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State Space Techniques In Approximation Theory With Applications To Design Of Recursive Optimal Estimators

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VOLUME I

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ABSTRACT

In many applications of recursive estimation theory to trajectory estimation the process dynamics are not used. In these cases one normally resorts to approximating the forces with fitting functions. Polynomials, exponentially weighted polynomials and trigonometric and their orthogonal counterparts Gram, Legendre, and Laguarre polynomials are developed in a vector space setting.

Matrices describing the connections between the bases and their velocity matrices and their pseudo-inverses are developed. Continuous and discrete function filtering dynamics are presented. These relations can serve as an aid to fitting functions filter design.

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INTRODUCTION

State Space Techniques For Approximation Theory With Applications To Optimal Estimation.

The applications of modern optimal estimation theory (Kalman Theory) to flight test trajectory estimation has brought about a need by those developing computer programs (the mathware) to obtain a broader systems analysis viewpoint of the total test procedure. All through the 1960's most of the personnel responsible for flight test data reduction at the national missile ranges were primarily mathematicians and statisticians. The advent and global acceptance of the Kalman Estimation Theory necessitated that data reducers also apply the tools of flight simulation groups, namely missile stochastic dynamical models augumented with instrumentation-system dynamics. In actual practice today most applications of the theory are suboptimal modifications since most missile flight processes are highly non-linear (mathematical descriptions highly dependent on wind tunnel data) with only partially known statistics of the process uncertainties and very poorly known non-linear measurement instrument dynamics and measurement instrument variances and biases.

The next best thing to a complete simulation missile model is to use approximation theory conditioned and constrained by a good hueristic engineering feel about the total process. The earlier applications by the ranges have been in power-series polynomials and orthogonal polynomials with a vast amount of literature available in this area of moving arc smoothing etc. Unfortunately most of the literature on approximation theory as well as published reports is steeped in the classical tedium of multiple nested-summations and countless indices running and stopping. The Kalman Theory based on the differential equations of the process is beautifully done in state-space mathematics.

The vast acceptance among engineering circles of the Kalman approach has brought about complete new curriculums in engineering training with big demands for vector space theory matrices and modern algebraic concepts to better understand the modern approach to algebraic system theory. The math departments should be forever endeared to Kalman.

The math modeling techniques for flight dynamics of aerospace vehicles for many interacting moving reference frames on which sensors are mounted can best be handled with the vectors and dyadics used by Gibbs but cast into a matrix setting as described in reference (69).

Thus modern estimation theory requires math models unified from the following disciplines:

Flight Mechanics
Instrument Dynamics
Statistics
Numerical Analysis
Matrix Numerical Methods
Approximation Theory

The primary concern of this report is to develop vector space and matrix relations useful in the design of recursive optimal estimations based on assumed fitting-function (approximation) models when the process dynamic model is not known.

This task necessitated a sizeable attempt to survey some of the vast amount of published literature on approximation theory in books and journals and establish some connections useful to the modern vogue of state-space (vector and matrix dynamic representations).

Householder in ref (39) says that orthogonal polynomials themselves have an extensive literature which he will not bother to cite, but that their association with matrices seems to be relatively new.

Davis says that the fields of interpolation and approximation have been cultivated for centuries. The amount of information available is truly staggering so he presents in his book the topics that caught his imagination.

He says that Bernstein polynomials yield smooth approximants and provide simultaneous approximation of the function and its derivatives; the mimic the behavior of the function to a remarkable degree but the price paid for their beautiful approximation properties is very slow convergence.

Davis published his book in 1963 and says that during the past few decades, the subject of interpolation and approximation has not been overly popular in American universities. Since the development of high speed computing machinery, the flame of interest in interpolation and approximation has burned brighter; and the realization that portions of the theory are best presented through functional analysis has added fuel to the flame.

Rice [The Approximation of Functions, 1964] says the main problem considered in his book is the approximation of a real continuous function x(t) by an approximating function, and that there are two major items in this problem to be stated. The first of these is the type of approximating function used, and the second is how one measures the "goodness" of an approximation. He says there is no scientific method of determining which of the many approximating functions normally available will lead to the most efficient approximation of x(t). The choice of approximation function has to be made on the basis of experience and intuition. The first three states (the function and the first and second derivative of the function) are needed in trajectory estimation. Rice says that the second major item to be specified is the measure of the goodness of the approximation to be used but is of less importance than the choice of the approximating function. The "measures of approximation" or "distance functions" are used to determine the distance of an approximation from x(t). Rice says that given a function x(t) to approximate, there is a rather fixed sequence of steps to be made, the first of which is to translate the intuitive or practical problem into a mathematically precise form. This means that one must choose the approximating function and the distance function. If the approximation problem is already presented in a mathematically precise form (as is often the case) then this step is missing. However, if such choices are to be made, it should be recognized that these choices are the most important of all the steps toward obtaining an approximation. Poor choices at this point can make severe difficulties unavoidable, no matter how talented one may be at mathematical analysis. He says that once the problem is given in a mathematically precise form there are a number of phases in its solution:

1. Choice of approximating function and distance function.

2. The existence of a solution.

3. The uniqueness of a solution.

4. The characteristic and other special properties of the solution.

5. The computation of the solution.

With respect to item 1 above, Rice says that there are some general guide-lines available. He says that in actual practice (as opposed to theoretical analysis) there is only a very small set of functions available to be used as approximating functions. This set includes polynomials, rational functions (the ratio of two polynomials), and trigonometric sums. In special situations, one may have analytical reasons for using special functions such as the logarithm function, the exponential function, Bessel functions, piecewise polynomials, etc. Even with the addition of these special functions, the total set of approximating functions is small. On the other hand, the particular choice from this small set may not be clear in many instances.

Rice says the key to efficient approximation is to find an approximating function which can take on the same nature of behavior as x(t). The approximating functions with limited range of behavior are, naturally, the most common and the simplest to use. These are the polynomials and trigonometric sums. A word which describes this behavior well is "roly-poly". These functions are unable to take on sharp bends followed by relatively flat behavior. Likewise, these approximating functions are unable to manage very large or infinite slopes followed by "normal" behavior. For functions of a roly-poly (or gently varying) nature, however, these approximating functions are very efficient.

Rice states further that one common method of circumventing the roly-poly nature of polynomials is to increase the degree. As the degree of a polynomial or trigonometric sum increases, the polynomial is more and more able to follow a function that is not of a roly-poly nature. This method has the obvious drawback of requiring a complicated approximating function and, frequently, of being difficult to evaluate, i.e., being numerically unstable.

Rice says a class of approximating functions which has a much greater degree of flexibility is the class of rational functions:

$$x(t) = \frac{\sum_{i=0}^{n} a_i t^i}{\sum_{i=0}^{m} b_i t^i}$$

A rational function of relatively low degree can take on a form which cannot be effectively approximated by polynomials of low or medium degree. These approximating functions have the additional advantage that they are simple to use. Their principal disadvantage is that it is considerbly more difficult to compute approximations by rational functions than by polynomials. However, new techniques and widespread availability of high-speed computing machines are rapidly reducing the importance of this disadvantage.

Rice states that another class of approximating functions which deserves consideration is the class of piecewise polynomial functions. These functions are defined by dividing the interval [0,1] into several intervals by a set of points called joints. The approximating function is then a polynomial of specified degree between the joints. The approximating function is either linear (in case the joints are given) or nonlinear (in case the joints are to be determined).

Rice claims that a particularly important subclass of these functions is the class of spline functions, which are piecewise polynomial functions of n^{th} degree joined smoothly so that they have n-1 continuous derivatives; i.e., there is a discontinuity possible only in the n^{th} derivative at the joints. When the joints are given, these approximating functions have three very desirable properties, namely, they are flexible, they are linear in the parameters (i.e., approximations are relatively easy to compute), and they are convenient to use in many applications. The problems associated with the "best joints" have not been investigated to any extent.

Rice says that it appears at this point that almost every aspect of the least-squares approximation problem has been resolved, and in theory this is true. In computation, however, there is still a possible difficulty which is touched upon here. This difficulty arises mainly from the fact that the coefficient matrix of the normal equations may be ill-conditioned. Space does not allow a full exposition of this property, but it suffices to say that matrices which are ill-conditioned are very difficult to invert. When one is doing hand computations involving four or five parameters, this ill-conditioning has little effect. The effect became of increased importance with the advent of high-speed computers, for it is now possible to make calculations involving a much larger number of parameters.

To illustrate this, Rice says we shall consider the most common example: namely, when $f_i(t)=t^{i-1}$. In this case, the coefficients of the metric matrix are

$$\int_0^1 t^{i-1}\, t^{j-1}\, dt = \frac{1}{i+j-1}$$

the coefficient matrix then is the Hilbert matrix:

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \cdots \\ \vdots & & & \end{bmatrix}$$

Rice says the inversion of a system of equations with this coefficient matrix is notoriously difficult. If there are ten parameters, then the inverse of this matrix has elements of the order of $3x10^{12}$. This means that an error of $1o^{-10}$ in the original system appears as an error of the order of 300 in the solution for the parameters of the best L_2-approximation. These difficulties may be avoided if one is able to use $f_i(t)$ approximately orthogonal. There are many systems of polynomials available, for example, for use on an interval or on finite point sets of approximately uniform distribution.

Morrison in "Introduction to Sequential Smoothing and Prediction" (1969) uses the discrete Lengendre and Laguerre polynomials. The Legendre polynomials form the basis (instead of the monomial basis) for the fixed memory polynomial filters and expanding memory polynomial filters. Morrison uses the discrete Laguarre polynomials to form the basis for the Fading Memory Polynomial Filters.

Morrison points out that Hildebrand considers briefly the question of orthogonality over a discretized interval and derives the Gram polynomials, sometimes also called the Chebyshev polynomials. These polynomials are related to Morrisons Legendre polynomials by a shift of origin. Morrison says that he has extended Hildebrand's approach to obtain the discrete Legendre and Laguerre polynomials; and that Milne has an alternate and extremely elegant derivation of the discrete Legendre polynomials in his book "Numerical Calculus". Morrison says that recursive parameter estimation over fixed data spans using the classical monomial base (power of t) yields matrices which are difficult to obtain the functional form of the inverse of anything much beyond a 2x2 is obviously out of the question; that although numerical inverses would give correct answers resorting to numbers would, (at the stage of discussion in his book) force the analysis to terminate, and the analysis has as yet hardly begun. [Many of the ranges do use second, third, and fourth degree polynomials for trajectory estimation and these inverses are included in this report.] Morrison states on page 238 that the discrete Legendre polynomials do not easily permit of differentiation; that any attempt to differentiate them completely fractures their structure, and all order is therefore irretrievably lost. He presents two solutions to this problem.

Lawson ("Survey of Computer Methods for Fitting Curves to Discrete Data or Approximating Continuous Functions", 1968) concentrates his discussion primarily upon the class of computational problems most commonly associated with the name "approximation theory", namely the approximation of continuous functions and curve fitting to discrete data. He views the problems from the point of view of practical scientific computation. Lawson considers the features and drawbacks of various forms of the approximating functions. He says that polynomials are in some sense the simplest forms and that there are a variety of parameterizations possible for a polynomial. When expressed as $a_i t_i$ which he calls the monomial basis parameterization. The matrix of basic function values is typically very poorly conditioned which can be significantly improved by translating the domain of the independent variable to be centered at zero. Exponent overflow is avoided by scaling to [-1,1]. He says that even with these precautions polynomials of degree higher than about seven in monomial basis form are essentially useless in eight decimal digit arithmetic. Other bases such as Chebyshev polynomials typically provide remarkable stability. A polynomial of degree n represented as a linear combination of Chebyshev polynomials can be evaluated in n multiplications and 2n additions, and he recommends the use of the Chebyshev basis in preference to the monomial basis wherever polynomials are used in computation unless there is some specific reason to use the monomial basis.

Lawson says that rational forms have various special properties such as remaining bounded at infinity, having poles, and having abrupt changes of curvature, which sometimes make them more useful than polynomials, that their parameters occur nonlinearly and their determination in fitting problems requires iterative procedures and there are various practical difficulties. The fact that best rational approximations on discrete sets do not always exist must be kept in mind. Lawson states that he believes that the use of rational functions for fitting discrete data will largely be supplanted by the use of spline polynomials. Rational functions have been very successfully used as approximating forms for some analytic functions, and the algorithms for computing these fits are well covered in the literature but the effective design of such approximations depends more upon a thorough understanding of the function being approximated, leading to the use of special identities and changes of variables than upon the actual computation of the approximation.

All of the polynomial parameterizations are candidates for use in rational function parameterization. There is also the possibility of using continued fraction forms, however, these are frequently unstable and must be tested for growth of rounding error in each individual case.

Lawson says Spline forms have been the subject of intensive study in recent years. Much more work needs to be done to evolve the best strategies for parameterizing and manipulating splines and understanding the problem with variable breakpoints. Splines combine extreme flexibility in changing curvature with the stability of low degree polynomials and linearity of coefficients (for fixed partition points). They provide a very attractive approach to general data fitting.

Lawson at the end of his paper makes a few "somewhat random observations". (1) He says that he believes that in the realm of advances in approximation algorithms the development and exploitation of the fast Fourier transform for harmonic analysis and synthesis must rank as the single most significant advance over previous methods. (2) The numerical superiority of modified Gram-Schmidt orthogonalization should become known to everyone and hopefully should even be mentioned in math texts on linear algebra. (3) The advantages of the Chebyshev basis relative to the monomial basis should become more widely known and exploited.

