (3n\rangle_0 = a\ (3n\rangle - D(yu)\, S^{*}_{(yu)}\, \mathcal{F}\ a\ (3n\rangle$$

$$= \left(\underset{3n\times 3n}{I} - \underset{3n\times n}{D(yu)}\ \underset{n\times 3m}{S^{*}_{(yu)}}\ \underset{3m\times 3n}{\mathcal{F}} \right) a\ (3n\rangle \qquad (67)$$

where the $y\ (3n\rangle_{opt}$ vector of Equation (67) corresponds to the $\hat{r}\ (n\rangle$ ranges and in $L^{(3)}$ the n vectors are shown in Figure (5) and designated as $y\ (3\rangle_{opt,i}$ as shown.

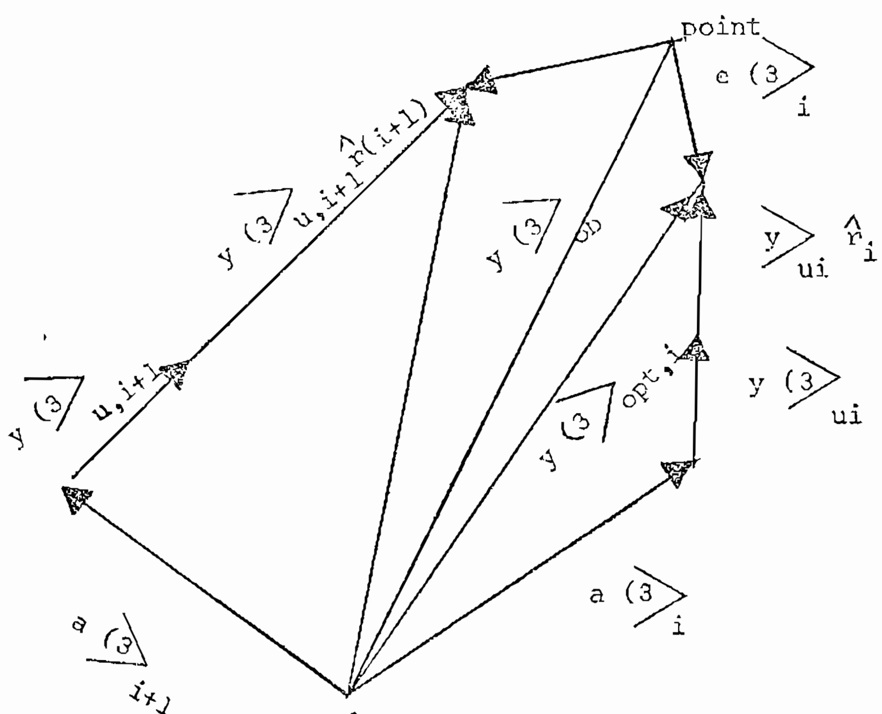

FIGURE (5)

POSITION VECTORS CORRESPONDING TO $\hat{r}\ (n\rangle$

If we consider the n position vectors of Figure (5) from the common origin 0 to n points along the unit vector rays corresponding to the $\hat{r}\,(n\rangle$ and take an arbitrary common point b, the n additive equations are

$$y\,(3\rangle_{opt,i} = y\,(3\rangle_{ob} + e\,(3\rangle_i \tag{68}$$

and package wise by Equation (68) and Equation (67)

$$\begin{aligned} y\,(3n\rangle_{opt} &= \underset{3n\times 3}{I_c}\; y\,(3\rangle_{ob} + e\,(3n\rangle \\ &= \left[\underset{3n\times 3n}{I} - D(yu)\, S^{*}_{(yu)}\, \mathcal{F} \right] a\,(3n\rangle \end{aligned} \tag{69}$$

Multiply Equation (69) by I_c^*

$$I_c^*\, y\,(3n\rangle_{opt} = y\,(3\rangle_{ob} + I_c^*\, e\,(3n\rangle = I_c^* \left[I - D(yu)\, S^{*}_{(yu)}\, \mathcal{F} \right] a\,(3n\rangle$$

If we select a point $\hat{y}\,(3\rangle_{ob}$ such that

$$I_c^*\, e\,(3n\rangle = 0\,(3\rangle \tag{70}$$

then unweighted mean $\hat{y}\,(3\rangle_{ob}$ is

$$\hat{y}\,(3\rangle_{ob} = \left[I_c^* - I_c^* D(yu)\, S^*_{(yu)}\, \mathcal{F} \right] a\,(3n\rangle \tag{71}$$

$$= I_c^*\, a\,(3n\rangle - I_c^* D(yu)\, S^*_{(yu)}\, \mathcal{F}\, a\,(3n\rangle \tag{72}$$

or by Equation (41) Equation (43) and Equation (66)

$$\hat{y}\,(3\rangle_{ob} = \mu\,(3\rangle_a + \underset{3xn}{Y_u}\,\hat{r}\,(n\rangle \tag{73}$$

Equation (73) is the same as Equation (44) with the exception of the interpretation of the unit sight-line vectors, hence Equation (73) by Equation (54) is

$$\hat{y}\,(3\rangle_{ob} = \mu\,(3\rangle_a + \left[\underset{3x3}{I} - \left(\underset{3x3}{\sum \tilde{P}_{ii}} \right)^{-1} \right] \left[\left(\underset{3x3}{\sum P_{ii}} \right) \mu\,(3\rangle_a - \underset{3x3}{\sum P_{ii}}\, a\,(3\rangle_i \right] \frac{1}{n} \tag{74}$$

where the projectors

$$P_{ii} = y\,(3\rangle_{ui}\,\langle 3)\,y_{ui} \tag{75}$$

<u>MATHEMATICAL PROBLEM (III.b)</u> Given n constant vectors $a\,(3\rangle_i$ with respect to a common origin 0 and n unit sight-line vectors $y\,(3\rangle_{ui}$ from the n different origins (the rays in general do not intersect). For any single 3 dimensional vector point at p. Figure (6) with respect to the common origin the n vectors from the n different origins to this single point can be expressed as a component along the unit sight-line vector $y\,(3\rangle_{ui}$ plus an additive component perpendicular to the ray. Find a single point p such that the sums of the squares of magnitudes of all the n components perpendicular to the sight-line is a minimum.

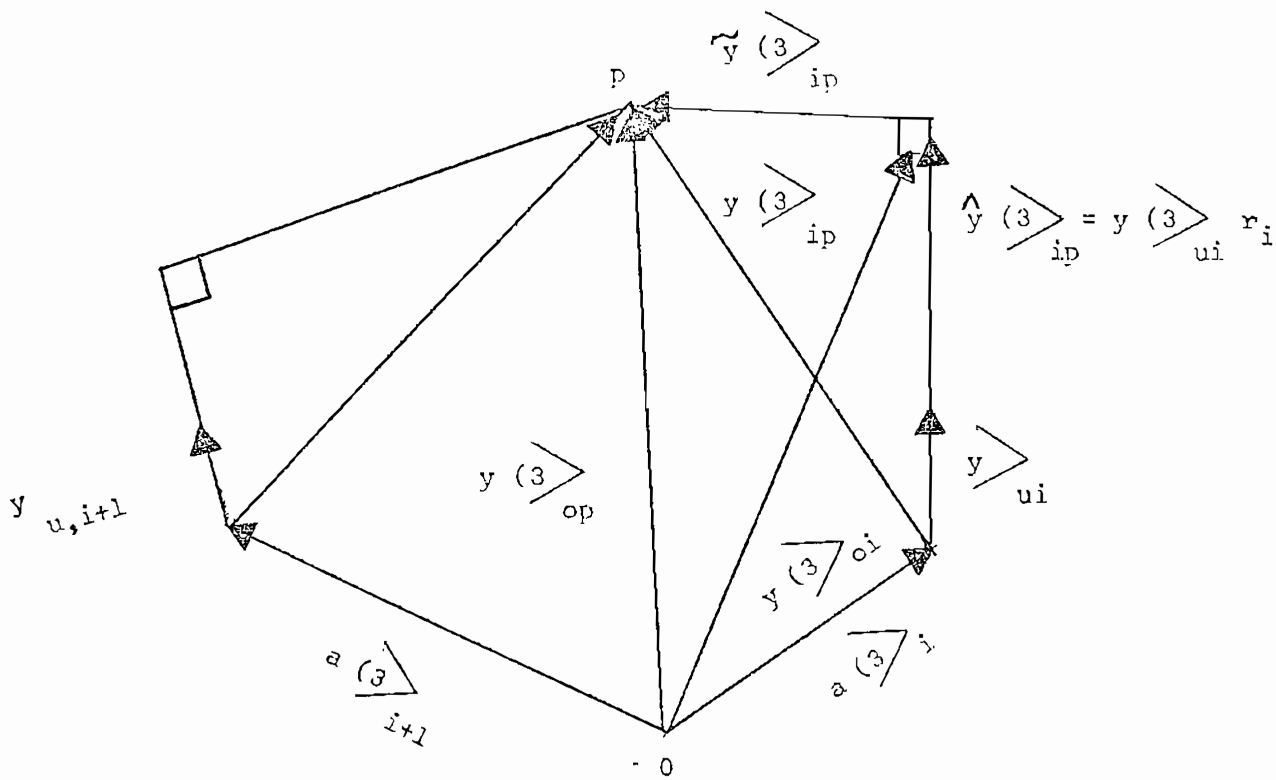

FIGURE (6)

ORTHOGONAL PROJECTION FROM ANY POINT p ONTO UNIT-SIGHT-LINE VECTORS

From Figure (6) it is clear that for any point p in $L^{(3)}$ one can decompose the vector into two components

$$y^{(3)}\rangle_{ip} = \hat{y}^{(3)}\rangle_{ip} + \tilde{y}^{(3)}\rangle_{ip} \qquad (76)$$

where the orthogonal projection onto the unit vector is

$$\hat{y}^{(3)}\rangle_{ip} = \underset{3\times3}{P_{ii}}\; y^{(3)}\rangle_{ip} = y^{(3)}\rangle_{ui}\; r_i \qquad (77)$$

and the orthogonal complement term is

$$\tilde{y}\ (3\rangle_{ip} = \tilde{P}_{ii}\ y\ (3\rangle_{ip} \tag{78}$$

where

$$\underset{3x3}{P_{ii}} = y\ (3\rangle_{ui} \langle 3)\ y_{ui} \tag{79}$$

and

$$\tilde{P}_{ii} = \underset{3x3}{I} - \underset{3x3}{P_{ii}} \tag{80}$$

By the geometry of Figure (6)

$$y\ (3\rangle_{op} = a\ (3\rangle_{i} + y\ (3\rangle_{ui}\ r_i + \tilde{y}\ (3\rangle_{ip} \tag{81}$$

or package wise

$$\underset{3nx3}{I_c}\ y\ (3\rangle_{op} = a\ (3n\rangle + \underset{3nxn}{D(yu)}\ r\ (n\rangle + \begin{bmatrix} \tilde{y}\ (3\rangle_{ip} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{y}\ (3\rangle_{np} \end{bmatrix} \tag{82}$$

Multiply Equation (82) by $\overset{*}{D}(yu)$ and solve for $r\ (n\rangle$ for any p

$$r\ (n\rangle = \overset{*}{D}(yu) \left[I_c\ y\ (3\rangle_{op} - a\ (3n\rangle \right] \tag{83}$$

since by construction

$$\overset{*}{D}(yu) \begin{pmatrix} \tilde{y}\ (3\rangle_{ip} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{y}\ (3\rangle_{np} \end{pmatrix} = 0\ (n\rangle \tag{84}$$

that is $\langle 3)_i\ y_u \tilde{y}\ (3\rangle_{ip} = 0$ (85)