J. Hiller in an article "Is Orthogonal Expansion Desirable?" addresses itself to an examination of the desirability of orthogonal expansion in identification and a class of sub-optimal control problems. He says that there is little difference between the orthogonal technique and a method based on the direct minimization of the (weighted) integral of error squared or some quadratic performance criterion. He shows that the integral approach is equivalent to a particular method of matrix inversion. Hiller points out that the identification problem in control has much in common with the approximation problem encountered in filter methods applied in identification. His paper addresses itself to an examination of one such technique-orthogonal expansion.

Hiller says it is common practice in design and also in signal measurement to determine the optimum set of weighting coefficients for a given set of functions of expansion by requiring that the integral of the error squared should be minimized. This criterion of optimality is often chosen for no other reason than that it leads to tractable mathematics whilst satisfying the rather vague and often subjective requirement that "error should be small". Close examination of many approximation procedures reveals that there is often little reason, other than convenience, involved in choice of criterion of optimality. He shows that in many cases the requirement of orthogonality is inconvenient and in all cases amounts in principle to specification of the manner in which a matrix is to be inverted. He uses a simple example to illustrate that orthogonal expansion is merely a particular way of carrying out matrix inversion, viz., the Cholesky method. Therefore, to answer the question: 'Is orthogonal expansion desirable?' we must in effect pose the query: 'Is inversion by the Cholesky method desirable?' If expansion of a real function is being carried out in terms of real component functions, the elements of the coefficient matrix, F, will be real. It may thus be hoped that all computation will involve only real functions. It has been shown by Wilkinson (1961) that if F is positive definite and symmetric, there exists a real lower triangular matrix L such that $F=LL^T$ the symbol T denoting transpose. In expansion problems symmetry is guaranteed but positive definiteness is not. Any attempt to carry out the Gram-Schmidt process on a system described by a non-positive definite F may give rise to the appearance of imaginary elements in L^{-1}.

Hiller enumerates the following features often claimed (Kautz 1954) for orthogonal expansions:

(i) Convergence is rapid and uniform, thereby providing an accurate approximation for any fixed number of terms.

(ii) The coefficients a_0, a_1, . . . etc. do not depend on the number of terms taken.

(iii) The calculation of the a_i is simple.

(iv) The mean square error is minimized for each set of items.

With respect to the first of these properties Hiller says that convergence of the orthonormal expansion method will only be rapid if the pole distribution of the approximating functions adequately spans the pole distribution of the function being approximated. It is not necessary that the pole locations should coincide for the error measure is insensitive to pole positioning provided that the two sets of poles lie in approximately the same region of the s plane.

The significance of properties (ii) and (iii) depends on the use planned for the expansion. Property (ii) relates to the additional work associated with increasing the number of terms used in an expanison. The main reason for considering the independence of the a_i on the number of terms as an advantage is the thought that alteration of the number of terms will involve only partial recalculation. In practice, however, it is unlikely that this method would offer significant advantages unless either the matrices were of large order or very fast inversion was necessary in an on-line control application. The fourth property of orthogonal expansion refers to minimization of the integral of error squared. This is of course achieved, in the direct approach.

Peter Swirling in his paper "Modern State Estimation Methods from the Viewpoint of the Method of Least Squares" states that his paper is intended to be a guide to the subject of modern state estimation from the viewpoint of the method of least squares, including an exposition of the relation of this subject to linear filter theory and a rather comprehensive account of recursive solutions. He says results will be stated, interpreted, and motivated, but not derived in detail. He states that one objective of his paper has been to state a number of substantive results which are insufficiently well known or (in some cases) about which there has been a great deal of confusion. He gives the following examples:

1. Many of the important results, including the fundamental theorems of recursive state estimation, do not require any statistical concepts or

assumptions either in their formulation or in their proof. For those results which do require statistical formulation, the great majority do not require the assumption of Gaussian statistics.

2. Even when the problem is formulated statistically, there is no essential difference in the treatment of problems where the state is stochastic and of problems where the state is non-stochastic or "deterministic": every problem in which the state is a stochastic process can easily be reforumlated as a problem of estimating a vector of non-stochastic parameters, yielding identical solutions.

3. Every problem in optimum linear filtering or prediction of random processes can be formulated as an exactly equivalent problem, yielding identical solutions, of estimating a vector of constant parameters by the method of least squares.

4. For purposes of deriving optimum recursive solutions to linear filtering and prediction problems, it is unnecessary to make several assumptions regarding the state equation which have been widely thought to be necessary.

Swirling says his second objective is to present a self contained and comprehensive development of the subject from a viewpoint which differs from, and is greatly superior to, the conventional approach. He says one may distinguish two approaches which have been employed in developing modern state estimation theory.

1. An approach in which the basic problem is taken to be optimum linear filtering and prediction [here Swirling references Kalman and Bucy's paper].

2. An approach in which the results are developed as elaborations of the classical methods of least squares [here he references his own paper].

Swirling says that most workers in the field have started from the "linear optimum filter" viewpoint, ...that the superiority of development from the method of least squares viewpoint is evident in two respects. First it easily avoids various widely prevalent confusions regarding the previous four examples (six stated in paper). All of the points of those examples are rather obvious when the subject is approached via least squares, but several of them have proved difficult to grasp from the other viewpoint. Second, he says, the development from the method of least squares viewpoint leads directly to much more general results. As an example very important in practice is the treatment of recursive solutions to non-linear problems which constitute the majority of practical applications. He says that it is well known that the algorithms resulting from the "optimum linear" viewpoint are not directly applicable to non-linear cases but must be modified or "extended" to be made applicable; these "extensions" he says are just the algorithms which result originally from the method of least squares development.

The last (16) sections present some continuous time dynamics for linear stochastic systems. The matrix Riccatti differential equation and adjoint dynamical systems are introduced. It is felt that a conceptual understanding of both the continuous and the discrete time processess should go hand in hand. In fact the design of multi variable digital filters could benefit from the aid of understanding how corresponding continuous filters behave (of course where one does have an analogy). Many mathematical operations can be built into special purpose hybrid computers or mini-computers, some of which are or can be analog devices.

As an example of the state vector techniques consider a three dimensional position vector with rectangular coordinates in time-varying body-axes, an ortho-normal base,

$$\bar{x}(t) = [\bar{b}_1, \bar{b}_2, \bar{b}_3(t)] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \langle \bar{b}(t) \; x(t) \rangle$$

and assume that each of the three coordinates are approximated by a d-dimensional set of fitting functions, that is for constant parameter $\langle a$

$$x_i(t) = \underset{i}{\langle} d) \; a \; f(t) \; (d \rangle$$

$$i = 1,2,3$$

or

$$x(t)\rangle = \underset{3xd}{A}\, f(t)\rangle$$

and

$$\bar{x}(t) = \langle \bar{b}(t)\ A\ f(t)\rangle$$

Taking the time derivative

$$\dot{\bar{x}} = \langle \dot{\bar{b}}\ A\ f(t)\rangle + \langle \bar{b}\ A\ \dot{f}(t)\rangle$$

where the velocity of the body-base vector is

$$\langle \dot{\bar{b}} = \langle \bar{b}\ V_b(t)$$

and the velocity of the fitting-function base is

$$\dot{f}(t)\rangle = V_f\ f(t)\rangle$$

Hence the velocity vector becomes

$$\dot{\bar{x}} = \langle \bar{b}(t)[\underset{3x3}{V_b(t)}\ \underset{3xm}{A} + \underset{3xm}{A}\ \underset{mxm}{V_f}]f(t)\rangle$$

The last few years have seen the emergence of a number of papers applying Spline functions to the trajectory estimation problem. These functions appear to be very good for post-flight processing where computational speed is not of prime importance as in real-time trajectory tracking. The bibliography lists a number of these spline publications.

For the real time applications one would expect the Walsh functions and Block-Pulse functions to come through in the lead from the standpoint of computational time. These functions are discussed briefly in section 40. They are natural functons for digital devices and have enjoyed much success in communications theory and in two-dimensional filtering.

Section 1

SOME VECTOR-SPACE METHODS FOR CONTINUOUS FUNCTION THEORY

This section applies some of the vector and matrix and dyadic analysis techniques from matricized Gibbs vector analysis methods. Since polynomials are vectors the base elements are separated from the coordinates and many operations are performed between rows and columns of the base fitting functions. Inner products between functions, and matrices of inner-products (or metric matrices) are used.

Orthogonal and orthogonal-compliment projectors are developed and applied to the recursive Gram-Schmidt process to obtain the orthogonal polynomials. The notion of polynomials in a given base and their coordinates in the "dual" or reciprocal base are introduced early and very naturally. The inverse of the Hilbert matrix is constructed in terms of the triangular Gram-Schmidt factors. The "weighted Gram-Schmidt" procedure is presented from the point of view of a Gram-Schmidt process in oblique bases with a symmetric metric matrix as weights.

Two parallel Gram-Schmidt procedures are shown, one for a continuous interval and the discrete analog for a finite set of points at which the fitting functions (base elements) are evaluated.

Consider a function $x(t)$ expressed as a linear combination of a set of <u>fitting-functions</u> $f_0(t) \cdots f_{d-1}(t)$

$$x(t) = \left\langle f(t)\ \ a(t) \right\rangle = \sum_{i=0}^{d-1} f_i a_i \tag{1}$$

where the row-tuple is

$$\left\langle f(t) = \left[f_0(t),\ f_1(t) \cdots f_{d-1}(t) \right]\right.$$

and the column-tuple is

$$\left. a(t)(d \right\rangle = [a_0(t), a_1(t), \cdots a_{d-1}(t)]^T \tag{2}$$

In general we will consider $\left. a(t) \right\rangle$ to be a constant vector. The product of two functions can be written as

$$x_1(t)x_2(t) = {}_1\left\langle a\ f(t) \right\rangle \left\langle f(t)\ a \right\rangle_2 \tag{3}$$

and the inner-product as

$$\left[x_1(t),x_2(t)\right] = \int_{t_b}^{t_f} x_1(t)\ x_2(t)dt \tag{4}$$

where t_b, and t_f refer to the back and front of the time span.

$$= {}_1\left\langle a \left[\int_{t_b}^{t_f} f(t)\right\rangle\left\langle f(t)dt\right] a \right\rangle_2 \tag{5}$$

$$\left[x_1,x_2\right] = {}_1\left\langle a\ M_{ff}(t_b t_f)\ a\right\rangle_2 \tag{6}$$

where the dxd matrix is

$$M_{ff}(t_1 t_2) = \int_{t_b}^{t_f} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{d-1} \end{pmatrix} \left[f_0,f_1,\ \cdots,\ f_{d-1}(t)\right] dt \tag{7}$$

and will be referred to as the metric-matrix. Note inner-product yields a scalar.

We also have the norm or distance function

$$||x|| \equiv \left[\int_{t_b}^{t_f} x^2(t)dt\right]^{\frac{1}{2}} . \tag{8}$$

If the metric-matrix of (Equation 7) is full rank (non-singular) the set of functions are linearly independent and from a basis, x(t) is called a vector and a_i the coordinates with respect to the base.

The dual base is given by

$$\langle f^{*}(t) = \langle f(t)\, M_{ff}^{-1}(t_b t_f) \tag{9}$$

the set of bi-orthogonal functions, since

$$I = \int_{t_b}^{t_f} f^{*}(t)\rangle\langle f(t) = M_{ff}^{-1} \int_{t_b}^{t_f} f(t)\rangle\langle f(t)\ dt \tag{10}$$

where I is the d square identity matrix. As an example of the utility of this concept suppose one is given x(t) continuous over the interval and asked to find the d coordinates by integration (analog computer) then

$$\int_{t_b}^{t_f} f(t)\rangle\, x(t)dt = \left[\int_{t_b}^{t_f} f(t)\rangle\langle f(t)dt\right] a\rangle \tag{11}$$

$$= M_{ff}\ a\rangle \tag{12}$$

and

$$a\rangle = M_{ff}^{-1} \int_{t_b}^{t_f} f(t)\rangle\, x(t)dt\ . \tag{13}$$

The solution of (Equation 13) assumed we have knowledge of the fitting functions and can generate them.

By Equation (9) in Equation (13)

$$a\rangle = \int_{t_b}^{t_f} f^{*}(t)\rangle\, x(t)dt\ . \tag{14}$$

The discreet approach is interesting to note. Given the function values $x(t_0), x(t_1), \cdots, x(t_{k-1})$ for $k \geq d$ time points, we have

$$\begin{bmatrix} x(t_0) \\ x(t_1) \\ x(t_2) \\ \vdots \\ x(t_{k-1}) \end{bmatrix} = \begin{bmatrix} \langle f(0) \\ \langle f(1) \\ \langle f(2) \\ \vdots \\ \langle f(k-1) \end{bmatrix} a(d)\rangle = \underset{kxd}{F} \; a(d)\rangle \tag{15}$$

Multiply (Equation 15) by F^T and

$$F^T x\rangle = F^T F \, a\rangle \tag{16}$$

and

$$(F^T F)^{-1} F^T x\rangle = a\rangle \tag{17}$$

The dxd symmetrix matrix is

$$F^T F = \left[f(0)\rangle , \; f(1)\rangle , \; \cdots , f(k-1)\rangle \right] \begin{bmatrix} \langle f(0) \\ \langle f(1) \\ \vdots \\ \langle f(k-1) \end{bmatrix}$$

$$= f(0)\rangle \langle f(0) + f(1)\rangle \langle f(1) + \cdots + f(k-1)\rangle \langle f(k-1)$$

$$= \sum_{i=0}^{k-1} f(i)\rangle \langle f(i) \tag{18}$$

Equation (18) is the discreet analog of the metric-matrix of Equation (12), thus one is lead to a matrix inversion either through integration or discreet rank-one (dyad) summation.