Substituting Equation (83) into Equation (82)

$$I_c\ y\ (3\rangle_{op} = a\ (3n\rangle + D(yu)\ \overset{*}{D}(yu)\ I_c\ y\ (3\rangle_{op}$$

$$- \ D(yu)\ \overset{*}{D}(yu)\ a\ (3n\rangle + \begin{bmatrix} \tilde{y}\ (3\rangle_{ip} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{y}\ (3\rangle_{np} \end{bmatrix} \tag{86}$$

or

$$\left(I - P \right)\ I_c\ y\ (3\rangle_{op} = \left(I - P \right)\ a\ (3n\rangle$$

$$+ \begin{bmatrix} \tilde{y}\ (3\rangle_{ip} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{y}\ (3\rangle_{np} \end{bmatrix} \tag{87}$$

where the matrix $\underset{3nx3n}{P}$ is Equation (B-126)

$$\underset{3nx3n}{P} = \begin{bmatrix} \underset{3x3}{P_{11}} & 0 & \cdots & 0 \\ 0 & \underset{3x3}{P_{22}} & 0 & \vdots \\ \vdots & & \ddots & 0 \\ 0 & & & \underset{3x3}{P_{nn}} \end{bmatrix} \tag{88}$$

and

$$\underset{3nx3n}{I} - \underset{3nx3n}{P} = \underset{3nx3n}{\tilde{P}} = \begin{bmatrix} \underset{3x3}{\tilde{P}_{11}} & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & \underset{3x3}{\tilde{P}_{nn}} \end{bmatrix} \tag{89}$$

Multiply Equation (87) by I_c^* and

$$\underset{3x3n}{I_c^*} \; \underset{3nc3n}{\tilde{P}} \; I_c \; y \left(3\right\rangle_{op} = I_c^* \tilde{P} \; a \left(3n\right\rangle + I_c^* \tilde{y} \left(3n\right\rangle \tag{90}$$

adjust $y\ (3\rangle_{op}$ such that

$$I_c^* \tilde{y}\ (3n\rangle = 0\ (3\rangle \tag{91}$$

and note that

$$I_c^* \tilde{P}\ I_c = \frac{1}{n} \sum_{i=1}^{n} \underset{3x3}{\tilde{P}} \tag{92}$$

then Equation (90) becomes

$$\hat{y}\ (3\rangle_{op} = \left(\sum_{i=1}^{n} \underset{3x3}{\tilde{P}_{ii}} \right)^{-1} \left[\mu\ (3\rangle_a - \sum_{i=1} P_{ii}\ a\ (3\rangle_i \frac{1}{n} \right] \tag{93}$$

The matrix of size 3x3 to be inverted is

$$\left(\underset{3x3}{\sum \tilde{P}} \right)^{-1} = \left(\underset{3x3}{In} - \underset{3x3}{\sum P_{ii}} \right)^{-1} . \tag{94}$$

CONCLUSIONS

The Odle solution requires fewer mathematical calculations than the Bodwell solution. If these solutions were to be used to obtain estimates of position points along a trajectory one could obtain the variance matrix equations and attempt to evalute the two techniques with respect to accuracy as well as mathematical computational complexity. However, most applications of these two solutions is to use these position measurements in recursive polynomial estimators to obtain position velocity and accelerations. Thus, the solutions here are merely mathematical definable criterion for combining redundant instruments to be fed as a composite measurement into a dynamical filter. The most commonly assumed dynamics are constant accelerations over a smoothing span etc., which is the same as second degree polynomial assumptions at the position level. The filter model errors for the non-linear missile dynamics are generally the dominant error sources especially during trajectory manuevers.

The problem of instrument selection can best be done with the help of all participating measurements mathematically weighted in the filter such that measurement residuals are very simply selected as is standard practice in Kalman type filters. Bear in mind that most recursive polynomial filters can be formulated -n an extended Kalman filter format. The modelling errors are the chief factor.

The error vectors and variance matrices due to the Bodwell or Odle technique as propogated through the dynamics of the polynomial filters in which coupling and correlations always occur have not been adequately studied. Such error math models for all practical cases must be done via simulation in which instrument dynamics and noises as well as timing errors etc., can be studied in a total system context, that is best filter, best instrument mixing, best geometry etc.

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APPENDIX A

MATRIX PACKAGING OF POINTS IN VECTOR SPACES AND SOME PROJECTORS

Consider an origin at point 0 in three space $L^{(3)}$ and the set of vectors considered as three dimensional column vectors

$$L^{(3)} = \left\{ y \; (3\rangle_0 \right\} \tag{1}$$

where

$$y \; (3\rangle_0 = \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}_0 \tag{2}$$

The position vectors of n points with respect to the origin 0 can be packaged as 3xn matrix (row of column vectors)

$$\underset{3xn}{Y}{}^0 = \left[y \; (3\rangle_1 \; . \; . \; . \; y \; (3\rangle_n \right]_0 \tag{3}$$

or as a column of column vectors

$$y \; (3n\rangle_0 = \begin{pmatrix} y \; (3\rangle^1 \\ \cdot \\ \cdot \\ \cdot \\ y \; (3\rangle^n \end{pmatrix}_0 \tag{4}$$

Both representations are used in the following analysis.

If in three space $L^{(3)}$ at least 3 of the n vectors are linlarly independent then the rank of the 3xn matrix of Equation (3) is said to be 3 or

$$\rho \underset{3xn}{Y_o} = 3 \qquad (5)$$

If all n of the vectors lie on a line then

$$\begin{aligned} y\rangle^2 &= y\rangle_1 \; \mu_2 \\ &\vdots \\ y\rangle_n &= y\rangle_1 \; \mu_n \end{aligned} \qquad (6)$$

and we have

$$\underset{3xn}{Y} = y\ (3\rangle_1\ (1,\ \mu,\quad \mu_n) \qquad (7)$$

or the rank-one decompositions of the matrix of Equation (3).

If the row vector of Equation (7) has unity coordinates

$$(1,\mu,\quad \mu_n\ ) = (1,1,\quad 1) \qquad (8)$$

then all n vectors are the same vector n times

$$Y_o = y\ (3\rangle_o\langle n)1 \qquad (9)$$

The row vector of ones (a 1xn matrix)

$$\langle n)1 = \underset{1xn}{(1,1,1,\ldots 1)} \tag{10}$$

its transpose

$$1(n\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{11}$$

and its generalized inverse is

$$\frac{1(n\rangle}{\langle 11\rangle} = 1^{*}\rangle \tag{12}$$

where

$$\langle 11\rangle = n$$

will be used extensively.

The inner product of Equation (10) and Equation (12) is the identity in one space $L^{(1)}$,

$$\langle 1 \quad 1^{*}\rangle = \frac{\langle 11\rangle}{\langle 11\rangle} = 1 \tag{13}$$

and the commuted or outer product is the rank -one orthogonal projector

$$\underset{n\times n}{P_{11}} = \frac{1\rangle\langle 1}{\langle 11\rangle} \tag{15}$$

with the idempotent index 2 property

$$P_{11}^{2} = P \tag{16}$$

The rank one projector P projects any vector $y\,(n\rangle$ onto the $1\,(n\rangle$ vector as shown in Figure (A-1)

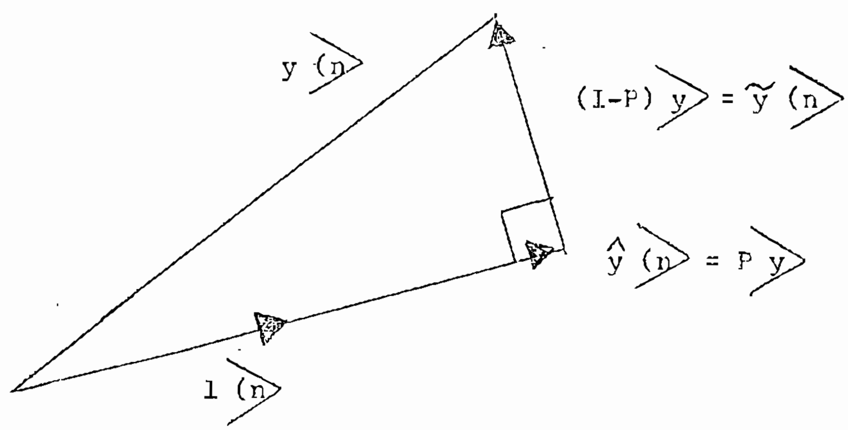

FIGURE (A-1)
PROJECTOR IN $L^{(n)}$

The rank (n-1) orthogonal complement projects is

$$\tilde{P}_{11} = \underset{n\times n}{I} - P_{11} \tag{17}$$

where

$$y\,(n)\rangle = \tilde{P}_{1i}\,y\rangle \tag{18}$$

as shown in the Figure, and

$$\tilde{P}^2=\tilde{P} \tag{19}$$

and

$$\tilde{P}P=0 \tag{20}$$

and

$$\langle \hat{y}\;\tilde{y}\rangle = 0 \tag{21}$$

If the 3xn matrix of Equation(3) is partitioned into its row-space

$$\underset{3xn}{Y_o} = \begin{bmatrix} {}^{1}\langle n)\; y \\ {}^{2}\langle n)\; y \\ {}^{3}\langle n)\; y \end{bmatrix}_0 \tag{22}$$

then we have three row vectors in $L^{(n)*}$ where the * implies the dual-space of dimension n (that is the space of row vectors). The distinction between spaces and their duals or adjoints will not be emphasized in this paper.

If the matrix of Equation (22) has rank three, then the three vectors are linearly independent (row rank equals column rank).

If the matrix has rank one, then the three vectors in $L^{(n)}$ are co-linear, for example Equation (9) becomes

$$\underset{3\times n}{Y_o} = \begin{pmatrix} y_o^1 \ \langle (n) \ 1 \\ y_o^2 \ \langle (n) \ 1 \\ y_o^3 \ \langle (n) \ 1 \end{pmatrix} \tag{23}$$

The column of column vectors of Equation (4) when all vectors are the same point can be written as

$$y \ (3n\rangle_o = \underset{3n\times 3}{\begin{pmatrix} \underset{3\times 3}{I} \\ \underset{3\times 3}{I} \\ \underset{3\times 3}{I} \end{pmatrix}} \ \underset{3\times 1}{y \ (3\rangle_0} = \underset{3n\times 3}{I_c} \ \underset{3\times 1}{y \ (3\rangle_0} \tag{24}$$

where

$$\underset{3n\times 3}{I_c} = \begin{pmatrix} \underset{3\times 3}{I} \\ \cdot \\ \cdot \\ \cdot \\ \underset{3\times 3}{I} \end{pmatrix} \tag{25}$$

is the analog of Equation (11).