By Equation (17) we can write

$$a(d\rangle = \underset{dxk}{F^{*}} \; x(k\rangle \tag{19}$$

and

$$F^{*} = (F^{T}F)^{-1}F^{T} \tag{20}$$

where the generalized inverse, psuedo inverse or (dual bases, in sense of column space vectors of F are a base, then row space vectors of F^{+} are reciprical, or biorthogonal) etc. for

$$\underset{dxk}{F^{*}} \; \underset{kxd}{F} = \underset{dxd}{I} \tag{21}$$

when F is full rank.

The commute of Equation (21) is a symmetric idempotent index two projector.

$$\underset{kxk}{FF^{*}} = P_{FF^{*}} \tag{22}$$

and

$$P^{2} = P \tag{23}$$

The orthogonal complement projector is

$$I - P = \tilde{P} \tag{24}$$

and

$$\tilde{P}^{2} = P \tag{25}$$

and

$$P\tilde{P} = 0 \tag{26}$$

Using Equation (19) in Equation (15)

$$x(k\Big\rangle = \underset{kxk}{\Gamma F^{+}} x(k\Big\rangle = P_{FF^{*}}\ x\Big\rangle \qquad (27)$$

and

$$x\Big\rangle - x\Big\rangle = 0\Big\rangle = (I-P)x\Big\rangle = \tilde{\Gamma}\ x\Big\rangle \quad . \qquad (28)$$

The continuous-function analog of Equation (27) is obtained by using Equation (14) in Equation (1)

$$x(t) = \Big\langle f(t) \int_{t_b}^{t_f} f^{*}(t)\Big\rangle\ x(t)\,dt \qquad (29)$$

or

$$x(t) = \Big\langle f^{*}(t) \int_{t_b}^{t_f} f(t)\Big\rangle\ x(t)\,dt \qquad (30)$$

The analog between Equation (27) and Equation (30) can be sharpened using a matrized-representations of Gibbs dyads.

Using bars over-vectors to mean the old-fashioned vector analysis of Eucleadean real space, embed Equation (27) in a particular basis which is oblique, that is the metric (where dot is inner-product or dot product)

$$\bar{e}\Big\rangle \cdot \Big\langle\bar{e} \quad = M_{ee} \neq I$$

$$M_{ee} = \begin{bmatrix} \bar{e}_1 \cdot \bar{e}_1 & \bar{e}_1 \cdot \bar{e}_2 & \cdots & \bar{e}_1 \cdot \bar{e}_K \\ \vdots & & & \\ \bar{e}_K \cdot \bar{e}_1 & \cdots & & \bar{e}_K \cdot \bar{e}_K \end{bmatrix} \tag{31}$$

and the duals or (recipicals)

$$\bar{e}^*\rangle = M_{ee}^{-1}\,\bar{e}\rangle \tag{32}$$

and

$$\bar{e}^*\rangle \cdot \langle\bar{e} = I \tag{33}$$

Multiply Equation (27) by $\langle\bar{e}$ or

$$\bar{x} = \langle\bar{e}\;x\rangle = \langle\bar{e}\;P\;x\rangle \tag{34}$$

Use Equation (33) in Equation (34)

$$\bar{x} = \langle\bar{e}\;P\;\bar{e}^*\rangle \cdot \langle\bar{e}\;x\rangle \tag{35}$$

$$\bar{x} = \bar{\bar{P}} \cdot \bar{x} = \langle\bar{e}\;(PP^*)\;\bar{e}^*\rangle \cdot \bar{x} \tag{36}$$

where

$$\bar{x} = \langle\bar{e}\;x\rangle = \bar{e}_1x^1 + \bar{e}_2x^2 + \dots \bar{e}_{k_1}x^{k_1} \tag{37}$$

and the operator in the $\langle\bar{e}$ base and its dual base is

$$\vec{P} = \left\langle \bar{e}\ P\ \bar{e}^{*} \right\rangle_{k_1 x k_1} = \left\langle \bar{e}\ FF^{*}\ \bar{e}^{*} \right\rangle \tag{38}$$

the double bar is in analogy to Gibbs use in three space for a base $\bar{i}, \bar{j}, \vec{k}$

$$\bar{i}\bar{i} = \bar{\bar{i}} = (\langle i, j, k) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1, 0\ 0) \begin{pmatrix} \bar{i} \\ j \\ k \end{pmatrix}$$

$$\bar{i}\bar{i} = (\bar{i}, \bar{j}, \bar{k}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{pmatrix} \tag{39}$$

etc.

If we now replace the bracketed-looking symbols of Equation (4) for inner product by $\textcircled{i}$ we have

$$x_1 \textcircled{i} x_2 \equiv \int_{t_b}^{t_f} x_1(t)\ x_2(t)dt . \tag{40}$$

and Equation (29) becomes

$$x(t) = p(t)\ \textcircled{i}\ x(t) \tag{41}$$

where the operator p(t) is

$$p(t) = \left\langle f(t)\ f^{*}(t) \right\rangle \tag{42}$$

Note

$$f^{*}(t)\rangle \; \textcircled{i} \; \langle f(t) = I = \int_{t_b}^{t_f} f^{*}(t)\rangle \langle f(t)\,dt \tag{43}$$

is analog of Equation (33).

$$\langle \bar{e}\ \bar{e}^{*}\rangle = \bar{\bar{I}} = \langle \bar{e}\ I\ \bar{e}^{*}\rangle \tag{44}$$

that is the operator whose matrix is the identity matrix in all bases. Note the binary operator $\textcircled{i}$ of Equation (41) operates on the two adjacents, on the left a package of vectors and on the right one vector. The Gibbs analog is

$$\bar{\bar{P}} \cdot \bar{x} = \langle \bar{e}\ P\ \bar{e}^{*}\rangle \cdot \bar{x} \tag{45}$$

$$= \left(\langle \bar{e}\ P\right)\ \bar{e}^{*}\rangle \cdot \bar{x} \tag{46}$$

where

$$\bar{e}^{*}\rangle \cdot \bar{x} = \begin{bmatrix} \bar{e}_1^{*} \cdot \bar{x} \\ \bar{e}_2^{*} \cdot \bar{x} \\ \cdot \\ \cdot \\ \cdot \\ \bar{e}_k^{*} \cdot \bar{x} \end{bmatrix} \tag{47}$$

and by Equation (41) and Equation (42)

$$x(t) = p(t) \; \textcircled{i} \; x(t) = \langle f(t) \left[f^{*}(t)\rangle \; \textcircled{i} \; x(t) \right] \tag{48}$$

$$= \langle f(t) \int_{t_b}^{t_f} f^{*}(t)\rangle\, x(t)\,dt \tag{49}$$

As in conventional vector analysis any invertable linear transformation yields a new basis say

$$\langle \ell(t) = \langle f(t)\; T_{f\ell}(t) \tag{50}$$

and

$$\langle f(t) = \langle \ell(t)\; T_{f\ell}^{-1} \tag{51}$$

Where the dxd matrix can be time-varying or constant.

For the constant matrix case

$$\int_{t_b}^{t_f} f^*(t)\rangle \langle \ell(t)\,dt = T_{f\ell} \quad . \tag{52}$$

or

$$T_{f\ell} = M_{ff}^{-1} \int_{t_b}^{t_f} f(t)\rangle \langle \ell(t))\,dt \quad . \tag{53}$$

$$T_{f\ell} = M_{ff}^{-1}\, M_{f\ell} \tag{54}$$

where

$$M_{f\ell} = \int_{t_b}^{t_f} f(t)\rangle \langle \ell(t)\,dt \tag{55}$$

The connection between the two metrics by transposing Equation (50) for constant $T_{f\ell}$ is

$$\int \ell\rangle \langle \ell\; dt = T_{f\ell}^{T} \int f\rangle \langle f\; dt\; T_{f\ell} \tag{56}$$

or

$$M_{\ell\ell} = T_{f\ell}^{T}\, M_{ff}\, T_{f\ell} \tag{57}$$

The coordinates of the vector in the two bases likewise is

$$x(t) = \left\langle f(t)\ \ a \right\rangle_f = \left\langle \ell(t)\ \ a \right\rangle_\ell \tag{58}$$

Using Equation (50) in Equation (58)

$$\left\langle f(t)\ \ a \right\rangle_f = \left\langle f(t)\ T_{f\ell}\ a \right\rangle_\ell \tag{59}$$

or

$$a\rangle_f = T_{f\ell}\ \ a\rangle_\ell \tag{60}$$

If the transformation matrix T of Equation (50) is rectangular instead of square say of size dxg

$$\langle g)\, \ell(t) = \langle d)\, f(t) \underset{dxg}{T} \tag{61}$$

then Equation (57) becomes

$$\underset{gxg}{M_{\ell\ell}} = \underset{gxd}{T^T}\ \underset{dxd}{M_{ff}}\ \underset{dxg}{T} \tag{62}$$

and the rank of $M_{\ell\ell}$ is less than or equal to the rank of M_{ff} (depending on rank of T). If M_{ff} and T are of rank d and g<d then the metric matrix $M_{\ell\ell}$ is fullrank and invertible, or form a basis for the subspace of dimension g.

As an example consider Equation (61) and

$$M_{f\ell} \equiv \int_{t_b}^{t_f} f(t)\rangle\langle \ell(t) = \underset{dxd}{M_{ff}}\ \underset{dxg}{T} \tag{63}$$

The matrices $M_{f\ell}$ and M_{ff} are

$$M_{f\ell} = \int_{t_b}^{t_f} \begin{bmatrix} f_0\ell_0 & \cdots & f_0\ell_{g-1} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ f_{d-1}\ell_0 & \cdots & f_{d-1}\ell_{g-1} \end{bmatrix} dt \tag{64}$$

and

$$M_{ff} = \int_{t_b}^{t_f} \begin{bmatrix} f_0f_0 & f_0f_1 & \cdots & f_0f_{d-1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ f_{d-1}f_0 & \cdot & \cdots & f_{d-1}^2(t) \end{bmatrix} dt \tag{65}$$

and called also "moment matrices" or matrices of inner-products.

Consider the exact vector partitioned

$$x(t) = \left(\underset{1}{\Big\langle} d_1 \Big) f \,,\; \underset{2}{\Big\langle} d_2 \Big) f \right) \begin{pmatrix} a \underset{1}{\Big\rangle} \\ a \underset{2}{\Big\rangle} \end{pmatrix} = \underset{1}{\Big\langle} f \; a \underset{1}{\Big\rangle} + x_r(t) \tag{66}$$

where

$$d = d_1 + d_2 \tag{67}$$

and the remainder $x_r(t)$ is

$$x_r(t) = \underset{2}{\Big\langle} d_2 \Big) f \quad a \underset{2}{\Big\rangle} \; . \tag{68}$$

If we are given x(t) and the first d_1 functions and seek to compute $a\rangle_1$ the best we can do is to approximate the d_1 vector, for

$$\int_{t_b}^{t_f} f(t)\rangle_1 \, x(t)\; dt = \left[\int_{t_b}^{t_f} f\rangle_1 {}^{1}\langle f \, dt\right] a\rangle_1 + \int_{t_b}^{t_f} f\rangle_1 \, x_r(t)\; dt \tag{69}$$

$$= M_{f_1 f_1} \, a\rangle_1 + \int_{t_b}^{t_f} f\rangle_1 \, x_r(t) dt \tag{70}$$

and since $a\rangle_2$ is unknown we can't solve for $a\rangle_1$.

Where the sub-metric $M_{f_1 f_1}$ is

$$\int_{t_b}^{t_f} \begin{pmatrix} f\rangle_1 \\ f\rangle_2 \end{pmatrix} \begin{bmatrix} {}_1\langle f & {}_2\langle f \, dt \end{bmatrix} = \begin{bmatrix} M_{f_1 f_1} & M_{f_1 f_2} \\ M_{f_2 f_1} & M_{f_2 f_2} \end{bmatrix} . \tag{71}$$

The discrete-analog problem by Equation (66) is

$$\begin{bmatrix} x(t_0) \\ x(t_1) \\ x(t_2) \\ \vdots \\ x(t_k) \end{bmatrix} = \begin{bmatrix} {}^{1}\langle f\,(t_o) \\ {}^{1}\langle f\,(t_1) \\ \cdot \\ \cdot \\ \cdot \\ {}^{1}\langle f\,(t_k) \end{bmatrix} a\rangle_1 + \begin{bmatrix} {}^{2}\langle f\,(t_o) \\ {}^{2}\langle f\,(t_1) \\ \cdot \\ \cdot \\ \cdot \\ {}^{2}\langle f\,(t_k) \end{bmatrix} a\rangle_2 \tag{72}$$

or

$$x\rangle = F_1\, a\rangle_1 + F_2\, a\rangle_2 \tag{73}$$

and

$$F_1^T\, x\rangle = F_1{}^T F_1\, a\rangle_1 + F_1^T F_2\, a\rangle_2 \tag{74}$$

or

$$a\rangle_1 = (F_1^T F_1)^{-1} F_1^T\, x\rangle - F_1{}^* F_2\, a\rangle_2 \tag{75}$$

and if $a\rangle_2$ is unknown we can not solve for $a\rangle_1$.

We can approximate $a\rangle_1$ and will show for both discrete and continuous case how one approximates in two different ways, (i) algebraic or orthoginal projections and (ii) partial derivatives. Consider the discrete case first. By Equation (66) let

$$x(t) = \hat{x}(t) + \tilde{x}(t) \tag{76}$$

$$= x_a(t) + x_r(t) \tag{77}$$

where

$$x_a(t) = {}_1\langle f\ (t)\ a\rangle_1 \tag{78}$$

and

$$\hat{x}(t) = {}_1\langle f(t)\ \hat{a}\rangle_1 \tag{79}$$

and

$$\tilde{x}(t) = x(t) - \hat{x}(t) \tag{80}$$

By Equation (76) and Equation (77) we have

$$\begin{bmatrix} x(t_0) \\ x(t_1) \\ x(t_2) \\ \vdots \\ x(t_k) \end{bmatrix} = \begin{bmatrix} \hat{x}(t_0) \\ \hat{x}(t_1) \\ \cdot \\ \vdots \\ \hat{x}(t_k) \end{bmatrix} + \begin{bmatrix} \tilde{x}(t_0) \\ \tilde{x}(t_1) \\ \cdot \\ \vdots \\ \tilde{x}(t_k) \end{bmatrix} = \begin{bmatrix} x_a(t_0) \\ x_a(t_1) \\ \cdot \\ \vdots \\ x_a(t_k) \end{bmatrix} + \begin{bmatrix} x_r(t_0) \\ \cdot \\ \cdot \\ \vdots \\ x_r(t_k) \end{bmatrix} \tag{81}$$

or in k+1 space

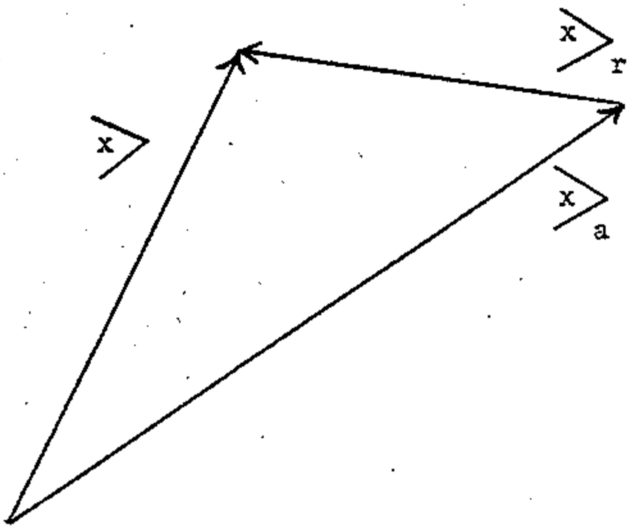

FIGURE 1 VECTOR TRIANGLE

If we select $\hat{x}\rangle$ to lie in space of $x\rangle_a$ or

$$\hat{x}\rangle = x\rangle_a \lambda \tag{82}$$

or

$$x\rangle = x\rangle_a \lambda + \tilde{x}\rangle \tag{83}$$

$${}_a\langle x\; x\rangle = {}_a\langle x\; x\rangle_a \lambda + {}_a\langle x\; \tilde{x}\rangle \tag{84}$$

Select $\tilde{x}\rangle$ perp to $\hat{x}\rangle = x\rangle_a \lambda$ hence

$$\frac{{}_a\langle x\; x\rangle}{{}_a\langle x\; x\rangle_a} = \hat{\lambda} \tag{85}$$

or

$$\hat{x}\rangle = x\rangle_a \hat{\lambda} = \frac{x\rangle_a \langle x_a}{{}_a\langle x\;\; x\rangle_a} \quad x\rangle \tag{86}$$

$$\hat{x}\rangle = P_{x_a x_a}\; x\rangle \tag{87}$$

But $x\rangle_a$ is not known hence we cannot compute $\hat{x}\rangle$ by Equation (86) which is the orthogonal projection of the only known vector $x\rangle$ onto the one-dimensional unknown space spanned by $x\rangle_a$.

If we solve Equation (81) for the approximation error vector $\tilde{x}\rangle$ then

$$\tilde{x}\rangle = x\rangle - \hat{x}\rangle \tag{88}$$

and

$$\langle \tilde{x}\, \tilde{x} \rangle = \langle x\, x \rangle - 2 \langle x\, \hat{x} \rangle + \langle \hat{x}\, \hat{x} \rangle$$

and taking the partial derivative with respect to the vector $\langle \hat{x}$

$$\langle \frac{\partial}{\partial \hat{x}} \langle \tilde{x}\, \tilde{x} \rangle = -2 \langle x + 2 \langle \hat{x} = \langle 0 \qquad (89)$$

or

$$\langle \hat{x} = \langle x \qquad (90)$$

Clearly Equation (90) minimizes the approximation error vector for it is exactly zero, that is

$$\tilde{x} \rangle = 0 \rangle \ . \qquad (91)$$

However the vector $x \rangle_a$ is a k+1 dimensional vector and by Equation (78)

$$x \rangle_a = F_1\, a \rangle_1 \qquad (92)$$

We assume the time points and the functions $_1\langle f(t)$ are known, hence F_I is known and $x \rangle_a$ has only $d_I \leq k$ unknowns that is $a(d \rangle_1$.

If we use Equation (79) and Equation (92) in Equation (81) we have

$$x(k \rangle = \underset{k_1 x d_1}{F_1}\, a \rangle_1 + F_2\, a \rangle_2 = F_1\, \tilde{a} \rangle_1 + \tilde{x} \rangle \qquad (93)$$

Partition F_1 into its column space

$$\underset{k_1 x d_1}{F_1} = \left[\phi \rangle_1 , \phi \rangle_2 , \cdot \cdot \cdot , \phi \rangle_{d_I} \right] \qquad (94)$$

of d_1 linearly independent vectors as shown in Figure 2.

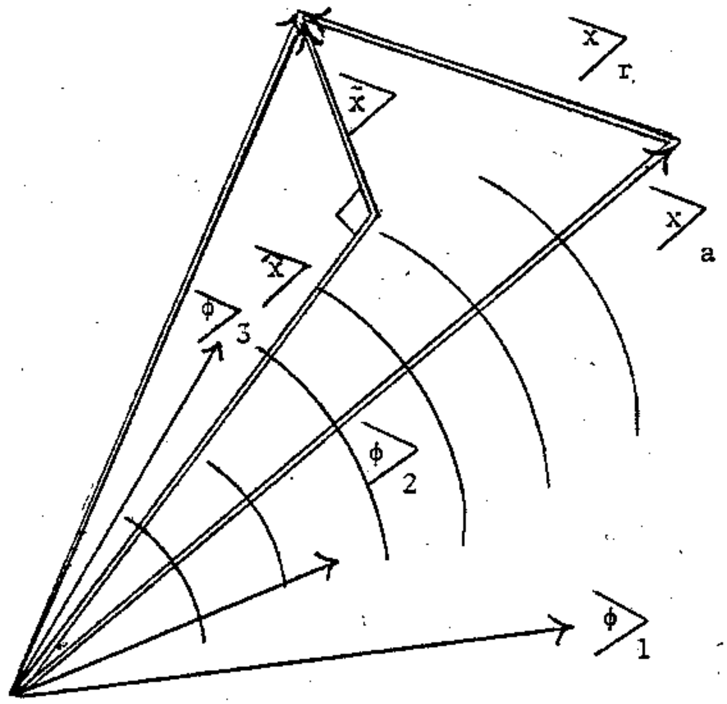

FIGURE 12

Multiply Equation (93) by F_1^T

$$F_1^T x\rangle = F_1^T F_1 \hat{a}\rangle + F^T \tilde{x}\rangle \tag{95}$$

$$= F_1^T F_1 \hat{a}\rangle_1 \tag{96}$$

where $a\rangle_1$ is selected such that

$$F_1^T \bar{x}\rangle = 0 \tag{97}$$

and by Equation (96)

$$\hat{a}\rangle_1 = F_1^{*} x\rangle \tag{98}$$

where the psuedo inverse is

$$\underset{d_1 x k_1}{F_1^{*}} = (F_1^T F_1)^{-1} F_1^T \tag{99}$$

The approximate vector is

$$\hat{x}\rangle = F_1 \hat{a}(d_1\rangle_1 = \underset{k_1 x k_1}{F_1 F_1^{*}} x\rangle \tag{100}$$

hence $\hat{x}\rangle$ is the orthogonal projection of the given data vector $x\rangle$ onto the subspace of F_1 for

$$P_{f_1 f_1} = F_1 F_1^{*} = P^2_{f_1 f_1} \tag{101}$$

also

$$F_1^{*} F_1 = \underset{d_1 x d_1}{I} \quad . \tag{102}$$

Using Equation (100) in Equation (81)

$$x\rangle = F_1 \hat{a}\rangle + \tilde{x}\rangle \tag{103}$$

$$= P_{f_1 f_1} x\rangle + \tilde{x}\rangle \tag{104}$$

we have

$$\tilde{x}\rangle = x\rangle - Px\rangle = (I-P)\;x\rangle \tag{105}$$

or

$$\tilde{x}\rangle = \tilde{P}_{f_1 f_1}\;x\rangle \tag{106}$$

Note that the rank of $P_{f_1 f_1}$ is d_1, that is

$$\rho \underset{k_1 x k_1}{P_{f_1 f_1}} = \rho(\underset{(k_1 x d_1)(d_1 x k_1)}{F_1 F_1^{*}}) = tr(F_1 F_1^{*}) = d_1 \tag{107}$$

the rank of a projector equals the trace of the projector, for

$$tr\;(F_1 F_1^{*}) = tr\left[\phi\rangle_{1}\,{}^{1}\langle\phi^{*} + \cdot\;\cdot\;\cdot\;\phi\rangle_{d_1}\,{}^{d_1}\langle\phi^{*}\right] = \tag{108}$$

$$= {}_{1}\langle\phi^{*}\;\phi\rangle_{1} + {}_{2}\langle\phi^{*}\;\phi\rangle_{2} + \cdot\;\cdot\;\cdot\;{}_{d_1}\langle\phi^{*}\;\phi\rangle_{d_1} \tag{109}$$

$$= 1 + \cdot\;\cdot\;\cdot + 1$$

$$= d_1$$

Note that the k+1 square matrix (projector of Equation 27) has rank d; whereas the (k+1) square projector of Equation (100) has rank $d_1 < d$.

By Equation (93) we see by multiplying by F_1^{*}

$$a\rangle_{1} + F_1^{*}F_2\;a\rangle_{2} = \hat{a}\rangle_{1} \tag{110}$$

since

$$F_1^* \tilde{x}\rangle = F_1^* \tilde{P}\, x\rangle = 0 \tag{111}$$

for

$$F_1^* \tilde{P} = 0 \tag{112}$$

and the error in the approximation of the parameter vector $a\rangle_1$ is

$$a\rangle_1 - \hat{a}\rangle_1 = \tilde{a}\rangle_1 = -F_1^* F_2\, a\rangle_2 \tag{113}$$

PARTIAL DERIVATIVE SOLUTION (DISCRETE CASE). The partial derivative of the inner-product of the error vector of Equation (95) is

$$\tilde{x}\rangle = x\rangle - F_1\, \hat{a}\rangle_1 \tag{114}$$

and

$$\langle\tilde{x} = \langle x - {}_1\langle\hat{a}\, F_1^T \tag{115}$$

$$\langle\tilde{x}\,\tilde{x}\rangle = \langle x\, x\rangle - 2\langle x\, F_1\, \hat{a}\rangle_1 + {}_1\langle\hat{a}\, F_1^T F_1\, \hat{a}\rangle \tag{116}$$

or

$$\langle\frac{\partial \langle\tilde{x}\,\tilde{x}\rangle}{\partial \hat{a}} = -2\langle x\, F_1 + 2\,{}_1\langle\hat{a}\, F_1^T F_1 = \langle 0 \tag{117}$$

or

$${}_1\langle\hat{a}\, (F_1^T F_1) = \langle x\, F_1 \tag{118}$$

and

$$\underset{1}{\Big\langle}\hat{a} = \Big\langle x\ F_1\ (F_1{}^T F_1)^{-1} \qquad (119)$$

which is the same as Equation (98) solution obtained via algrbra.

ALGEBRAIC SOLUTION (continuous case)

By Equation (66) and Equation (76)

$$x(t) = x_a(t) + x_r(t) = \hat{x}(t) + \tilde{x}(t) \qquad (120$$

and analogy with Equation (82)

$$\hat{x}(t) = x_a(t)\ \lambda \qquad (121)$$

$$\int x_a(t)\ x(t)\ dt = \lambda \int x_a^2(t)\ dt$$

or

$$\lambda = \frac{\int x_a(t)\ x(t)\ dt}{\int x_a{}^2(t)\ dt} \qquad (12$$

where

$$\int x_a\ \tilde{x}(t) = 0 = \int \hat{x}(t)\ \tilde{x}(t)\ dt \qquad (12$$

or

$$\hat{x}(t) = \frac{x_a(t) \int x_a(t)\ x(t)\ dt}{\int x_a{}^2(t)\ dt} = P_{x_a x_a}(t)\ \textcircled{i}\ x \qquad (12$$

Which is the analog of Equation (87); but since $x_a(t)$ is unknown as before we can't compute $\hat{x}(t)$ in this manner.