The transpose of (25) is

$$\underset{3x3n}{I_c^T} = (\underset{3x3}{I}, \ldots \underset{3x3}{I}) \tag{26}$$

and the generalized inverse is

$$\underset{3x3n}{I_c^*} = \left(\underset{3x3}{I_c^T \quad I_c} \right)^{-1} \underset{3xn}{I_c^T} = \left(\frac{1}{n} \right) \underset{3x3n}{I_c^T} \tag{27}$$

since the inner - Grammian matrix

$$\underset{3x3}{I_c^T \quad I_c} = n \underset{3x3}{I} \tag{28}$$

The two communted projectors associated with I_c are

$$I_c^* \, I_c = \underset{3x3}{I} \tag{29}$$

and

$$\underset{3nx3n}{P_{11}} = \underset{(3nx3)(3x3n)}{I_c \quad I_c^*} = \begin{pmatrix} \underset{3x3}{I} \\ I \\ \cdot \\ \cdot \\ \cdot \\ \underset{3x3}{I} \end{pmatrix} \begin{pmatrix} \underset{3x3}{I} & \cdots & \underset{3x3}{I} \end{pmatrix} \frac{1}{n} \tag{30}$$

The three dimensional subspace of 3n space $L^{(3n)}$ spanned by the three linearly independent column vectors of I_c are

$$\underset{3n\times 3}{I_c} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 1 \end{pmatrix} \right] = \left[c(3n\rangle_1 \quad c(3n\rangle_2 \quad c(3n\rangle_3 \right] \tag{31}$$

and the orthogonal projector of Equation (30) projects a vector $y(3n\rangle$ onto the three dimensional subspace or

$$\hat{y}(3n\rangle = \underset{3n\times 3n}{P_{II}} \; y(3n\rangle \tag{32}$$

The orthogonal complement projector is

$$\underset{3n\times 3n}{\tilde{P}_{II}} = \underset{3n\times 3n}{I} - \underset{3n\times 3n}{P_{II}} \tag{33}$$

with the large space picture of Figure (A-2)

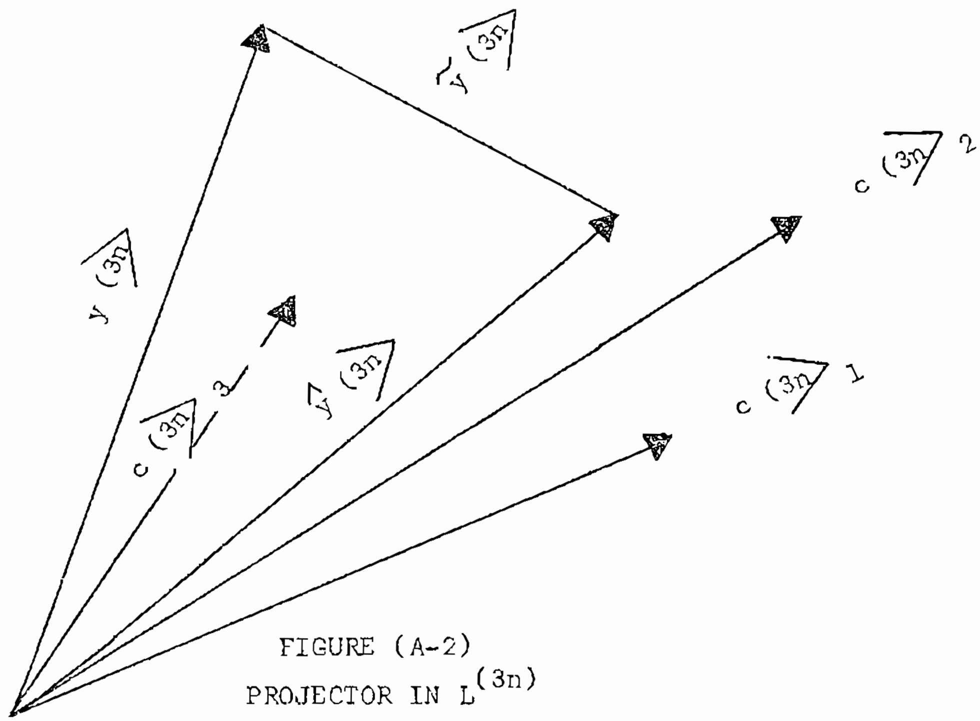

FIGURE (A-2)
PROJECTOR IN $L^{(3n)}$

Using Equation (31) in Equation (24) we have

$$y\,|3n\rangle = I_c\, y\,|3\rangle_0 = c\,|3n\rangle_1\, y_o^1 + c\,|3n\rangle_2\, y_o^2 + c\,|3n\rangle_3\, y_o^3 \qquad (34)$$

or the vector in $L^{(3n)}$ lies in a three dimensional subspace of 3n space. One can also consider Equation (24) or I_c as a linear transformation that maps a vector in $L^{(3)}$ onto a vector in $L^{(3n)}$.

The n points in 3 space of Equation (3) referenced to the origin 0 when referenced to an origin say at point P of Figure (A-3)

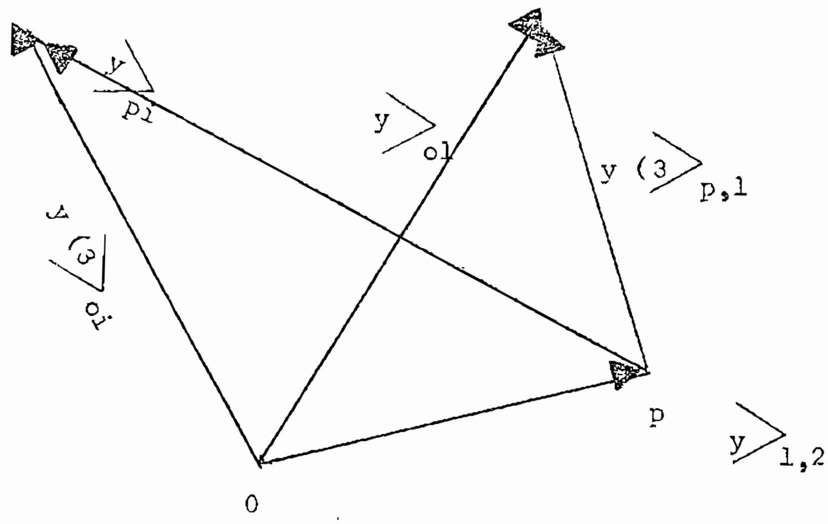

FIGURE (A-3)
CHANGE OF ORIGIN IN $L^{(3)}$

can be expressed as

$$y\ (3\rangle_{oi} = y\ (3\rangle_{op} + y\ (3\rangle_{pi} \tag{35}$$

or for n vectors

$$\underset{3xn}{Y_o} = y\ (3\rangle_{op} \langle n)\ 1 + \underset{3xn}{Y_p} \tag{36}$$

If Y_o has rank 3 in general Y_p will have rank 3.

If Y_o has rank 3 and Y_p has rank 2 then all points of Y_p are co-planar where the plane does not pass through the origin 0. If Y_o has rank 3 and Y_p has rank 1, the points all line on a line which does not pass through the origin, etc. These singular or near-singular conditions on position vectors or their derivates, or their discrete differences as enter in least squares computations are factors in computational problems.

The "uniweighted-mean" of the sequence of n vectors in 3 space satisfies

$$\underset{3xn}{Y_o}\, 1^*(n\rangle = y\,(3\rangle_{op} + Y_p 1^*\rangle \tag{37}$$

$$= y\,(3\rangle_{op} \tag{38}$$

where

$$Y_p\, 1^*(n\rangle = 0\,(3\rangle \tag{39}$$

If Y_o has rank 3 then Y_p has rank 2 and one can consider the vector $1\,(n\rangle$ as lying in the null space of the matrix Y_p ; or the 3 vectors in the row space of Y_p are perpendicular to the vector $1\,(n\rangle$; or partitioning Equation (39) into its column space

$$Y_p\, 1\,(n\rangle \frac{1}{n} = \left(y\,(3\rangle_{p1} + \dots + y\,(3\rangle_{pn} \right) \frac{1}{n} = 0\,(3\rangle \tag{40}$$

that is in 3 space the point p is a point of symmetry (that is an origin with respect to which the sums of all of the vectors is zero). In statistical problems the jargon is residual vectors.

One can also find the mean via the tools of gradients and partial derivatives as

$$\underset{3\times3}{Y_p\, Y_p^T} = \left[Y_o - y\,(3\rangle_p \langle 1 \right] \left[Y_o^T - 1\,(n\rangle_p \;\langle 3y \right] \tag{41}$$

and

$$tr\; Y_p\, Y_p^T = tr \left[y\,(3\rangle_1\, {}^{1}\langle 3\, y + \; . \; . \; . \; + y\,(3\rangle_n\, {}^{n}\langle 3)\, y \right]_p \tag{42}$$

$$= tr \sum_{i=i}^{n} \left(y\,(3\rangle_i \;\; {}^{i}\langle 3)\, y \right)_p \tag{43}$$

$$= \left({}_{1}\langle 3)\, y \quad y\,(3\rangle_1 + \; . \; . \; . \; + {}_{n}\langle y \quad y\,(3\rangle_n \right)_p$$

$$tr \left(Y_p\, Y_p^T \right) = \sum_{i=i}^{n} \left({}_{i}\langle 3)\, y\, y\,(3\rangle_i \right)_p \tag{44}$$

and equating the gradient to zero

$$\langle 3)\, \frac{\partial}{\partial\, y_p}\; tr \;\; Y_p\, Y_p^T = \langle 3)\; 0 \tag{45}$$

The trace of Equation (33) is

$$tr\; Y_p\, Y_p^T = \Big[tr \;\underset{3\times3}{Y_o\, Y_o^T} - y\,(3\rangle_p \langle 1\; Y_0^T$$

$$- Y_o\, 1\,(n\rangle\; {}_{p}\langle 3)\, y + y\,(3\rangle_p \;\; {}_{p}\langle 3)\, y\, n \Big] \tag{46}$$

$$= \mathrm{tr}\; Y_o Y_o^T - 2 \left\langle 1\; Y_o^T\; y\; (3 \right\rangle_p + \left\langle 3)_p\; y \quad y\; (3 \right\rangle_p n \tag{47}$$

and the gradient is

$$\left\langle 3) \frac{\partial}{\bar{y}}\right._p \mathrm{tr}\left(Y_p Y_p^T \right) = -2 \langle 1\; Y_o^T + n2 \langle 3)_p\; y = \langle 0 \tag{48}$$

or

$$\langle 3)_0\; y = \frac{\langle 1\; Y_o^T}{n} \tag{49}$$

and transposing Equation (49)

$$y\; (3\rangle_p = \frac{Y_o\; 1\; (n\rangle}{n} \tag{50}$$

Packaging Equation (27) in $L^{(3n)}$ we have

$$y\; (3n\rangle_o = I_c\; y\; (3\rangle_{op} + y\; (3n\rangle_p \tag{51}$$

and

$$I_c^{*} y\ (3n\rangle_0 = y\ (3\rangle_{op} = I_c^{T} y\ (3n\rangle_0 \left(\frac{1}{n}\right) \qquad (52)$$

where

$$I_c^{*} y\ (3n\rangle_p = \left(\frac{1}{n}\right)\ I_c^{T}\ y\ (3n\rangle_p = 0\ (3\rangle \qquad (53)$$

In summary about the simple process of finding the mean of a sequence of n vectors we have the following interpretations;

(i) In $L^{(3)}$ find an origin or a point of vector symmetry such that the vector sum of the sequence of vectors as seen from that origin is zero, as shown in Equation (40)

(ii) In $L^{(n)}$ to decompose the 3 vectors in the row space of Y_o into their components in the one dimensional subspace spanned by $\langle n)\ 1$ and components perpendicular to $\langle n)\ 1$. By Equation (38) substituted into EQUATION (36)

$$Y_o = Y_o\ \frac{1\ (n\rangle\langle n)\ 1}{n} + Y_p \qquad (54)$$

we see that

$$y\ (3\rangle_{op}\langle n)\ 1 = Y_o\ \frac{1\ (n\rangle\ \langle n)\ 1}{n} \qquad (55)$$

or using Equation (15) in Equation (55)

$$y\,(3\rangle_{op}\,\langle n)\,1 = Y_o \underset{n\times n}{P_{11}} = \underset{3\times n}{\hat{Y}_o} \tag{56}$$

and

$$Y_p = Y_o \left(\underset{n\times n}{I} - P_{11} \right) = Y_o\, \tilde{P}_n = \tilde{Y}_o \tag{57}$$

By Equation (57)

$$y\,(3\rangle_{op} = \frac{\hat{Y}_o\, 1\, (n\rangle}{n} \tag{58}$$

thus the three coordinates of the three row vectors in the 1 dimensional subspace of n-space for the orthogonal decomposition stated is characterized by Equation (38) and Equation (58).

(iii) In $L^{(3n)}$ to decompose the single vector $y\,(3n\rangle_o$ into a component in the three dimensional subspace spanned by three column vectors of I_c of Equation (26) and a component perpendicular to the subspace of I_c. By Equation (52) in Equation (51)

$$y\,(3n\rangle_0 = I_c\, I_c^{*}\, y\,(3n\rangle_0 + y\,(3n\rangle_p \tag{59}$$

where

$$I_c \, I_c^* \, y \, (3n\rangle_o = \underset{3n\times 3n}{P_{II}} \, y \, (3n\rangle_o = \hat{y} \, (3n\rangle_o \tag{60}$$

and by Equation (60) in (59)

$$\left(\underset{3n\times 3n}{I} - \underset{3n\times 3n}{P_{II}} \right) y \, (3n\rangle_0 = y \, (3n\rangle_p = \underset{3n\times 3n}{P_{II}} \, y \, (3n\rangle_0 = \tilde{y} \, (3n\rangle_0 \tag{61}$$

(iv) The gradient approach which minimizes the gradient of the trace of a symmetric matrix "varance matrix " and is related to the minimum-varance approach or the minimum of the sums of squares of residuals, etc.

<u>WEIGHTED MEANS</u>. Consider the expression of Equation (28) in which the rank 3 matrix is expressed as a rank one matrix (dyad) plus a second matrix, that is

$$\underset{3\times n}{Y_o} = y \, (3\rangle_{op} \langle n) \, 1 + \underset{3\times n}{Y_p} \tag{62}$$

and seek a weighted mean or a weighted point of symmetry. For any non-zero vector $w \, (n\rangle$ in $L^{(n)}$ three exists a $y \, (3\rangle_{op}$ such that

$$Y_o \, w \, (n\rangle = y \, (3\rangle_{op} \langle 1 \;\, w\rangle + Y_p w\rangle \tag{63}$$

and

$$Y_p \; w \, (n\rangle = 0 \, (3\rangle \tag{64}$$

Using the constraints of Equation (64) in Equation (63)

$$\frac{Y_o w \, (n\rangle}{\langle 1 \; w \rangle} = y \, (3\rangle_{opw} \tag{65}$$

Using Equation (65) in Equation (62)

$$Y_o = Y_o \; \frac{w \, (n\rangle \; \langle n) \, 1}{\langle 1w \rangle} + Y_p \tag{66}$$

$$Y_o = \hat{Y}_{ow} + \tilde{Y}_{ow} \tag{67}$$

where

$$\hat{Y}_{ow} = Y_o \; P_{w1} \tag{68}$$

and the oblique projector P_{w1} is

$$P_{w1} = \frac{w \, (n\rangle \; \langle n) \, 1}{\langle w1 \rangle} = P^2_{w1} \tag{69}$$

and the complement projector is

$$\tilde{P}_{w1} = I - P_{w1} = \left(\tilde{P}_{w1}\right)^2 \tag{70}$$

By Equation (68) in Equation (67)

$$\tilde{Y}_{w1} = Y_p = Y_o \left(I - P_{w1}\right) \tag{71}$$

$$\tilde{Y}_{w1} = Y_o \tilde{P}_{w1} \tag{72}$$

Most often in statistical analysis one selects an $\langle w$ such that the additional constraint is satisfied

$$\langle w1 \rangle = 1 \tag{73}$$

This report will not use the weighted mean except for the special case

$$\langle w = \frac{\langle 1}{\langle 11 \rangle} \tag{74}$$

however in statistical estimation applied to the geometrical problems of the range weights are used which are geometrical as well as statistical. For example one may geometrically weight data inversely to the range; or statistically one weights data as a function of the noise variances, many estimators are designed that weight data exponentially.

The two complementary oblique projectors P_{w1} and $\tilde{P}_{w1}$ have the properties of "orthogonality"

$$\tilde{P}_{w1}\, P_{w1} = 0 \tag{75}$$

and their sums

$$P_{w1} + \tilde{P}_{w1} = \underset{n\times n}{I} \tag{76}$$

The weighted mean consideration in $L^{(3n)}$ by Equation (51)

$$y\,(3n\rangle_0 = \underset{3n\times 3}{I_c}\; y\,(3\rangle_{op} + y\,(3n\rangle_p \tag{77}$$

and there exist any infinity of ${}_{3\times 3n}W$ matrixes having rank 3 with necessary properties and goodies, that is

$$W\,y\,(3n\rangle_0 = W\,\underset{3\times 3}{I_c}\; y\,(3\rangle_{op} + W\,y\,(3n\rangle_p \tag{78}$$

and

$$\underset{3x3n}{W}\; y\,(3n\rangle_p = 0\,(3\rangle \qquad (79)$$

and in Equation (78) inverting the 3x3 matrix

$$\left(\underset{3x3}{WI_c}\right)^{-1} \underset{3x3n}{W}\; y\,(3n\rangle_o = y\,(3\rangle_{opw} \qquad (80)$$

Using Equation (80) in Equation (77)

$$y\,(3n\rangle_o = \underset{3nx3}{I_c} \left(\underset{3x3}{WI_c}\right)^{-1} \underset{3x3n}{W}\; y\,(3n\rangle_0 \qquad (81)$$

$$+\; y\,(3n\rangle_p$$

$$= \hat{y}\,(3n\rangle_0 + \tilde{y}\,(3n\rangle_0 \qquad (82)$$

The oblique projection in $L^{(3n)}$ is

$$\hat{y}\,(3n\rangle_0 = P_{Iw}\, y\,(3n\rangle_0 \qquad (83)$$

$$\underset{3n\times 3n}{P_{Iw}} = \underset{3n3x}{I_c} \underset{3x3}{\left(WI_c \right)^{-1}} \underset{3x3n}{W = P_{Iw}^2} \tag{84}$$

and

$$\tilde{P}_{Iw} = \underset{3n\times 3n}{I - P_{Iw}} \tag{85}$$

and

$$\tilde{Y} \left(3n\right\rangle_0 = \tilde{P}_{Iw}\, y \left(3n\right\rangle_0 = y \left(3n\right\rangle_p \tag{86}$$

From the foregoing we have developed the use of orthogonal and oblique matrix projectors and applied them to the simple process of finding the weighted point of symmetry for a sequence of n vectors in 3 space.

Thus we see there are a fruitful multitude of techniques available for the analysis of point processes. This report will show some of the relations between the different techniques.

A brief glimplse at the geometrical meanings of these projectors is given in Figure (A-2) when $\langle y$ lies in subspace spanned by $\langle 1$ and $\langle w$.

The transpose of the non-symmetric rank one projector of Equation (61) is

$$P_{w1}^{T} = \frac{1\rangle\langle w}{\langle 1w \rangle} = P_{1w} = P_{1w}^{2} \qquad (87)$$

The two rank one symmetric projectors are P_{11} and P_{ww} where

$$P_{ww} = \frac{w\rangle\langle w}{\langle ww \rangle} = P_{ww}^{2} \qquad (88)$$

and

$$\tilde{P}_{ww} = I - P_{ww} \qquad (89)$$

The following decompositions define the vectors of the figure

$$\left.\begin{aligned} y &= \langle y\, P_{11} + \langle y\, \tilde{P}_{11} \\ &= \langle y\, P_{1w} + \langle y\, \tilde{P}_{1w} \\ &= \langle y\, P_{w1} + \langle y\, \tilde{P}_{w1} \\ &= \langle y\, P_{ww} + \langle y\, \tilde{P}_{ww} \end{aligned}\right\} \qquad (90)$$

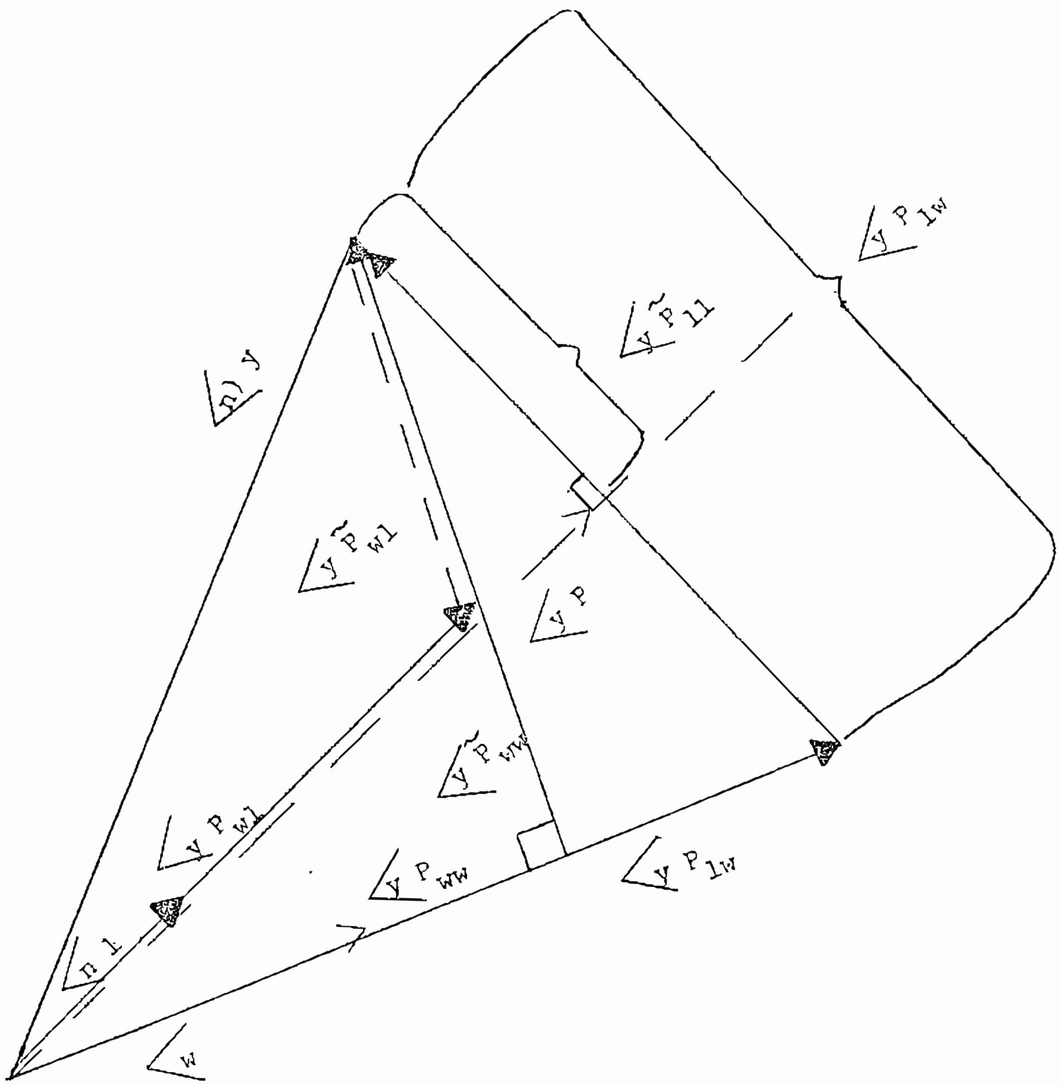

FIGURE (A-4)

VECTOR DECOMPOSITIONS FOR CO-PLANAR VECTORS

From the fore going considerations it is intuitively clear that when $\langle y$ does not lie in the subspace spanned by $\langle 1$ and $\langle w$, then we must add a vector to each of the decompositions of Equation (90). For example

$$\underset{n\times 2}{A_{1w}} = \left[1\,(n\rangle \;,\; w\,(n\rangle \right] \tag{91}$$

is a rank 2 matrix and the "reciprocal vectors" are

$$\underset{2xn}{A^*} = \left(A^T A\right)^{-2} A^T = \begin{bmatrix} {}^1\langle n)\ a^* \\ {}^2\langle n)\ a^* \end{bmatrix} \qquad (92)$$

and

$$AA^* = \underset{nxn}{P} \qquad (93)$$

which is the projector onto the two dimensional subspace spanned by the column vectors of Equation (92).

Any vector in $L^{(n)}$ can be expressed as

$$y\ (n\rangle = P\ y\rangle + \widetilde{P}\ y\rangle \qquad (94)$$

where

$$P\ y\rangle = \hat{y}\rangle \qquad (95)$$

Further geometry will not be pursued here.

APPENDIX B

DIFFERENCE VECTOR OPERATORS (MATRICES)

Many of the mathematical and or statistical minimization techniques involve difference vectors. Some useful matrices and their properties are presented in this section.