PARTIAL DERIVATIVE (continuous case).

By Equation (120)

$$\tilde{x}(t) = x(t) - \hat{x}(t) \tag{125}$$

and if we attempt to duplicate Equation (89) we have

$$\tilde{x}(t) \; \textcircled{i} \; \tilde{x}(t) = \int_{t_b}^{t_f} \tilde{x}^2(t)\, dt$$

$$= \int_{t_b}^{t_f} \; x(t) - \hat{x}(t) \quad {}^2 dt \tag{126}$$

and to take the partial derivative of Equation (126) with respect to $\hat{x}(t)$

$$\frac{\partial}{\partial \hat{x}(t)} \; \tilde{x}(t) \, \textcircled{i} \; \tilde{x}(t) = \frac{\partial}{\partial \hat{x}(t)} \int_{t_b}^{t_f} \left[x(t) - \hat{x}(t) \right]^2 dt \tag{127}$$

Using Equation (79) for $\hat{x}(t)$, Equation (126) becomes

$$\tilde{x}(t) \, \textcircled{i} \, \tilde{x}(t) = \int_{t_b}^{t_f} \left[x(t) - \; {}_1\langle f(t) \; \hat{a} \rangle_1 \right]^2 dt \tag{128}$$

$$= \int_{t_b}^{t_f} \left[x^2(t) - 2x(t) \; {}_1\langle f(t) \; \hat{a} \rangle_1 + {}_1\langle \hat{a} \; f \rangle_1^{\;1} \langle f \; \hat{a} \rangle_1 \right] dt \tag{129}$$

Taking the partial derivative with respect to the constant vector $a\rangle_1$ we have

$$ {}_1\langle\frac{d}{d\bar{a}}(\tilde{x}(t) \enclose{circle}{i} \tilde{x}(t)) = \int_{t_b}^{t_f}\left[-2x(t)\ {}_1\langle f(t) + 2\ {}_1\langle\hat{a}\ f\rangle_1\ {}^1\langle f\right]dt \qquad (130) $$

Equating the above qradient to zero we have

$$ -2\int_{t_b}^{t_f} x(t)\ {}_1\langle f(t)dt + 2\ {}_1\langle\hat{a}\int_{t_b}^{t_f} f(t)\rangle_1\ {}^1\langle f(t)\ dt = \langle 0 \qquad (131) $$

or

$$ {}_1\langle\hat{a} = M_{f_1 f_1}^{-1}\int_{t_b}^{t_f} x(t)\ {}_1\langle f(t)dt \qquad (132) $$

and using Equation (132)

$$ \hat{x}(t) = {}_1\langle f(t)\ \hat{a}\rangle_1 = {}_1\langle f(t)\ M_{ff}^{-1}\int_{t_b}^{t_f} x(t)\ f(t)\rangle_1\ dt \qquad (132) $$

$$ \hat{x}(t) = {}_1\langle f^{*}(t)\int_{t_b}^{t_f} x(t)\ f(t)\rangle_1\ dt \qquad (134) $$

Equations (132) and (134) are the analogs of Equation (119) and Equation (100).

<u>ALGEBRAIC SOLUTION.</u> We can sole for the parameters $\hat{a}\rangle_1$ without the partial notion, thus

$$ x(t) = {}_1\langle f(t)\ a\rangle_1 + x_r(t) = {}_1\langle f(t)\ \hat{a}\rangle_1 + \tilde{x}(t) $$

Taking the inner-product of x(t) by $f^*(t)\rangle_1$

$$\int_{t_b}^{t_f} f^*(t)\rangle_1 \, x(t)dt = \left[\int_{t_b}^{t_f} f^*(t)\rangle_1 \, {}^1\langle f(t)dt\right] \hat{a}\rangle_1 = \hat{a}\rangle_1 \tag{135}$$

where

$$\int_{t_b}^{t_f} f^*(t)\rangle_1 \, \tilde{x}(t)dt = 0\rangle \tag{136}$$

and

$$f^*(t)\rangle_1 = M^{-1}_{f_1 f_1} \, f(t)\rangle_1 \tag{137}$$

and

$$\int_{t_b}^{t_f} f^*\rangle_1 \, {}^1\langle f(t)dt = I_{d_1 x d_1} \quad , \tag{138}$$

that is, biorthogonality in the subspace.

The parameter-vector error is

$$a\rangle_1 - \hat{a}\rangle_1 = -\int_{t_b}^{t_f} f^*(t)\rangle_1 \, {}^2\langle f(t) \, a\rangle_2 \, dt \tag{139}$$

or

$$\tilde{a}\rangle_1 = -\underset{d_1 x d_1}{M_{f_1 f_1}^{-1}} \; \underset{d_1 x d_2}{M_{f_1 f_2}} \; \underset{d_2 x 1}{a\rangle_2} \tag{140}$$

which is the continuous analog of Equation (113).

<u>LINEARLY DEPENDENT FITTING FUNCTIONS.</u> If the fitting functions are linearly depedent or if one performs the transformation of Equation (50) where

$$\langle g)\, \ell(t) = \langle d)\, f(t) \underset{dxg}{T} \tag{141}$$

where g>d and $\langle f(t)$ are linearly independent then

$$\langle f(t) = \langle g)\, \ell(t) \underset{gxd}{T^*} \tag{142}$$

since

$$\underset{d(g)d}{T\;T^*} = \underset{dxd}{I} \tag{143}$$

and the commute is the projector

$$\underset{gxg}{T^*T} = P_{T^*T} = P^2 \tag{144}$$

The psuedo-duals of Equation (141)

$$\ell^*\rangle = T^*\, f^*\rangle \tag{145}$$

and

$$\int_{t_b}^{t_f} \ell^{*}\rangle\langle \ell = T^{*} \int_{t_b}^{t_f} f^{*}\rangle\langle f \, dt \, T \tag{146}$$

or

$$\ell^{*}(t)\rangle \;\; \text{ⓘ} \;\; \langle \ell(t) = T^{*}T = P_{T^{*}T} \tag{147}$$

The psuedo-metric of Equation (141) is

$$\int_{t_b}^{t_f} \ell(t)\rangle\langle \ell(t) dt = M_{\ell\ell} = T^{T}_{gxd} \, M_{ff} \, T_{dxg} \tag{148}$$

and the psuedo inverse of the metric, since the right hand side of Equation (148) has three full rank factors, is

$$M^{*}_{\ell\ell} = T^{*} \, M^{-1}_{ff} \, T^{T*} \tag{149}$$

A <u>psuedo-similarity</u> matrix relation.

We can now express the psuedo-dual of Equation (145) as

$$\ell^{*}\rangle = M^{*}_{\ell\ell} \, \ell\rangle \tag{150}$$

<u>WEIGHTED INNER PRODUCT</u>. If we use the standard notation for inner product following some of the techniques of Erdelyi (for real functions of real variables)

$$f_d(t) = f(t,d) \tag{151}$$

where t is the "polynominal argument" and d is the degree of the fitting functions, and a weight $w(t)$ which is non-negative on the interval (t_b, t_f), we may associate the inner product

$$(f_{d_1}(t),\ f_{d_2}(t)) = \int_{t_b}^{t_f} f_{d_1}(t)\ f_{d_2}(t)\ w(t)dt \quad . \tag{152}$$

if

$$f_{d_1}(t) = 1 \quad \text{and } d_1 = d_2 \tag{153}$$

then

$$\int_{t_b}^{t_f} w(t) > 0 \tag{154}$$

is the standard requirement on the weight function.

Package wise one can extend the scalar product to

$$(f(t)\rangle ,\ \langle f(t)) \equiv \int_{t_b}^{t_f} w(t)\ f(t)\rangle\langle f(t)dt = M_{tt_w}(t_b,t_f) \tag{155}$$

hence on a base or a system of fitting functions one can consider (alias-alibi concept) a base change occurred.

The teminology of the <u>fitting functions</u> f(t,d) and <u>base functions</u> will be used interchangeably.

Clearly the weighted metric of Equation (155) can be considered a base change

$$\langle \ell(t) = \langle f(t)\ w(t)^{\frac{1}{2}} \quad . \tag{156}$$

If the metric matrix $M_{\ell\ell_W}$ of Equation (155) is diagonal the system or set of functions (bases) are said to be orthogonal, if the matrix is the identity the system is said to be an ortho-normal system, that is

$$M_{\ell\ell}(t_1t_2) = D \text{ (diagonal) orthogonal} \tag{157}$$

$$M_{\ell\ell}(t_1t_2) = I \text{ ortho-normal} \tag{158}$$

Every orthogonal system can be normalized by replacing

$$\underset{du}{\ell(t)} = \frac{\ell_d(t)}{||\ell_d||} \qquad (159)$$

where

$$||\ell_d(t)|| = \left[\int_{t_1}^{t_2} \ell_d^2(t)\,dt\right]^{\frac{1}{2}} \quad . \qquad (160)$$

Element-wise the orthonormial condition of Equation (158) is

$$(f_i(t),\ f_j(t)) = \begin{bmatrix} 0 & \text{if} & i \neq j \\ 1 & \text{if} & i = j \end{bmatrix} \quad . \qquad (161)$$

We find alias, alibi confusion in effect in the standard liturature of linear vector spaces when the base vectors are surpressed and a vector is represented as a column vector or the transpose as a row vector. Consider the equation

$$y(n\rangle = \underset{nxn}{A}\ x(n\rangle \qquad (162)$$

where (1) $y\rangle$ can be the same vector (or point) with coordinates in a new base, or (2) $y\rangle$ can be a different vector, the image of $x\rangle$ under the transformation A as shown in Figure (3).

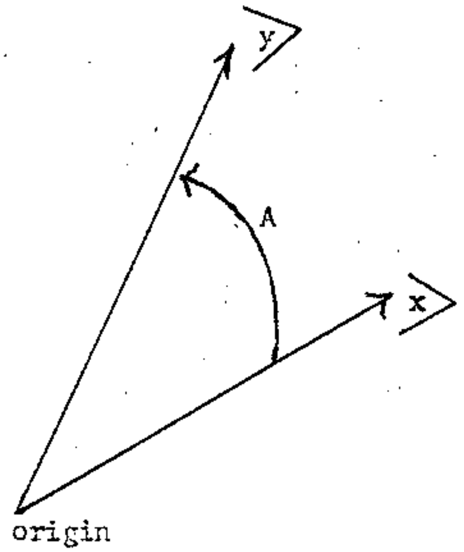

FIGURE (3)

This ambiguity can be eliminated by computing the base vectors as in Equations (31) through (49), with the added advantage when derivatives are taken with respect to dynamical bases the matricized-dyadic representations of Gibbs retains order instead of confusion as in the classical tensor analysis. The n-tuple representation also requires pulling a suppressed metric-matrix out of the hat when one wants to introduce different inner-products; for example given two column vectors $x\rangle$ and $y\rangle$, (in an oblique base), the inner product is

$$\langle y\ Q\ x\rangle = (x,y) \tag{163}$$

and length is

$$||x\ || = (\langle x\ Q\ x\rangle)^{\frac{1}{2}} . \tag{164}$$

In aerospace systems analysis problems the explicit relationships between vectors, bases, etc. can be simplified by explicity operating on the bases instead of supressing them. More on this theme will follow when the bases are polynomial functions etc. For developments when the vectors are the classical Gibbsian vectors of classical physis see references (69 ....)

SIMPLE SETS OF POLYNOMIALS. Throughout the report the definition of Rainville (page 147) will be used. A set of polynomials $\{f_d(t)\}; d=0,1,2 \cdots$ is called a simple set if $f_d(t)$ is of degree precisely d in t so that the set contains one polynomial of each degree. One immediate result of the definition of a simple set of polynomial is that any polynomial can be expressed linearly in terms of the elements of that simple set. Rainville proves the following theorem (not proved here).(73)

THEOREM. If $\{f_d(t)\}$ is a simple set of polynomials and if x(t) is a polynomial of degree m, there exist constants C_k such that

$$x(t) = \sum_{k=0}^{m} C_k \; f_k(t) \; .$$

where $m \leq d$.

ORTHOGONAL POLYNOMIALS, VECTORS, AND MATRICES, VIA GRAHM SCHMIDT AND TRIANGULAR DECOMPOSITIONS. This section will consider some procedures for taking any set of vectors and obtaining an orthogonal set. In some cases the vectors are polynomial vectors as linear combinations of base polynomials that is

$$x(t) = \left\langle f(t) \;\; a \right\rangle$$

where x(t) is a vector, $f_1(t)$, $f_2(t)$ $\cdots$ are considered vectors. The set of vectors lead to symmetric "metric" matrices. Hence considerations of Gram -Schmidt and related orthogonal procedures for matrices is part of the same considerations. Rather than restrict the vectors to be polynomials or classical Gibbsian vectors of Equation (31) through Equation (48) the inner-product symbol will be at times taken to be ⓘ, and arbitrary types of vectors designated with a single bar on top.

GRAM-SCHMIDT ORTHOGONALIZATION is a classical mathematical technique for solving the following problem: Given a sequence of vectors (linearly independent or linearly dependent) $\{f_1, \bar{f}_2, \bar{f}_3, \cdots \bar{f}_d\}$ produce a mutually orthogonal set of linearly independent vectors $\{\bar{s}_1, \bar{s}_2, \bar{s}_3, \cdots \bar{s}_g\}$ where $g \leq d$ such that for $k=1,2, \cdots d$, the set $\{\bar{f}_1, \bar{f}_2, \bar{f}_3, \cdots \bar{f}_k\}$ having rank ℓ where $\ell \leq k$ spans the same ℓ-dimensional subspace as the mutual orthogonal set $\{\bar{s}_1, \bar{s}_2, \cdots \bar{s}_\ell\}$ (having full rank).