Consider the n points in $L^{(3)}$ (or n position vectors with respect to the point 0 as origin) and the difference vector $\Delta y \,(3\rangle_{i,i+1}$ as shown in Figure (B-1). The subscript 0 as origin designator is suppressed.

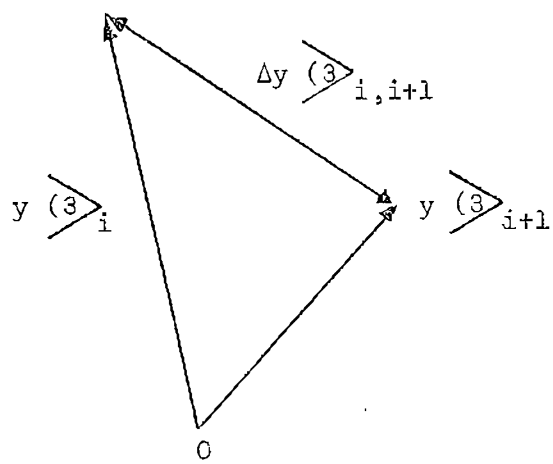

FIGURE (B-1)
DIFFERENCE VECTOR

The difference vector can be written as

$$\Delta y \,(3\rangle_{i,i+1} = y \,(3\rangle_{i+1} - y \,(3\rangle_{i} = \left[\, y\rangle_{i} \; y\rangle_{i+1} \right] \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad (1)$$

If one considers a "dyadic product on a differencing relation"

$$\begin{bmatrix} \bar{y}^1 \\ \bar{y}^2 \\ \vdots \\ \bar{y}^n \end{bmatrix} \textcircled{\Delta} \left(\bar{y}_1 \cdots \bar{y}_n \right) = \begin{bmatrix} \Delta\bar{y}_{11} & \Delta\bar{y}_{12} & \cdots & \Delta\bar{y}_{1n} \\ \vdots & & & \vdots \\ \Delta\bar{y}_{n1} & & \cdots & \Delta\bar{y}_{nn} \end{bmatrix}_{n\times n} \tag{2}$$

where

$$\left.\begin{aligned} \Delta\bar{y}_{11} &= \bar{y}_1 - \bar{y}_1 = \bar{0} \\ \Delta\bar{y}_{12} &= \bar{y}_2 - \bar{y}_1 \\ &\vdots \\ \Delta\bar{y}_{1n} &= \bar{y}_n - \bar{y}_1 \\ &\text{etc.} \end{aligned}\right] \tag{3}$$

and in terms of the column vectors, for example

$$\Delta y \; (3\rangle_{in} = y \; (3\rangle_n - y \; (3\rangle_i \tag{4}$$

There are n^2 elements in the matrix of Equation (2), eliminating the n diagonal zero vectors, and using the symmetry

$$\Delta y\rangle_{12} = -\Delta y\rangle_{21} \tag{5}$$

we have

$$\frac{n^2-n}{2} = \frac{n(n-1)}{2} = m \qquad (6)$$

difference vectors or the number of combinations of n things taken two at a time.

For example if n=4 we have

$$m = \frac{4(4-1)}{2} = 6 \qquad (7)$$

or six difference vectors.

$$\left[\Delta y\rangle_{1,2},\ \Delta y\rangle_{1,3},\ \Delta y\rangle_{1,4},\ \Delta y\rangle_{2,3},\ \Delta y\rangle_{2,4},\ \Delta y\rangle_{3,4} \right] = \underset{3\times 6}{\Delta Y} \qquad (8)$$

We shall be concerned with the rank of the 3×6 matrix of difference vectors of Equation (8) when we know the rank of the 3×4 matrix of position vectors

$$\underset{3\times 4}{Y} = \left[y\,(3\rangle_1,\ y\,(3\rangle_2,\ y\,(3\rangle_3,\ y\,(3\rangle_4 \right] \qquad (9)$$

For example we know that three linearly independent vectors in three space have three coplanar difference vectors as shown in Figure (B-2)

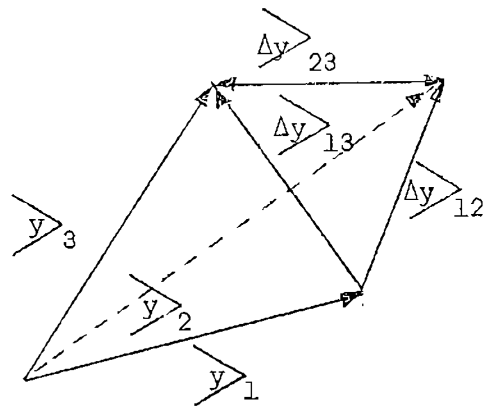

FIGURE (B-2)

LINEARLY DEPENDENT DIFFERENCE VECTORS

This section will obtain; and develop some of the properties of linear transformations that map the position vectors to the difference vectors and utilize these transformations in the main body of the report.

We seek the connection matrix $\mathcal{F}$ of Equation (8) and Equation (9)

$$\underset{3\times 6}{\Delta Y} = \underset{3\times 4}{Y} \underset{(4\times 6)}{\mathcal{F}} \tag{10}$$

The matrix $\mathcal{F}$ can be written by inspecting or by the differencing definition, that is

$$\Delta y \, (3\rangle_{12} = \left[y\rangle_1, y\rangle_2, y\rangle_3, y\rangle_4 \right] \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = Y \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{11}$$

$$\Delta y \, (3\rangle_{13} = \underset{3\ 4}{Y} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vdots$$

$$\Delta y\rangle_{3,4} = Y \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

or Packaging Equation (11)

$$\underset{3\times 6}{\Delta Y} = \left\{ Y \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, Y \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, Y \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdots Y \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\} \tag{12}$$

or "factoring out" Y

$$\underset{3\times 6}{\Delta Y} = \underset{3\times 4}{Y} \; \underset{4\times 6}{\mathcal{F}} \tag{13}$$

where

$$\underset{4\times 6}{\mathcal{F}} = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \tag{14}$$

The question now arises as to the rank of d' and will be considered later in this section.

If we have n vectors and m combinations of differences

$$\underset{3\times m}{\Delta Y} = \left[\Delta y\rangle_{1,2}, \cdots \Delta y\rangle_{1,n}, \Delta y\rangle_{2,3}, \cdots \Delta y\,(3\rangle_{n-1,n} \right] \tag{15}$$

and the transpose

$$\underset{3\times m}{\Delta Y^{T}} = \begin{bmatrix} \langle 3)\Delta y \\ {}_{1,2} \\ \langle 3)\Delta y \\ {}_{1,3} \\ \vdots \\ \langle \Delta y \\ {}_{n-1,n} \end{bmatrix} \tag{16}$$

and the inner-Grammian

$$\underset{3(m)3}{\Delta Y \Delta Y^{T}} = \left[\Delta y\rangle_{1,2}, \cdots \Delta y\rangle_{n-1,n} \right] \begin{bmatrix} {}^{1,2}\langle \Delta y \\ \vdots \\ \langle \Delta y \\ {}_{n-1,n} \end{bmatrix} \tag{17}$$

$$= \Delta y\rangle_{1,2} \, {}^{1,2}\langle \Delta y + \Delta y\rangle_{1,3} \, {}^{1,3}\langle \Delta y + \cdots + \Delta y\rangle_{n-1,n} \, {}^{n-1,n}\langle \Delta y \tag{18}$$

$$\sum_{\alpha,\beta=1}^{n} \Delta y \; (3 \rangle_{\alpha,\beta}^{\alpha,\beta} \langle 3) \; \Delta y \tag{19}$$

$$\alpha \neq \beta \tag{20}$$

the trace of matrix Equation (18) is

$$\operatorname{tr}\left(\underset{3\times3}{\Delta Y \Delta Y^T}\right) = {}_{1,2}\langle \Delta y \; \Delta y \rangle_{1,2} + \cdots + {}_{n-1,n}\langle \Delta y \; \Delta y \rangle_{n-1,n} \tag{21}$$

or the sums of the squares of the magnitudes of the difference vectors between all combinations of points. The gradient of the scalar function of Equation (21) is used often as a function to be minimized.

By Equation (10) and its transpose in Equation (17)

$$\underset{3(6)3}{\Delta Y \Delta Y^T} = \underset{3\times4}{Y} \; \underset{4\times6}{\mathcal{F}} \; \underset{6\times4}{\mathcal{F}} \; \underset{4\times3}{Y^T} \tag{22}$$

also the commuted product of Equation (23) or the outer-Grammian is

$$\underset{6\times3}{\Delta Y^T} \; \underset{3\times6}{\Delta Y} = \underset{6\times4}{\mathcal{F}^T} \; \underset{4\times3}{Y^T} \; \underset{3\times4}{Y} \; \underset{4\times6}{\mathcal{F}} \tag{23}$$

The traces of the Grammians of Equation (22) and Equation (23) are equal

$$\operatorname{tr}\underset{3\times3}{(\Delta Y \Delta Y^T)} = \operatorname{tr}\underset{6\times6}{(\Delta Y^T \Delta Y)} \tag{24}$$

$$= \mathrm{tr}\ \underset{4\times4}{Y^T Y \mathcal{F} \mathcal{F}^T} = \mathrm{tr}\ \underset{4\times6}{\mathcal{F}}\ \underset{6\times4}{\mathcal{F}^T}\ \underset{4\times4}{Y^T Y} \tag{25}$$

The 4×4 Grammian by Equation (22) is by Equation (14) and its transpose

$$\underset{4\times4}{\mathcal{F}\mathcal{F}^T} = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{26}$$

$$\underset{4(6)4}{\mathcal{F}\mathcal{F}^T} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \tag{27}$$

If we add and subtract the identity in 4 space to Equation (27)

$$\underset{4\times4}{\mathcal{F}\mathcal{F}^T} = \mathcal{F}\mathcal{F}^T + I - I = 4\left[I - I\ (4\rangle\langle 4)\ 1\right] \tag{28}$$

$$= 4\ \underset{4\times4}{\tilde{P}_{11}} \tag{29}$$

The rank of $\mathcal{F}$ equals the rank of its Grammian Equation (28) which equals the rank of its factor $\tilde{P}$ or

$$\rho\mathcal{F} = \rho\mathcal{F}\mathcal{F} = \rho\ \underset{4\times4}{\tilde{P}} = 3 \tag{30}$$

since

$$\rho P_{11} = \mathrm{tr}\ \frac{1\rangle\langle 1}{4} = 1 \tag{31}$$

and

$$\rho \tilde{P} = \rho(I-P_{11}) = \rho I - \rho P_{11} = 3 \tag{32}$$

Note that Equation (27) is a Toeplitz-matrix and its generalized inverse is useful in the computation of the pseudo inverse of $\mathcal{F}$, that is

$$\underset{6\times 4}{\mathcal{F}^{*}} = \underset{6\times 4}{\mathcal{F}^{T}} \underset{4\times 4}{(\mathcal{F}\mathcal{F}^{T})^{*}} = \underset{6\times 6}{(\mathcal{F}^{T}\mathcal{F})^{*}} \underset{6\times 4}{\mathcal{F}^{T}} \tag{33}$$

The m (all) combinations of differences of n vectors taken two at a time is generated by

$$\underset{n\times m}{\mathcal{F}} = \begin{bmatrix} -1 & -1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & -1 & & \vdots \\ 0 & 1 & \vdots & 1 & & \vdots \\ 0 & 0 & \vdots & 0 & & 0 \\ \vdots & \vdots & 0 & \vdots & & -1 \\ \vdots & \vdots & 1 & \vdots & \cdots & 1 \end{bmatrix} \tag{34}$$

and the inner-Grammian is

$$\underset{n(m)n}{\mathcal{F}\mathcal{F}^{T}} = n\left(\underset{n\times n}{I} - \underset{n\times n}{P_{11}} \right) = n \underset{n\times n}{\tilde{P}_{11}} \tag{35}$$