Consider the case where the set of vectors $\langle d)\bar{f}$ is full rank hence forms a base, then for any full rank matrix B we have a new base $\langle \bar{b}$ or

$$\langle \bar{b} = \langle \bar{f} \underset{d\times d}{B} \tag{165}$$

we have d^2 free choices for the coordinates of $\langle \bar{b}$ in the $\langle \bar{f}$ base as entries in the B matrix (subject to condition B^{-1} exist). If we desire a new base such that the connection matrix is triangular we are constraining $\frac{(d^2-d)}{2}$ of the new coordinates to be zero, or

$$\langle \bar{b} = \langle \bar{f} \begin{bmatrix} b^1_{\cdot 1} & b^1_{\cdot 2} & b^1_{\cdot 3} & \cdots & b^1_{\cdot d} \\ 0 & b^2_{\cdot 2} & b^2_{\cdot 3} & \cdots & b^2_{\cdot d} \\ 0 & 0 & b^3_{\cdot 3} & & \cdot \\ 0 & 0 & 0 & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & \cdot & \cdot & & \cdot \\ 0 & & 0 & \cdots 0 & b^d_{\cdot d} \end{bmatrix}_{\Delta} \tag{166}$$

or

$$\bar{b}_1 = (\bar{f}_1, \bar{f}_2, \bar{f}_3, \ldots \bar{f}_d) \begin{pmatrix} b^1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_1 \tag{167}$$

$$\bar{b}_2 = (\bar{f}_1, \bar{f}_2, \cdots \bar{f}_d) \begin{pmatrix} b^1 \\ b^2 \\ 0 \\ 0 \\ \vdots \end{pmatrix}_2 \tag{168}$$

etc.

Transposing Equation (165) for triangular B_Δ

$$\bar{b}\rangle = B_\Delta^T \; f\rangle \tag{169}$$

and forming the symmetric matrix of inner-products

$$\bar{b}\rangle \textcircled{i} \langle\bar{b} = B_\Delta^T \; \bar{f}\rangle \textcircled{i} \langle\bar{f} \; B_\Delta \tag{170}$$

or

$$M_{bb} = B_\Delta^T \; M_{ff} \; B_\Delta \quad . \tag{171}$$

Now a symmetric matrix can have $(d^2+d)\frac{1}{2}$ independent entries. If we put the further constraint that the symmetric metric-matrix be diagonal, that is that the $\langle\bar{b}$ vectors of Equation (169) be orthogonal we have d (non-zero) choices on the main diagonal; and the remaining upper triangular entries of Equation (166) are uniquely determined.

For example there exist many 2x2 upper triangular matrices B_Δ of Equation (166) such that B_Δ is upper triangular and full rank and such that the new base vectors $\langle b$ are not orthogonal. Suppose the metric of the $\langle f$ vectors is

$$M_{ff} = \begin{pmatrix} 1 & \cos 30^\circ \\ \cos 30^\circ & \textcircled{1} \end{pmatrix} \tag{172}$$

as shown in Figure (1)

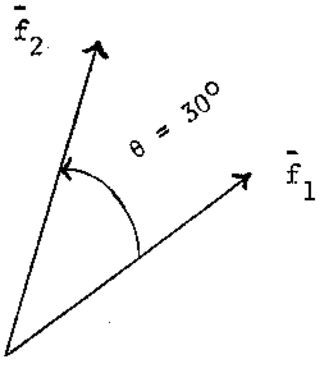

FIGURE 4 UNIT BASE VECTORS

and construct the new base as

$$\bar{b}_1 = 2\bar{f}_1 \qquad (173)$$

$$\bar{b}_2 = 2\bar{f}_1 + \bar{f}_2$$

or

$$\langle\bar{b} = \langle\bar{f} \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} = \langle\bar{f}\, B_\Delta \qquad (174)$$

then the metric of the $\langle\bar{b}$ base is

$$\bar{b}\rangle \cdot \langle\bar{b} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & C30^\circ \\ C30^\circ & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \qquad (175)$$

or

$$M_{bb} = \begin{bmatrix} 4 & 2(2+C30^\circ) \\ 2(2+C30^\circ) & 5+4C30^\circ \end{bmatrix} \qquad (176)$$

which is non-diagonal.

Note that in Equation (174) we assigned the four values to the triangular matrix B.

In general for the 2x2 matrix we have

$$\langle\bar{b} = \langle\bar{f} \begin{bmatrix} b^1_{\cdot 1} & b^1_{\cdot 2} \\ b^2_{\cdot 1} & b^2_{\cdot 2} \end{bmatrix} \qquad (177)$$

on four choices of b_{ij}; the triangular constraint

$$b^{2}_{\cdot 1} = 0 \tag{178}$$

leaves 3 free choices for the B_{Δ} matrix, or

$$\langle \bar{b} = \langle \bar{f} \begin{bmatrix} b^{1}_{\cdot 1} & b^{1}_{\cdot 2} \\ 0 & b^{2}_{\cdot 2} \end{bmatrix} \tag{179}$$

and

$$M_{bb} = \begin{bmatrix} b^{1}_{\cdot 1} & 0 \\ b^{2}_{\cdot 2} & b^{2}_{\cdot 2} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix}_{ff} \begin{bmatrix} b^{1}_{\cdot 1} & b^{1}_{\cdot 2} \\ 0 & b^{2}_{\cdot 2} \end{bmatrix} \tag{180}$$

$$M_{bb} = \begin{bmatrix} g_{11}(b^{1}_{\cdot 1})^{2} & b_{\cdot 1}(g_{11}b^{1}_{\cdot 2}+g_{12}b^{2}_{\cdot 2}) \\ b^{1}_{\cdot 1}(g_{11}b^{1}_{\cdot 2}+g_{12}b^{2}_{\cdot 2}) & (b^{1}_{\cdot 2})^{2}g_{11}+2(b^{1}_{\cdot 2}b^{2}_{\cdot 2})g_{12}+(b^{2}_{\cdot 2})^{2}g_{22} \end{bmatrix} \tag{181}$$

where we are given

$$M_{ff} = \begin{bmatrix} g_{ij} \end{bmatrix}_{ff} = \begin{bmatrix} g_{ij} \end{bmatrix} \tag{182}$$

the f subscrips supressed.

Let the grammian matrix

$$M_{bb} = \begin{bmatrix} g_{ij} \end{bmatrix}_{bb} \tag{183}$$

The diagonal constraint or M_{bb}, equating off-diagonal elements is

$$g_{12bb} = 0 = g_{21bb} = b^{1}_{\cdot 1}(g_{11}b^{1}_{\cdot 12} + g_{12}b^{2}_{\cdot 12}) \tag{184}$$

or(clearly $b^{1}_{\cdot 1}$ cannot be zero)

$$0 = g_{11}b^{1}_{\cdot 12} + g_{12}b^{2}_{\cdot 2} \tag{185}$$

or

$$b^{1}_{\cdot 2} = \frac{g_{12}}{g_{11}}\, b^{2}_{\cdot 2} \tag{186}$$

(clearly $g_{11} \neq 0$). For full rank $b^{1}_{\cdot 1}$ and $b^{2}_{\cdot 2}$ must be different from zero, hence by Equation (186) $b^{1}_{\cdot 2}$ can be zero only when $g_{12}=0$, or the $\langle\bar{f}$ base is orthogonal. Thus the constraint of Equation (184) plus the constraint of Equation (178) leaves two free choices of coordinates in the b matrix, namely $b^{1}_{\cdot 1}$ and by Equation (185) $b^{2}_{\cdot 2}$ (or $b^{1}_{\cdot 2}$) when M_{ff} is non-diagonal-the case of interest.

We thus have ($\langle\bar{b} = \langle\bar{g}$)

$$\langle\bar{g} = \langle\bar{f} \begin{bmatrix} b^{1}_{\cdot 1} & \frac{g_{12}}{g_{11}}\, b^{2}_{\cdot 2} \\ 0 & b^{2}_{\cdot 2} \end{bmatrix}_{g} \tag{187}$$

which is a function of $b^{1}_{\cdot 1}$ and $b^{2}_{\cdot 2}$, and

$$\bar{g}\rangle \cdot \langle\bar{g} = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix} \tag{188}$$

or by Equation (187)

$$\langle\bar{g} = \langle\bar{f}\, \underset{2\mathrm{x}2}{B_g} \quad . \tag{189}$$

The $\langle\bar{g}$ vectors are orthogonal, the only unspecifieds are their magnitudes. If $b^1_{\cdot 1}$ and $b^2_{\cdot 2}$ are chosen the magnitudes or d_{11} and d_{22} (squares of magnitudes) are uniquely determined. Likewise if the d_{11} and d_{22} are chosen the unique upper triangular matrix B_g can be determined.

For a full rank set of $\langle\bar{f}$ and $\langle\bar{g}$ the Gram-Schmidt constructions procedure that is for all subsets $(\bar{f}_1, \dots \bar{f}_\ell)$ and $(\bar{g}_1, \bar{g}_2, \dots \bar{g}_\ell)$ for $\ell = 1,2, \dots d$ span the same subspaces quantities one can also write

$$\begin{aligned}
\bar{f}_1 &= f^1_{\cdot 1}\, \bar{g}_1 = \langle\bar{g} \begin{pmatrix} f^1_{\cdot 1} \\ 0 \\ 0 \end{pmatrix} \\
\bar{f}_2 &= f^1_{\cdot 2}\, \bar{g}_1 + f^2_{\cdot 2}\, \bar{g}_2 = \langle\bar{g} \begin{pmatrix} f^1_{\cdot 2} \\ f^2_{\cdot 2} \\ 0 \\ \vdots \end{pmatrix} \\
&\;\;\vdots \\
\bar{f}_d &= \langle\bar{g} \begin{pmatrix} f^1_{\cdot d} \\ f^2_{\cdot d} \\ \vdots \\ f^d_{\cdot d} \end{pmatrix}
\end{aligned} \tag{190}$$

or

$$\langle \bar{f} = \langle \bar{g}\; F_\Delta \tag{191}$$

By Equation (189) in Equation (191)

$$\langle \bar{f} = \langle \bar{f}\; B_g\; F_\Delta = \langle \bar{f} \tag{192}$$

or

$$B_g\; F_\Delta = I \tag{193}$$

or

$$F_\Delta = B_g^{-1} \quad . \tag{194}$$

By Equation (190) and (194) we seee that the inverse of the upper triangular Gram-Schmidt coordinate matrix is also upper triangular.

As an exercise one can show that the B_g matrix of Equation (174)

$$B_g = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \tag{195}$$

has the inverse

$$B_g^{-1} = \begin{bmatrix} \frac{1}{2} & -1 \\ 0 & 1 \end{bmatrix} \tag{196}$$

Transposing Equation (191) using (194)

$$\bar{f} \rangle = B_g^{-T}\; \bar{g} \rangle \tag{197}$$

and forming the symmetric metric

$$\bar{f} \rangle \textcircled{i} \langle \bar{f} = B_g^{-T}\; \bar{g} \rangle \textcircled{i} \langle \bar{g}\; B_g^{-1} \tag{198}$$

or

$$M_{ff} = (B_g^{-1})^T D (B_g^{-1}) \tag{199}$$

$$M_{ff} = F_\Delta^T \; D \; F \tag{200}$$

Hence the full-rank metric matrix is congruent to a diagonal matrix where the diagonal matrix D is

$$g\rangle \textcircled{i} \langle g = M_{gg} = \begin{pmatrix} d_{11} & 0 & 0 & \\ 0 & d_{22} & & \\ & & \ddots & \\ & & & d_{dd} \end{pmatrix} \tag{201}$$

The symmetric "metric matrices" or matrices of inner-products of the vectors with themselves are called Gram matrices, moment matrices etc. The matrices M_{ff}, M_{bb} and M_{gg} are positive definite since they represent inner-products (reals).

Positive definite. A matrix M is said to be positive definite if for all non-zero vectors $\bar{x}$ or $x\rangle$

$$\langle x \; M \; x\rangle > 0 \tag{202}$$

where > is greater than symbol, also written in most texts books as

$$x^T \; Mx > 0 \tag{203}$$

where x is understood to be a column vector and superscript T is transpose.

Wendroff proves a theorem on page 126 of his book (stated here but not proved since most of theorems become geometrically obvious). (89)

THEOREM. If M is symmetric matrix and positive definite there exists an upper triangular matrix U such that

$$M = U^T U \tag{204}$$

Clearly if we partition the upper triangular matrix U into its column space

$$U = \begin{bmatrix} u^1_{\cdot 1} & u^1_{\cdot 2} & u^1_{\cdot 3} & \cdots & u^1_{\cdot d} \\ 0 & u^2_{\cdot 2} & u^2_{\cdot 3} & & u^2_{\cdot d} \\ 0 & 0 & u^3_{\cdot 3} & & \cdot \\ \cdot & \cdot & 0 & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & & u^d_{\cdot d} \end{bmatrix} = \left[u\rangle_1, \; u\rangle_2 \cdots u\rangle_d \right] \tag{205}$$

and transposing Equation (205) into a column of row vectors

$$U^T = \begin{bmatrix} {}^1\langle u \\ {}^2\langle u \\ \cdot \\ \cdot \\ \cdot \\ {}^d\langle u \end{bmatrix} \tag{206}$$

Using Equation (205) and (206) in Equation (204)

$$M = U^T U = \begin{bmatrix} {}^1\langle u \\ {}^2\langle u \\ \cdot \\ \cdot \\ \cdot \\ {}^d\langle u \end{bmatrix} \left[u\rangle_1 \; u\rangle_2 \; \cdots \; u\rangle_d \right] \tag{207}$$

Define the row of Gibbsian - type vectors (not n-type column vectors) as

$$\langle \bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3, \ . \ . \ . \ \bar{u}_d) = \langle \bar{e}\ U$$

$$= [\langle e\ u\rangle_1, \langle e\ u\rangle_2, \ . \ . \ . \langle e\ u\rangle_d] \qquad (208)$$

and transposing Eq (208)

$$\bar{u}\rangle = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{u}_d \end{bmatrix} = U^T \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{e}_d \end{bmatrix} = U^T\ \bar{e}\rangle \qquad (209)$$

where the background $\langle \bar{e}$ base is

$$\bar{e}\rangle \cdot \langle \bar{e} = I \qquad (210)$$

that is orthonomal.

By Eq (208),(209) and (210)

$$U^T U = \bar{u}\rangle \cdot \langle \bar{u} = U^T\ \bar{e}\rangle \cdot \langle \bar{e}\ U \qquad (211)$$

Using Eq (211) in Eq (202)

$$\langle x\ U^T U\ x\rangle = \langle x\ \bar{u}\rangle \cdot \langle \bar{u}\ x\rangle = \bar{x} \ \cdot \ \bar{x} \qquad (212)$$

where the column vector $\bar{x}$ in the $\langle \bar{u}$ base has coordinates given by Eq (213).

$$\bar{x} = \langle \bar{u} \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^d \end{pmatrix} = \langle \bar{u}\; x \rangle \qquad (213)$$

or transposing

$$\bar{x} = \langle x\; \bar{u} \rangle \qquad (214)$$

and the inner-product

$$\begin{aligned} \bar{x} \cdot \bar{x} &= \langle x\; \bar{u} \rangle \cdot \langle \bar{u}\; x \rangle \\ &= \langle x\; M_{uu}\; x \rangle \end{aligned} \qquad (215)$$

SEQUENTIAL GENERALIZED GRAM-SCHMIDT PROCEDURE. Consider a sequence of linearly indepedent vectors

$$(\bar{f}_1,\ \bar{f}_2,\ \bar{f}_3,\ \bar{f}_4\ \cdot\ \cdot\ \cdot\ \bar{f}_d) = \langle d)\,\bar{f} \qquad (216)$$

where we desire to construct the matrix B_s of Equation (189) or

$$\langle \bar{s} = \langle \bar{f}\; B_s\ . \qquad (217)$$

we will construct an orthonormal sequence. Let the first unit magnitude vector be

$$\bar{s}_1 = \frac{\bar{f}_1}{(\bar{f}_1\ \textcircled{i}\ \bar{f}_1)^{\frac{1}{2}}} \qquad (218)$$

or

$$\bar{f}_1 = \bar{s}\ f_{\cdot 1} = \langle\bar{s} \begin{bmatrix} f^1_{\cdot 1} \\ 0 \\ 0 \\ \vdots \end{bmatrix} \tag{219}$$

or

$$\bar{s}_1 = (\bar{f}_1, \bar{f}_2 \cdot \cdot \cdot \bar{f}_d) \begin{pmatrix} (\bar{f}_1 \textcircled{i} \bar{f}_1)^{\frac{1}{2}} \\ 0 \\ 0 \end{pmatrix} = \langle\bar{f} \begin{pmatrix} b^1_{\cdot 1} \\ 0 \\ \vdots \end{pmatrix}$$

The second vector $\bar{s}_2$ is to be perpendicular to $\bar{s}_1$ and lie in same subspace as $\bar{f}_1$ and $\bar{f}_2$ as shown in Figure 2.

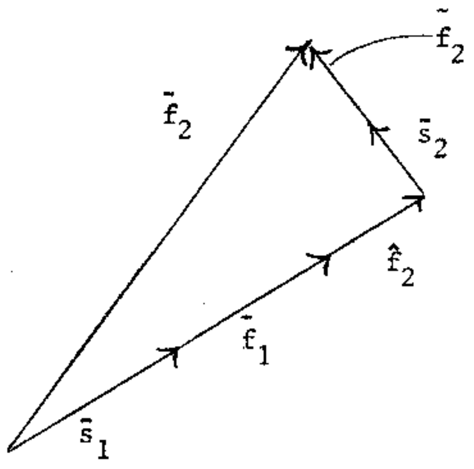

FIGURE 5

By Figure 2 there exists a scalar $\lambda^{1}_{\cdot 2}$ such that

$$\bar{f}_2 = \bar{f}_1 \, \lambda_{\cdot 2} + \tilde{f}_2 \tag{220}$$

or

$$\tilde{f}_2(1) = \bar{f}_2 - \bar{f}_1 \, \lambda_{\cdot 2} = (\bar{f}_1, \bar{f}_2) \begin{pmatrix} -\lambda^{1}_{\cdot 2} \\ 1 \end{pmatrix} \tag{221}$$

and we take the second orthonormial vector

$$\bar{s}_2 = \frac{\tilde{f}_2}{(\tilde{f}_2 \textcircled{i} \tilde{f}_2)^{\frac{1}{2}}} \quad . \tag{222}$$

By Equation (221) there is one unknown coordinate $\lambda^{1}_{\cdot 2}$ of the vector $\tilde{f}_2$ in the two dimensional subspace spanned by f_1 and $\bar{f}_2$. Taking the inner-product of Equation (221) with $\bar{f}_1$

$$\bar{f}_1 \textcircled{i} \tilde{f}_2 = 0 = \left[\bar{f}_1 \, (i) \, \bar{f}_1 , \; \bar{f}_1 \textcircled{i} \bar{f}_2 \right] \begin{pmatrix} -\lambda^{1}_{\cdot 2} \\ 1 \end{pmatrix} \tag{223}$$

or

$$\lambda^{1}_{\cdot 2} = \frac{f_1 \textcircled{i} \bar{f}_2}{\bar{f}_1 \textcircled{i} \bar{f}_1} \tag{224}$$

and Equation (224) in Equation (221)

$$\tilde{f}_2(1) = (\bar{f}_1, \bar{f}_2) \begin{pmatrix} - \dfrac{\bar{f}_1 \textcircled{i} \bar{f}_2}{\bar{f}_1 \textcircled{i} \bar{f}_1} \\ \\ 1 \end{pmatrix} \tag{225}$$

or

$$\tilde{f}_2(1) = \bar{f}_1 \left(\frac{-\bar{f}_1 \textcircled{i} \bar{f}_2}{\bar{f}_1 \textcircled{i} \bar{f}_1} \right) + \bar{f}_2 \tag{226}$$

$$= \bar{\bar{I}} \textcircled{i} \bar{f}_2 - \frac{\bar{f}_1 \bar{f}_1 \textcircled{i} \bar{f}_2}{\bar{f}_1 \textcircled{i} \bar{f}_1} \tag{227}$$

or factoring out $\bar{f}_2$,

$$\tilde{f}_2(1) = \left(\bar{\bar{I}} - \frac{\bar{f}_1 \bar{f}_1}{\bar{f}_1 \textcircled{i} \bar{f}_1} \right) \textcircled{i} \bar{f}_2 \tag{228}$$

Where the operator $\bar{\bar{I}}$ is the "idempotent" operator with respect to the inner-product used, that is

$$\bar{\bar{I}} \textcircled{i} \bar{f}_2 = \bar{f}_2 \tag{229}$$

We need the rank-one projector

$$\bar{\bar{P}}(1,1) = \frac{\bar{f}_1 \bar{f}_1}{\bar{f} \textcircled{i} \bar{f}_1} = \bar{f}_1 \bar{f}_1^{*} \tag{230}$$

where

$$\bar{f}_1^{*} = \frac{\bar{f}_1}{\bar{f}_1 \textcircled{i} \bar{f}_1} \tag{231}$$

which is idempotent for all vectors lying in the $\bar{f}_1$ subspace since

$$\bar{\bar{P}}(1,1) \textcircled{i} \bar{f}_1 \lambda = \bar{f}_1 \bar{f}_1^{*} \textcircled{i} \bar{f}_1 \lambda = \bar{f}_1 \lambda \tag{232}$$

(that is the operator P leaves the vectors $\bar{f}_1\lambda$ alone), since the inner-product of the base vector and its dual is

$$\bar{f}_1^{*} \cdot \bar{f}_1 = 1 \ . \tag{233}$$

The operator is called idempotent (index 2) since (see Equation (144)) for example)

$$\bar{\bar{P}} \textcircled{i} \bar{\bar{P}} \equiv \bar{\bar{P}}^2 = \bar{\bar{P}} \ . \tag{234}$$

Since $\bar{\bar{P}}_{11}$ is a rank one linear operator or transformation and every linear transformation has a matrix of (real) coordinates, imbed f_1 in the $\langle \bar{f}$ base or

$$\left\langle \bar{f}\ e \right\rangle_1 = \bar{f}_1 \tag{235}$$

where

$${}_1\langle e = (1, 0, 0 \cdot \cdot \cdot) \tag{236}$$

and the dual base with respect to the $\langle \bar{f}$ base (recipical base vectors) is

$$\begin{aligned} \bar{f}^{*}\rangle &= (\bar{f}(d\rangle \textcircled{i} \langle d)\bar{f})^{-1} \bar{f}\rangle \\ &= M_{ff}^{-1} \bar{f}\rangle \end{aligned} \tag{237}$$

or

$$\bar{f}_1^{*} = {}_1\langle e\ \bar{f}^{*}\rangle \tag{238}$$

Hence in the $\langle \bar{f}$ base and its dual the matrix of the operator is

$$\bar{\bar{P}}_{11} = \langle \bar{f}\; e\rangle_1 \;\; {}^1\langle e\; \bar{f}^*\rangle \tag{239}$$

where the dxd matrix (dyad) is

$$e\rangle_1 \; {}^1\langle e = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} (1,0,0 \cdots) = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ & 0 & 0 & & \\ & \cdot & & & \\ & \cdot & & & \\ & \cdot & & & \\ & 0 & & & 0 \end{bmatrix} \tag{240}$$

With respect to the d-dimensional subspace or (full space) spanned by the vectors of the sequence of Equation (217), the $\bar{\bar{I}}$ operator can be written as

$$\bar{\bar{I}} = \langle \bar{f}\; \underset{dxd}{I}\; \bar{f}^*\rangle \tag{241}$$

where

$$\bar{f}^*\rangle \; \textcircled{i} \; \langle \bar{f} = \underset{dxd}{I} \tag{242}$$

Now it is clear that

$$\begin{aligned} \bar{\bar{I}} \; \textcircled{i} \; \bar{x} &= \langle \bar{f}\; I\; \bar{f}^*\rangle \;\; \textcircled{i} \; \langle \bar{f}\; x\rangle \\ &= \langle \bar{f}\; x\rangle = \bar{x} \end{aligned} \tag{243}$$

for any vector expressed in the $\left\langle \bar{f} \right.$ base as

$$\bar{x} = \left\langle \bar{f}\ \bar{\bar{x}} \right\rangle \qquad . \tag{244}$$

The <u>orthogonal complement</u> projector (with respect to the $\bar{\bar{P}}_{(1,1)}$ projector is from Equation (228)

$$\overset{\approx}{\bar{P}} = \bar{\bar{I}} - \bar{\bar{P}} = \bar{\bar{I}} - \bar{f}_1\bar{\bar{f}}_1^{\,*} \tag{245}$$

and is idempotent

$$\overset{\approx}{\bar{P}} \textcircled{i} \overset{\approx}{\bar{P}} = \overset{\approx}{\bar{P}} \ , \tag{246}$$

the orthogonal complement projector is related to the projector $\overset{\approx}{\bar{P}}$ as

$$\overset{\approx}{\bar{P}} \textcircled{i} \bar{\bar{P}} = \bar{\bar{0}} \quad . \tag{247}$$

Returning now to Equation (228)

$$\tilde{f}_2(1) = \overset{\approx}{\bar{P}}_{(1,1)} \textcircled{i} \ \tilde{f}_2 \tag{248}$$

and transposing

$$\tilde{f}_2^{\ T} = \tilde{f}_2 = \bar{f}_2^{\ T} \textcircled{i} \ \overset{\approx}{\bar{P}}_{(1,1)} \tag{249}$$

since the projectors are symmetric that is

$$\overset{\approx}{\bar{P}}^T = \overset{\approx}{\bar{P}} \tag{250}$$

Taking the inner-product

$$\tilde{f}_2 \textcircled{i} \tilde{f}_2 = \bar{f}_2^T \textcircled{i} \bar{\bar{P}}_{11} \textcircled{i} \tilde{\bar{\bar{P}}}_{11} \textcircled{i} \bar{f}_2 \tag{251}$$

$$\tilde{f}_2 \textcircled{i} \tilde{f}_2 = \bar{f}_2 \textcircled{i} \tilde{\bar{P}} \textcircled{i} \bar{f}_2 \tag{252}$$

$$= \bar{f}_2 \textcircled{i} \left(\bar{\bar{I}} - \frac{\bar{f}_1 \bar{f}_1}{\bar{f}_1 \textcircled{i} \bar{f}_1} \right) \textcircled{i} \bar{f}_2$$

$$= \bar{f}_2 \textcircled{i} \bar{f}_2 - \frac{\bar{f}_2 \textcircled{i} \bar{f}_1 \bar{f}_1 \textcircled{i} \bar{f}_2}{\bar{f}_1 \textcircled{i} \bar{f}_1} \tag{253}$$

By Equation (225), Equation (248) and Equation (245)

$$\tilde{f}_2 = \bar{f}_2 - \left(\frac{\bar{f}_1 \textcircled{i} \bar{f}_2}{\bar{f}_1 \textcircled{i} \bar{f}_1} \right) \bar{f}_1 \tag{254}$$

and by Equation (222)

$$\bar{s}_2 = \tilde{f}_2 \, (\tilde{f}_2 \textcircled{i} \tilde{f}_2)^{-\frac{1}{2}} \quad . \tag{255}$$

Note that when $\tilde{f}_2$ is small in magnitude we are in computational trouble.