Using Equation (35) in Equation (22) we see that for n vectors

$$\underset{3(m)3}{\Delta Y \Delta Y^T} = \underset{3\times n}{Y} (n\tilde{P}_{11}) Y^T = n \underset{3\times 4}{Y\tilde{P}} \underset{4\times 3}{\tilde{P}Y^T} \tag{36}$$

$$= n \underset{3\times 3}{\tilde{Y}\tilde{Y}^T} \tag{37}$$

where

$$\underset{3\times 4}{\tilde{Y}} = \underset{3\times 4}{Y} \underset{4\times 4}{\tilde{P}_{11}} \tag{38}$$

The interpretation in $L^{(n)}$ when Y is partioned into its row space of 3 vectors in N space as

$$\overset{\alpha}{\langle n)} y = \overset{\alpha}{\langle n)} \hat{y} + \overset{\alpha}{\langle n)} \tilde{y} \tag{39}$$

$$\alpha \approx 1,\ 2,\ 3 \tag{40}$$

Package wise Equation (39) is

$$Y = \hat{Y} + \tilde{Y} \tag{41}$$

$$\hat{Y} = \underset{3\times n}{Y} \underset{n\times n}{P_{11}} = \mu\ (3\rangle \langle n)\ 1 = \begin{bmatrix} \mu_1 \langle n)\ 1 \\ \mu_2 \langle n)\ 1 \\ \mu_3 \langle n)\ 1 \end{bmatrix} \tag{42}$$

by Equation (A-30), and

$$\tilde{Y} = Y\tilde{P} = Y(I-P) \tag{43}$$

$$\tilde{Y} = \underset{3\times n}{Y} - \mu \, (3\rangle\langle n) \, 1 \tag{44}$$

The three row vectors in $L^{(n)}$ and their wiggles are shown in Figure (B-3)

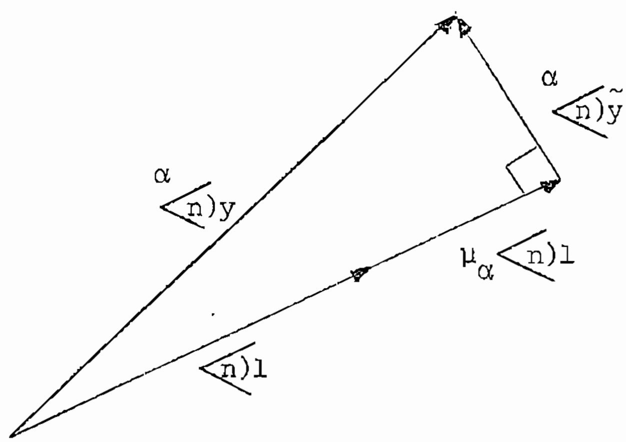

FIGURE (B-3)
RESIDUAL VECTORS IN $L^{(n)}$

Equation (36) can be written as

$$n \underset{3\times 3}{\tilde{Y}\tilde{Y}^T} = n \begin{bmatrix} {}^{1}\langle n) \, \tilde{y} \\ {}^{2}\langle n) \, \tilde{y} \\ {}^{3}\langle n) \, \tilde{y} \end{bmatrix} \left[\tilde{y} \, (n\rangle_1 , \; \tilde{y} \, (n\rangle_2 , \; \tilde{y} \, (n\rangle_3 \right]$$

$$= n \begin{bmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \tilde{g}_{13} \\ \tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{23} \\ \tilde{g}_{31} & \tilde{g}_{32} & \tilde{g}_{33} \end{bmatrix} \tag{45}$$

and the trace of Equation (45) is

$$\operatorname{tr} n\, \tilde{Y}\tilde{Y}^T = n\left(\tilde{g}_{11}+\tilde{g}_{22}+\tilde{g}_{33} \right) \tag{46}$$

By Equation (46) and Equation (37) we see the equivalent results: the trace function which is the sums of the squares of the magnitudes of all m combinations of the discrete difference vectors in $L^{(3)}$ is equal to the three sums of the squares of the magnitudes of the components of the three vectors in $L^{(n)}$ which are perpendicular to $\langle n)1$.

<u>Difference Operator in $L^{(3n)}$</u>. We next consider a useful matrix for the differencing of the representation of the sequence of n three dimensional vectors in $L^{(3n)}$. By Equation (A-4) and Equation (B-16).

$$\begin{bmatrix} \Delta y(3\rangle_{12} \\ \Delta y(3\rangle_{13} \\ \vdots \\ \Delta y\rangle_{23} \\ \vdots \\ \Delta y\rangle_{n-1,n} \end{bmatrix} = \Delta y(3m\rangle = \underset{3m\times 3n}{\mathcal{F}}\, y(3n\rangle \tag{47}$$

where

$$
\underset{3m\times 3n}{\mathcal{F}} = \begin{bmatrix} \underset{3\times3}{-I} & \underset{3\times3}{I} & 0 & \cdot & \cdot & \cdot & 0 \\ \underset{3\times3}{-I} & 0 & I & & & & 0 \\ \underset{3\times3}{-I} & 0 & 0 & 0 & & & \cdot \\ \vdots & & & & & & \vdots \\ 0 & \cdot & \cdot & \cdot & & -I & I \end{bmatrix} \tag{48}
$$

and

$$
\underset{(3n\times 3m)}{\mathcal{F}^T} \; \underset{(3m\times 3n)}{\mathcal{F}} = n \underset{3n\times 3n}{\tilde{P}_{II}} \tag{49}
$$

where $\tilde{P}_{II}$ is given by Equation (A-33).

The transpose of Equation (47) is

$$
\langle_{(3m)}\Delta y = \langle_{(3n)}y \, \mathcal{F}^T \tag{50}
$$

and the scalar product in $L^{(3n)}$ of Equations (49) and (50) is

$$
\begin{aligned}
\langle_{(3m)}\Delta y \; \Delta y_{(3m)}\rangle &= \langle_{(3)}\overset{1}{\Delta y} \; \overset{1}{\Delta y}_{(3)}\rangle + \cdots + \langle_{(3)}\overset{m}{\Delta y} \; \overset{m}{\Delta y}_{(3)}\rangle \\
&= \langle_{(3n)}y \, \mathcal{F}^T \mathcal{F} \; y_{(3n)}\rangle
\end{aligned} \tag{51}
$$

Using Equation (49) in Equation (51)

$$\langle 3m)\Delta y \;\; \Delta y(3m\rangle = n \, \langle 3n) y \underset{3n\times 3n}{\tilde{P}} y(3n\rangle$$

$$= n \, \langle 3n)\tilde{y} \;\; \tilde{y}(3n\rangle \qquad (52)$$

The inner product of the single vector in $L^{(3m)}\langle 3m)\Delta y$ is equal to n times the inner-product of the vector in $L^{(3n)}\langle 3n)\tilde{y}$. Note that the trace of the matrix of Equation (22) is equal to the inner product of the vector of Equation (50).

Another operation that is used in the main report is obtained from a consideration of Equation (47)

$$\Delta y \; (3m\rangle = \underset{3m\times 3n}{\mathcal{F}} \; y \; (3n\rangle \qquad (53)$$

When the n position vector in the three space are expressed in terms of n three-dimensional unit sight line vectors and an n-dimensional vector of ranges, that is

$$y \; (3n\rangle = \underset{3n\times n}{\begin{bmatrix} y\,(3\rangle_{u1} & 0\rangle & \cdots & 0\,(3\rangle \\ 0\,(3\rangle & y\,(3\rangle_{u2} & & \vdots \\ \vdots & & & \vdots \\ 0\,(3\rangle & 0\,(3\rangle & \cdots & y\,(3\rangle_{un} \end{bmatrix}} \begin{bmatrix} r^1 \\ \vdots \\ r_n \end{bmatrix} \qquad (54)$$

$$= \underset{3n\times n}{D(y_u)} \; r \; (n\rangle \qquad (55)$$

Substituting Equation (55) into Equation (53)

$$\Delta y\ (3m\rangle = \underset{3m\times 3n}{\mathcal{F}}\ \underset{3n\times n}{D(y_u)}\ \underset{n\times 1}{r\ (n\rangle} \qquad (56)$$

$$\Delta y\ (3m\rangle = \underset{3m\times n}{S_{yu}}\ r\ (n\rangle \qquad (57)$$

where

$$\underset{3m\times n}{S_{yu}} = \underset{3m\times 3n}{\mathcal{F}}\ \underset{3n\times n}{D(y_u)}$$

$$= \underset{3m\times n}{\begin{bmatrix} -y(3\rangle_{u1} & y\rangle_{u2} & & \cdots & & 0\rangle \\ -y(3\rangle_{u1} & 0\rangle & y\rangle_{u3} & & \cdots & \\ \vdots & & & & & \vdots \\ 0\rangle & 0\rangle & & \cdots & -y(3\rangle_{un-1} & y(3\rangle_{un} \end{bmatrix}}$$

The pseudo inverse of Sy_u is needed and is

$$\underset{n\times 3m}{S^{*}_{yu}} = (\underset{n\times 3m}{S^{T}_{yu}}\ \underset{3m\times n}{S_{yu}})^{-1}\ \underset{n\times 3m}{S^{T}_{yu}} \qquad (59)$$

multiply Equation (57) by S^{*}_{yu}

$$\underset{n\times 3m}{S^{*}_{yu}}\ \Delta y\ (3m\rangle = \underset{n\times n}{S^{*}_{yu}S_{yu}}\ r(n\rangle \tag{60}$$

But the rank of S_{yu} is n

$$\rho\ S_{yu} = n$$

hence

$$S^{*}_{yu}S_{yu} = \underset{n\times n}{I} \tag{61}$$

and

$$S^{*}_{yu}\ \Delta y\ (3m\rangle = r\ (n\rangle = S^{*}_{yu}\ \delta\ y\ (3n\rangle \tag{62}$$

Using Equation (62) in Equation (57)

$$\Delta y\ (3m\rangle = \underset{3m\times 3m}{S_{yu}S^{*}_{yu}}\ \Delta y\ (3m\rangle \tag{63}$$

Equation (63) is obvious for by Equation (57) which states that the $\Delta y(3m\rangle$ vector lies in the n dimensional subspace spanned by the column -- vectors of S_{yu} and $r(n\rangle$ are the n coordinates.

By Equation (62) we can say S^{*}_{yu} maps $\Delta y(3m\rangle$ from $L^{(3m)}$ to $L^{(n)}$ or to a vector in a lower dimensional space. For any other difference vector in $L^{(3m)}$ say

$$\Delta z \left(3m\right\rangle = \underset{3m\times 3n}{\sigma} \; z \left(3n\right\rangle \tag{64}$$

S^{*}_{yu} maps $\Delta z \left(3m\right\rangle$ to

$$\underset{n\times 3m}{S^{*}_{yu}} \; \Delta z \left(3m\right\rangle = S^{*}_{yu} \, \sigma \; z \left(3m\right\rangle = x \left(n\right\rangle \tag{65}$$

From the foregoing expressions namely Equations (62), (63) and (65) the following transformations are of interest:

$$\underset{n\times 3m}{S^{*}_{yu}}$$

$$\underset{3m\times 3m}{S_{yu} S^{*}_{yu}}$$

$$\underset{n\times 3n}{S^{*}_{yu} \sigma}$$

The pseudo inverse S^{*}_{yu} will be obtained first. The invertible grammian matrix of Equation (59) deserves first consideration.

The transpose of Equation (58) is

$$S^{T}_{yu} = D^{T}(y_{u}) \; \sigma^{T} \tag{66}$$

and

$$S^T_{yu} S_{yu} = D^T(y_u) \; \Phi^T \Phi \; D(y_u) \tag{67}$$

By Equation (35)

$$\underset{3n \times 3n}{\Phi^T \Phi = \tilde{P}_{II}} \; n \tag{68}$$

hence

$$S^T_{yu} S_{yu} = \underset{n \times 3n}{D^T(y_u)} \; \tilde{P}_{II} \; \underset{3n \times n}{D(y_u)} \; n \tag{69}$$

and

$$\left(S^T_{yu} S_{yu} \right)^{-1} = \left[D^T(y_u) \; \tilde{P}_{II} \; D(y_u) \right]^{-1} \frac{1}{n} \tag{70}$$

and by Equation (70) in Equation (59)

$$S^*_{yu} = \frac{\left[D^T(y_u) \; \tilde{P}_{II} \; D(y_u) \right]^{-1}}{n} \; D^T(y_u) \; \Phi^T \tag{71}$$

The product of the three matrices within the inverse brackets of Equation (70) are not full rank factors, the inverse is more difficult to obtain.

The inverse of the Grammian matrix of Equation (59) is obtained from the transpose of Equation (58)

$$\underset{n\times 3m}{S^T_{y_u}} = \begin{bmatrix} -\langle \overset{1}{y}_u & -\langle \overset{1}{y}_u & \cdots & -\langle (3)\overset{1}{y}_u & \cdots & \langle 0 \\ \langle \overset{2}{y}_u & \langle 0 & & \langle 0 & & \langle 0 \\ \langle 0 & \langle (3)\overset{3}{y}_u & & & & \vdots \\ \vdots & \langle 0 & & & & -\langle \overset{n-1}{y}_u \\ \langle 0 & \langle 0 & \cdots & \langle \overset{n}{y}_u & \cdots & \langle \overset{n}{y}_u \end{bmatrix}_{n\times 3m} \tag{72}$$

and

$$\underset{n\times n}{S^T_{yu} S_{yu}} = \begin{bmatrix} (n-1)g_{11} & -g_{12} & \cdots & -g_{1n} \\ -g_{21} & (n-1)g_{22} & & -g_{2n} \\ \vdots & & & \vdots \\ -g_{n1} & & \cdots & (n-1)g_{nn} \end{bmatrix} \tag{73}$$

and since the vectors are unit magnitude

$$\langle \overset{i}{y}_u \, \overset{i}{y}_u \rangle = 1 = g_{ii} \qquad \text{and} \qquad g_{ij} = \cos\theta_{ij} \tag{74}$$

Note the common features of the matrix of Equations (27) and (73). Equation (73) can be split

$$\underset{n\times n}{S^T_{yu} S_{yu}} = \underset{n\times n}{nI} + \begin{bmatrix} -1 & -g_{12} & \cdots & -g_{1n} \\ -g_{21} & & & \vdots \\ \vdots & & \cdots & -1 \end{bmatrix}_{n\times n} \tag{75}$$

The n×n matrix of inner products of the unit sight line vectors in $L^{(3)}$ has rank three, that is

$$\begin{bmatrix} 1 & g_{12} & \cdots & g_{1n} \\ g_{21} & & & \vdots \\ g_{n1} & & & 1 \end{bmatrix} = \underset{n\times 3}{Y^T_u} \; \underset{3\times n}{Y_u} \tag{76}$$

where the matrix of unit vectors is

$$\underset{3\times n}{Y_u} = \left[y \big\rangle^{(3)}_{u1} \cdots y \big\rangle^{(3)}_{un} \right] \tag{77}$$

Using Equation (76) in Equation (75) and Equation (69)

$$\underset{n\times n}{S^T_{yu} S_{yu}} = \underset{n\times n}{nI} - \underset{n\times n}{Y^T_u Y_u} = D^T \underset{n\times n}{\tilde{P}_{II}} Dn \tag{78}$$

The inverse of the n×n matrix of Equation(78) can be reduced to the inversion of a 3×3 matrix using the inversion lemma of Householder , Reference (   ).

$$\left(n \underset{n\times n}{I} - Y_u^T Y_u \right)^{-1} = \left(\underset{n\times n}{S_{yu}^T S_{yu}} \right)^{-1}$$

$$= \left[\underset{n\times 3n}{D^T(y_u)} \; \underset{3n\times 3n}{\tilde{P}_{II}} \; \underset{3n\times n}{D(y_u)} \right]^{-1} \left(\frac{1}{n} \right)$$

$$= \left(\frac{1}{n} \right) \left\{ I + \left(\frac{1}{n} \right) \underset{n\times 3}{Y_u^T} \left[\underset{3\times 3}{I} - \underset{3\times 3}{Y_u Y_u^T} \left(\frac{1}{n} \right) \right]^{-1} \underset{3\times n}{Y_u} \right\} \quad (79)$$

Using Equation (79) in Equation (59)

$$\underset{n\times 3n}{S_{yu}^*} = \left(nI - \underset{n\times n}{Y_u^T \; Y_u} \right)^{-1} S_{yu}^T \quad (80)$$

$$S_{yu}^* = \frac{S_{yu}^T}{n} + \frac{1}{n^2} \underset{n\times 3}{Y_u^T} \left[\underset{3\times 3}{I} - \underset{3\times 3}{Y_u Y_u^T} \left(\frac{1}{n} \right) \right]^{-1} Y_u \, S_{yu}^T \quad (81)$$

Note by Equation (72) and Equation (77)

$$\underset{3\times n}{Y_u} \; \underset{n\times 3m}{S^T_{yu}} = \left[y\rangle_{u1} \cdots y\,(3\rangle_{un} \right] \underset{n\times 3m}{\begin{bmatrix} -\,{}^1\langle y_u & -\,{}^1\langle y_u & \cdots & \langle 0 \\ {}^2\langle y_u & \langle 0 & & \vdots \\ \langle 0 & {}^3\langle y_u & & \\ \vdots & & & -\,{}^{n-1}\langle y_u \\ \cdots & \langle 0 & & {}^{n}\langle y_u \end{bmatrix}} \tag{82}$$

$$= \left[-\,y\rangle_{u1}\,{}^{u1}\langle y \;+\; y\rangle_{u2}\,{}^{2}\langle y_u,\; -\,y\rangle_{u1}\,{}^{1}\langle y_u \;+\; y\rangle^{3}_{u}\,{}^{3}\langle y_u,\; \cdots \right.$$

$$\left. \cdots \;-\, y\rangle_{un-1}\,{}^{n-1}\langle y_u \;+\; y\rangle^{n}_{u}\,{}^{n}\langle y_u \right] \tag{83}$$

or in terms of the 3×3 projectors onto the unit sight line vectors

$$\underset{3\times 3m}{Y_u S^T_{y\,u}} = \left[-P_{11}+P_{22}, -P_{11}+P_{33}, \cdots\; -P_{11}+P_{n,n}, \cdots, -P_{n-1,n-1}+P_{n,n} \right] \tag{84}$$

$$= \underset{3\times 3}{-P_{11}} \left(I,\; \underset{3\times 3}{I} \cdots \underset{(n-1)}{I} \right) + \left(\underset{3\times 3(n-1)}{P_{22}, P_{33} \cdots P_{nn}} \right)$$

$$\underset{3\times 3}{-P_{22}} \; \underset{3\times 3(n-2)}{(I \cdots I)} \;+\; \cdots$$

$$- \underset{3\times 3}{P_{n-1,n-1}} + \underset{3\times 3}{P_{n,n}} \tag{85}$$

where

$$y\,(3\rangle_{ui}\,\langle 3)\,y_{ui} = \underset{3\times 3}{P_{ii}} = \underset{3\times 3}{P^2_{ii}} \tag{86}$$

The 3×3 matrix of Equation (81) is

$$\left(\underset{3\times 3}{I} - \frac{Y_u Y_u^T}{n}\right)^{-1} = n\left(n\,\underset{3\times 3}{I} - \underset{3\times 3}{Y_u Y_u^T}\right)^{-1} \tag{87}$$

The sums of the 3×3 projectors is

$$\underset{3\times n}{Y_u}\,Y_u^T = \left[y\rangle_{u1} \cdot y\rangle_{un}\right]\begin{bmatrix} {}^{1}\langle y_u \\ \\ {}^{n}\langle y_u \end{bmatrix} \tag{88}$$

$$= y\rangle_{u1}\,{}^{1}\langle y_u + \cdots + y\rangle_{un}\,{}^{n}\langle y_u$$

$$= \underset{3\times 3}{P_{11}} + \cdots + \underset{3\times 3}{P_{nn}}$$

$$= \left(P_{11}, P_{22} \cdots P_{nn}\right) \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix} \tag{89}$$

$$= \sum_{i=1}^{n} \underset{3\times 3}{P_{ii}} \tag{90}$$

and

$$n \underset{3\times 3}{I} - \underset{3\times 3}{Y_u Y_u^T} \tag{91}$$

$$= I - P_{11} + I - P_{22} + \cdots + I - P_{nn}$$

$$= \tilde{P}_{11} + \cdots + \tilde{P}_{nn} \tag{92}$$

or

$$nI - \underset{3\times 3}{Y_u Y_u^T} = \left(\tilde{P}_{11}, \tilde{P}_{22}, \cdots \tilde{P}_{nn}\right) \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix} \tag{93}$$

$$= \sum_{i=1}^{n} \underset{3\times 3}{\tilde{P}_{ii}} \tag{94}$$

and

$$\left(\underset{3\times3}{nI} - \underset{3\times3}{Y_u Y_u^T}\right)^{-1} = \left(\sum \underset{3\times3}{\tilde{P}_{ii}}\right)^{-1} \tag{95}$$

and Equation (94) in Equation (87)

$$\left(\underset{3\times3}{I} - \frac{Y_u Y_u^T}{n}\right)^{-1} = n\left(\sum_{i=1}^{n} \underset{3\times3}{\tilde{P}_{ii}}\right)^{-1} \tag{96}$$

Hence we have using Equation (96) and Equation (84) in Equation (81)

$$\underset{n\times3m}{S^*_{yu}} = \left(\frac{1}{n}\right)\underset{n\times3m}{S^T_{yu}}$$

$$+ \frac{1}{n}\underset{n\times3}{Y_u^T}\left(\sum_{i=1}^{n}\underset{3\times3}{\tilde{P}_{ii}}\right)^{-1}\Big[\underset{3\times3}{-P_{11}-P22};-P_{11}+P_{33};\cdots-P_{11}\pm P_{n,n};-P_{22}+P_{33};$$

$$\cdots-P_{n-1,n-1}+P_{n,n}\Big] \tag{97}$$

The n×n projector (inner projector) in $L^{(n)}$

$$S^*_{yu}S_{yu} = \underset{n\times n}{I} \tag{98}$$

since by Equation (71) and Equation (58)

$$S^*_{yu} S_{yu} = \frac{\left[D^T(y_u)\tilde{P}_{II} D^T(y_u)\right]^{-1}}{n} \; \text{[illegible]}\, y_u) \, \mathcal{F}^T \mathcal{F} \; D(y_u) \tag{99}$$

and by Equation (68) in Equation (99)

$$S^*_{yu} S_{yu} = \frac{\left[D^T \tilde{P}_{II} D^T\right]^{-1}}{n} \; D^T \, \tilde{P}_{II} \, D(y_u) \; n = \underset{n\times n}{I} \tag{100}$$

The 3m×3m projector (outer projector in $L^{(3m)}$) is

$$\underset{3m\times 3m}{S_{yu} S^*_{yu}} = \mathcal{F} \; D(u_u) \; \frac{\left[D^T(y_u)\tilde{P}_{II} D(y_u)\right]^{-1}}{n} \; D^T \mathcal{F}^T \tag{101}$$

The product map $S^*_{yu} \mathcal{F}$ of Equation (65) is by Equation (71)

$$S^*_{yu} \mathcal{F} = \frac{\left[D^T(y_u)\tilde{P}_{II} D(y_u)\right]^{-1}}{n} \; D^T(y_u) \; \mathcal{F}^T \mathcal{F} \tag{102}$$

or by Equation (68)

$$S^*_{yu} \mathcal{F} = \left[D^T \, \tilde{P}_{II} \, D(y_u)\right]^{-1} D^T_{yu} \; \tilde{P}_{II} \tag{103}$$

$$S^*_{yu} \mathcal{F} = \left[D^T_{yu} \; \tilde{P}_{II} \; D_{yu}\right]^{-1} D^T_{yu} \; \tilde{P}_{II} \tag{104}$$

Consider the last matrix product term of Equation (104) by Equation (A-33)

$$
\tilde{P}_{II} = \frac{1}{n}\begin{bmatrix}
(n-1)\,\underset{3\times 3}{I} & -\underset{3\times 3}{I} & \cdots & -I \\
-\underset{3\times 3}{I} & & & \vdots \\
\vdots & & & \\
-I & & & -I \\
-\underset{3\times 3}{I} & \cdots & -I & (n-1)\,\underset{3\times 3}{I}
\end{bmatrix} \tag{105}
$$

and by Equation (54) in Equation (105)

$$
D^T_{yu}\,\tilde{P}_{II} = \underset{n\times 3n}{\begin{bmatrix}
\overset{1}{\langle 3)y_u \rangle} & \langle 0 \rangle & \langle 0 \rangle & \cdots \\
\langle 0 \rangle & \overset{2}{\langle y_u \rangle} & \langle 0 \rangle & \\
 & & \overset{3}{y_u} & \\
\vdots & & & \vdots \\
\langle 0 \rangle & \cdots & & \overset{n}{\langle y_u \rangle}
\end{bmatrix}} \times
$$

$$\underset{3n\times 3n}{\begin{bmatrix} (n-1)I & -I & \cdots & -I \\ -I & & & \vdots \\ \vdots & & & \\ & & & -I \\ -I & \cdots & -I & (n-1)I \end{bmatrix}} \frac{1}{n}$$

$$= \underset{n\times 3n}{\begin{bmatrix} (n-1)\,{}^{1}\langle y_u & -{}^{1}\langle y_u & \cdots & -{}^{1}\langle y_u \\ -{}^{2}\langle y_u & (n-1)\,{}^{2}\langle y_u & & -{}^{2}\langle y_u \\ \vdots & & & \vdots \\ -{}^{n}\langle y_u & & \cdots & (n-1)\,{}^{n}\langle y_u \end{bmatrix}} \frac{1}{n} \tag{106}$$

The action of Equation (106) on a vector $a(3n\rangle$ in $L^{(3n)}$ is

$$\underset{n\times 3n}{D^T_{yu}}\ \underset{3n\times 3n}{\tilde{P}_{II}}\ \underset{3n\times 1}{a(3n\rangle} = f(n\rangle \tag{107}$$

where

$$f\,(n\rangle = \begin{bmatrix} (n-1)\,{}^{1}\langle y\; a\rangle_{1} - \langle y_1\; a\rangle_{2} \cdots {}^{1}\langle y\;\; a(3\rangle_{n} \\ -\,{}^{2}\langle y_u\; a\rangle_{1} + (n-1)\,{}^{2}\langle y\; a\rangle_{2} - {}^{2}\langle y\; a\rangle_{3} - {}^{2}\langle y\; a\rangle_{n} \\ \vdots \qquad\qquad\qquad \vdots \\ -\,{}^{n}\langle y\; a\rangle_{1} - {}^{n}\langle y\; a\rangle_{2} \quad \cdots \quad +(n-1)\,{}^{n}\langle y\; a\rangle_{n} \end{bmatrix} \frac{1}{n} \tag{108}$$

$$= \begin{bmatrix} g_{11} - {}^{1}\langle y_u\;\; \mu(3\rangle_{a} \\ g_{22} - {}^{2}\langle y_u\;\; \mu\rangle_{a} \\ \vdots \\ g_{nn} - {}^{n}\langle y_u\;\; \mu(3\rangle_{a} \end{bmatrix} = g(n\rangle_{yua} - \underset{n\times 3}{Y^{T}_{u}}\; \mu(3\rangle_{a} \tag{109}$$

where the scalar products are

$$g_{ii} = {}^{i}\langle 3)y_u\;\; a(3\rangle_{i} \tag{110}$$

and

$$g(n\rangle_{yua} = \begin{bmatrix} {}^{1}\langle y_u \; a(3\rangle_1 \\ \vdots \\ {}^{n}\langle y_u \; a(3\rangle_n \end{bmatrix} \tag{111}$$

also Equation (107) can be considered as

$$D^T_{yu} \tilde{P}_{II} \; a(3n\rangle = D^T_{yu} \; \tilde{a}(3n\rangle \tag{112}$$

where

$$\tilde{a}(3n\rangle_I = \tilde{P}_{II} \; a(3n\rangle = \left(I - \frac{I\rangle\langle I}{n} \right) a(3n\rangle \tag{113}$$

$$= a(3n\rangle - \underset{3n\times 3}{I_c} \; \mu(3\rangle_a = \begin{bmatrix} a(3\rangle_1 - \mu(3\rangle_a \\ \\ a(3\rangle_n - \mu(3\rangle_a \end{bmatrix} \tag{114}$$

where

$$\frac{\langle I \; a(3n\rangle}{n} = \mu(3\rangle_a \tag{115}$$

and the subscript I on Equation (113) designates orthogonal projections in $L^{(3n)}$.

The product becomes

$$\underset{n\times 3n}{D^T_{yu}} \tilde{a}(3n\rangle = \begin{bmatrix} \langle {}^1y_u \tilde{a}(3\rangle_1 \\ \langle {}^2y_u \tilde{a}(3\rangle_2 \\ \vdots \\ \langle {}^ny_u \tilde{a}(3\rangle_n \end{bmatrix} = D^T_{yu} \tilde{P}_{II}\, a(3n\rangle \tag{116}$$

with

$$\tilde{a}(3\rangle_i = a(3\rangle_i - \mu(3\rangle_a \tag{117}$$

Using Equation (79) and Equation (116) in Equation (104) as a transformation on a vector in $L^{(3n)}$ to $L^{(n)}$ as in Equation (65)

$$\underset{n\times 3n}{S^*_{yu}}\sigma\; a(3n\rangle = \left\{ \underset{n\times n}{I} + \left(\frac{1}{n}\right) Y^T_u \left[\underset{3\times 3}{I - Y_u\, Y^T_u \left(\frac{1}{n}\right)}\right]^{-1} Y_u \right\} D^T_{yu}\, \tilde{a}(3n\rangle \tag{118}$$

and using Equation (96) in Equation (118)

$$\underset{(n\times 3n)}{S^*_{yu}}\sigma\; \underset{(3n\times 1}{a(3n\rangle} = \left\{ \underset{n\times n}{I} + \underset{n\times 3}{Y^T_u} \left(\sum_{i-1}^{n} \underset{3\times 3}{\tilde{P}_{ii}} \right)^{-1} Y_u \right\} D^T_{yu}\, \tilde{a}(3n\rangle \tag{119}$$

The product $Y_u D^T_{yu}$ of Equation (119) is

$$Y_u D^T_{yu} = \left[y(3\rangle_{u1}, \cdots y(3\rangle_{un} \right] \begin{bmatrix} \langle 3)y_{u1} & \langle 0 & \cdots & \langle 0 \\ \langle 0 & \langle^2 y_u & & \\ \vdots & & & \vdots \\ \langle 0 & \cdots & & {}^n\langle 3)y_u \end{bmatrix}$$

$$= \underset{3\times 3n}{\left[P_{11}, P_{22}, \cdots P_{nn} \right]} \tag{120}$$

Using Equation (120) in Equation (119)

$$\underset{n\times 1}{S^*_{yu}} \alpha F a(3n\rangle = D^T_{yu}\, \tilde{a}(3n\rangle + Y^T_u \left(\sum_{i=1}^{n} \underset{3\times 3}{\tilde{P}_{ii}} \right)^{-1} \left(\underset{3\times 3}{P_{11}} \cdots P_{nn} \right) \tilde{a}(3n\rangle_I \tag{121}$$

Where $\tilde{a}(3n\rangle_I$ is given by Equation (117).

Another matrix whose generalized inverse is needed is D(y) of Equation (54) that is

$$y(3n\rangle = \underset{n\times 3n}{D(y_u)}\; y(3n\rangle \tag{122}$$

and

$$r(n\rangle = \underset{n\times 3n}{D^{*}_{y}}\ y(3n\rangle \tag{123}$$

where

$$\underset{n\times 3n}{D^{*}_{y}} = \left(\underset{n\times n}{D^{T}_{y} D_{y}}\right)^{-1} \underset{n\times 3n}{D^{T}(y)} = \underset{n\times 3n}{D^{T}} \tag{124}$$

since

$$\underset{n\times n}{D^{T}_{y} D_{y}} = \begin{bmatrix} {}^{1}\langle 3)y_u y(3\rangle_u & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & {}_{n}\langle 3)y_u y(3\rangle^{n}_{u} \end{bmatrix} = \underset{n\times n}{I} \tag{125}$$

The projector in $L^{(3n)}$ is

$$\underset{3n\times n}{D(y_u)}\ \underset{n\times 3n}{D(y_u)^{*}} = D(y)D^{T}(y) = \begin{bmatrix} \underset{3\times 3}{P_{11}} & 0 & \cdots & \underset{3\times 3}{0} \\ \vdots & & & \vdots \\ \underset{3\times 3}{0} & & & \underset{3\times 3}{P_{nn}} \end{bmatrix} \tag{126}$$

where the 3×3 projectors are

$$\underset{3\times 3}{P_{ii}} = y\rangle_{ui}\ {}^{i}\langle y_u \tag{127}$$

APPENDIX C

TERMINOLOGY

$$y\,(3\rangle_{om} = \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}_{om}$$

Column vector in three space $L^{(3)}$ from origin at 0 to point at m.

$$y\,(3\rangle_{im} = y\rangle_{uim}\, r_{im} \quad .$$

Position vector in $L^{(3)}$ from i^{th} instrument origin to point at m. The u subscript designates a unit vector; the range is r_{im}, a scalar.

$$a\,(3\rangle_i$$

Constant, known position vector from origin 0 to surveyed station origin.

$$a\,(3m\rangle = \begin{pmatrix} a\,(3\rangle^1 \\ \cdot \\ \cdot \\ \cdot \\ a\,(3\rangle^n \end{pmatrix}$$

Column vector of column vector to the n stations.

$$r\,(n\rangle = \begin{pmatrix} r^1 \\ r^2 \\ \cdot \\ \cdot \\ \cdot \\ r^r \end{pmatrix}$$

Column vector of ranges from the n stations.

$$\mu\,(3\rangle_a = \sum_{i=1}^{n} a\,(3\rangle_i \frac{1}{n}$$

the mean of the n vectors.

$$\langle n)\,1 = (1, 1, \ldots 1)$$

row vector of 1's.

$$\underset{3x3n}{I_c^T} = (\underset{3x3}{I}, \underset{3x3}{I}, \ldots \underset{3x3}{I})$$

row vector of 3x3 unit matrices.

$\underset{3xn}{Y}$ In general capitol letters indicate matrices with size designated by 3xn, for example the matrix of unit vectors

$$\underset{3xn}{Y_u} = \left[y\,(3\rangle_{ui}, \ldots y\,(3\rangle_{un} \right] .$$

$$\underset{3n\times n}{D_{yu}} = D(yu) = \begin{bmatrix} y\,(3\rangle_{u1} & 0\rangle & \cdots & 0\rangle \\ 0\rangle & y\,(3\rangle_{u2} & & \\ \vdots & & & \\ & & & y\,(3\rangle_{un} \end{bmatrix}$$

is a disgonal matrix of 3 dimensional column unit vectors.

$$S(yu) = S\,(y\rangle_{u}) = \begin{bmatrix} -y\,(3\rangle_{u1} & -y\,(3\rangle_{u1} & \cdots & 0\rangle \\ y\rangle_{u1} & 0\rangle & & \\ 0\rangle & y\,(3\rangle_{u3} & & \\ \vdots & 0\rangle & & \\ & \vdots & & -y\,(3\rangle_{u,n-1} \\ 0\rangle & 0\rangle & & y\,(3\rangle_{un} \end{bmatrix}$$

all of the remaining symbols are described in the text of the report.