Using Equation (253) and (254) in Equation (255)

$$\bar{s}_2 = \frac{\bar{f}_2 - \left(\frac{\bar{f}_1 \textcircled{i} \bar{f}_2}{\bar{f}_1 \textcircled{i} \bar{f}_1} \right) \bar{f}_1}{\left[\bar{f}_2 \textcircled{i} \bar{f}_2 - \frac{(\bar{f}_2 \textcircled{i} \bar{f}_1)^2}{\bar{f}_1 \textcircled{i} \bar{f}_1} \right]^{\frac{1}{2}}} \tag{256}$$

$$= \frac{\bar{f}_1 \textcircled{i} \bar{f}_1)\ \bar{f}_2 - \bar{f}_1 \textcircled{i} \bar{f}_2\ \ \bar{f}_1}{\left(\bar{f}_1 \textcircled{i} \bar{f}_1\right)^{\frac{1}{2}} \left[\bar{f}_2 \textcircled{i} \bar{f}_2 - \bar{f}_2 \textcircled{i} \bar{f}_1{}^{2}\right]^{\frac{1}{2}}} \tag{257}$$

$$\bar{s}_2 = (\bar{f}_1, \bar{f}_2) \begin{bmatrix} \dfrac{-\bar{f}_1 \textcircled{i} \bar{f}_2}{\left[(\bar{f}_2 \textcircled{i} \bar{f}_2) - (\bar{f}_2 \textcircled{i} \bar{f}_1)^2\right]^{\frac{1}{2}}} \\ \\ \dfrac{\bar{f}_1 \textcircled{i} \bar{f}_1}{\left[(\bar{f}_2 \textcircled{i} \bar{f}_2) - (\bar{f}_2 \textcircled{i} \bar{f}_1)^2\right]^{\frac{1}{2}}} \end{bmatrix} \frac{1}{(\bar{f}_1 (i) \bar{f}_1)^{\frac{1}{2}}} \tag{258}$$

By Equation (258) and (218) we see that the coordinates are functions of the Grammian matrix of the $<f$, that is

$$\bar{f}\ (d> \textcircled{i} <d)\bar{f} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1d} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ g_{d1} & & & g_{dd} \end{bmatrix}_{ff} \tag{259}$$

Using Equation (259) in Equation (258)

$$\bar{s}_2 = (\bar{f}_1, \bar{f}_2) \begin{bmatrix} \dfrac{-g_{12}}{(g_{22}g_{11}-g_{12}^{2}{}^{\frac{1}{2}}} \\ \\ \dfrac{g_{11}}{(g_{22}g_{11}-g_{12}^{2})^{\frac{1}{2}}} \end{bmatrix} \frac{1}{g_{11}^{\frac{1}{2}}} \tag{260}$$

Consider next the third vector $\bar{s}_3$ such that

$$\begin{pmatrix} \bar{s}_1 \\ \bar{s}_2 \\ \bar{s}_3 \end{pmatrix} \textcircled{i} (\bar{s}_1, \bar{s}_2, \bar{s}_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{261}$$

Since $\bar{s}_3$ is to lie in the three dimensional subspace spanned by $(\bar{f}_1, \bar{f}_2, \bar{f}_3)$ and must be perpendicular to $\bar{f}_1$ and $\bar{f}_2$ we have by Figure 3

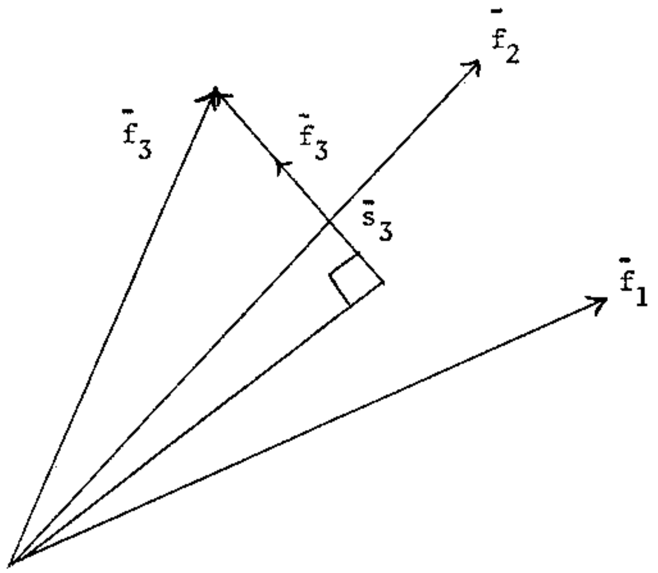

FIGURE 6

$$\bar{f}_3 = \bar{f}_1 \lambda^{1}_{\cdot 3} + \bar{f}_2 \lambda^{2}_{\cdot 3} + \tilde{f}_3(1,2) \tag{262}$$

where the vector $\tilde{f}_3(1,2)$ is the perpendicular component to subspace of $\bar{f}_1$ and f_2

$$\tilde{f}_3(1,2) = (\bar{f}_1,\bar{f}_2,\bar{f}_3) \begin{pmatrix} -\lambda^{1}_{\cdot 3} \\ -\lambda^{2}_{\cdot 3} \\ 1 \end{pmatrix} \tag{263}$$

and we define the unit magnitude vector

$$\bar{s}_3 = \frac{\tilde{f}_3(1,2)}{||\tilde{f}_3(1,2)||} \tag{264}$$

where the norm or the square of the magnitude is

$$||\tilde{f}_3(1,2)|| = \left[\tilde{f}_3(1,2) \textcircled{i} \tilde{f}_3(1,2) \right]^{\frac{1}{2}}. \tag{265}$$

There are two unknowns in Equation (263), namely $\lambda^{1}_{\cdot 3}$ and $\lambda^{2}_{\cdot 3}$. Take the package inner-product of Equation (263)

$$\begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \end{pmatrix} \textcircled{i} \tilde{f}_3(1,2) = \begin{pmatrix} \bar{f}_1 \textcircled{i} \tilde{f}_3(1,2) \\ \bar{f}_2 \textcircled{i} \tilde{f}_3(1,2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{266}$$

$$= \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \end{pmatrix} \textcircled{i} (\bar{f}_1,\bar{f}_2,\bar{f}_3) \begin{pmatrix} -\lambda^{1}_{\cdot 3} \\ -\lambda^{2}_{\cdot 3} \\ 1 \end{pmatrix} \tag{267}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{bmatrix} \begin{pmatrix} -\lambda^{1}_{\cdot 3} \\ -\lambda^{2}_{\cdot 3} \\ 1 \end{pmatrix} \tag{268}$$

or

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} \lambda 1 \\ \lambda^{2} \end{pmatrix}_{3} = \begin{pmatrix} g_{13} \\ g_{23} \end{pmatrix} \tag{269}$$

solving for $\lambda(2 >_{3}$

$$\begin{pmatrix} \lambda^{1} \\ \lambda^{2} \end{pmatrix}_{3} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} g_{13} \\ g_{23} \end{pmatrix} \tag{270}$$

The inverse matrix is

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} = \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix} \frac{1}{g_{11}g_{22}-g_{12}^{2}} \tag{271}$$

using Equation (271) in Equation (270)

$$\begin{pmatrix} \lambda^{1} \\ \lambda^{2} \end{pmatrix}_{3} = \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix} \begin{pmatrix} g_{13} \\ g_{23} \end{pmatrix} \frac{1}{g_{11}g_{22}-g_{12}^{2}} \tag{272}$$

$$= \begin{pmatrix} g_{22}g_{13} - g_{12}g_{23} \\ -g_{12}g_{13} + g_{11}g_{23} \end{pmatrix} \frac{1}{(g_{11}g_{22}-g_{12}^{2})} \tag{273}$$

Using Equation (273) in Equation (263)

$$\tilde{f}_3(1,2) = (\bar{f}_1, \bar{f}_2, \bar{f}_3) \begin{bmatrix} \dfrac{-g_{22}g_{13} + g_{12}g_{23}}{g_{11}g_{22} - g_{12}^{2}} \\ \\ \dfrac{g_{12}g_{13} - g_{11}g_{23}}{g_{11}g_{22} - g_{12}^{2}} \\ \\ 1 \end{bmatrix} \tag{274}$$

The introduction of the g_{ij} elements obscured the projective aspects of the problem, hence returning to Equation (263) and taking the inner-product with $f^{*}(2\rangle$ we have by partitioning

$$\bar{f}^{*}(2\rangle \textcircled{i} \tilde{f}_3(1,2) = \bar{f}^{*}(2\rangle \textcircled{i} \left[\langle 2)\bar{f}, \bar{f}_3 \right] \begin{pmatrix} -\lambda(2\rangle \\ 1 \end{pmatrix} \tag{275}$$

or

$$0(2\rangle = \left[I_{2x2}, \bar{f}^{*}(2\rangle \textcircled{i} \bar{f}_3 \right] \begin{pmatrix} -\lambda(2\rangle \\ 1 \end{pmatrix} \tag{276}$$

where

$$\bar{f}^{*}(2\rangle = \left(\bar{f}(2\rangle \textcircled{i} \langle 2)\bar{f} \right)^{-1} \bar{f}(2\rangle \tag{277}$$

and $\tilde{f}_3(1,2)$ is perpendicular to both $\bar{f}_1$ and $\bar{f}_2$ and their duals. By Equation (276)

$$\lambda(2\rangle = \bar{f}^*(2\rangle \textcircled{i} \bar{f}_3 \tag{278}$$

Using Equation (278) in Equation (262)

$$\tilde{f}_3(1,2) = \bar{f}_3 - (\bar{f}_1,\bar{f}_2)\ \lambda(2\rangle \tag{279}$$

or Equation (278) in Equation (279)

$$\tilde{f}_3(1,2) = \bar{f}_3 - \langle 2)\bar{f}\ \bar{f}^*(2\rangle \textcircled{i} \bar{f}_3$$

$$\tilde{f}_3(1,2) = \left[\bar{\bar{I}} - \langle 2)\bar{f}\ \bar{f}^*(2\rangle\right] \textcircled{i} \bar{f}_3$$

$$\tilde{f}_3(1,2) = \tilde{\bar{P}}(1,2) \textcircled{i} \bar{f}_3 \tag{280}$$

where

$$\tilde{\bar{\bar{P}}}(1,2) = \bar{\bar{I}} - \bar{\bar{P}}(1,2) \tag{281}$$

and

$$\bar{\bar{P}}(1,2) = \langle \bar{f}\bar{f}^*(2\rangle = \bar{f}_1\bar{f}_1^*(1,2) + \bar{f}_2\bar{f}_2^*(1,2) \tag{282}$$

As before Equation (281) and Equation (282) are projectors.

The matrix of the rank two projector $\bar{\bar{P}}(1,2)$ in the full $\langle\bar{f}$ base can be written a number of ways, for example the two vectors in the full base are

$$\bar{f}_1 = \langle d)\bar{f}\ e(d\rangle_1 = \langle 2)\bar{f}\ e(2\rangle_1$$

$$\bar{f}_2 = \langle d)\bar{f}\ e(d\rangle_2 = \langle 2)\bar{f}\ e(2\rangle_2 \tag{283}$$

and the two reciprocal vectors are

$$\bar{f}^{*}(1,2)(2\rangle = (\bar{f}(2\rangle \textcircled{i} \langle 2)f)^{-1}\ \bar{f}(2\rangle \qquad (284)$$

$$\bar{f}^{*}(1,2)(2\rangle = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1}\ \bar{f}(2\rangle = \underset{2x2}{M^{-1}}\ \bar{f}(2\rangle \qquad (285)$$

Packaging Equation (283) in the two dimensional subspace spanned by $\langle 2)f$

$$(\bar{f}_1\bar{f}_2) = \langle 2)\bar{f}\ \underset{2x2}{I} \qquad (286)$$

or in the full d-dimensional space spanned by all the vectors $\langle d)\ \bar{f}$

$$\langle 2)\bar{f} = \langle d)\bar{f} \begin{bmatrix} \underset{2x2}{I} \\ \\ \underset{(d-2)x2}{0} \end{bmatrix} = \langle d)\bar{f}\ \underset{dx2}{E_1} \qquad (287)$$

Using Equation (284) in Equation (282)

$$\bar{\bar{P}}(1,2) = \langle 2)\bar{f}\ \underset{2x2}{I}\ \bar{f}^{+}(1,2)(2\rangle = \langle 2)\bar{f}\ \underset{2x2}{M^{-1}}\ \bar{f}(2\rangle \qquad (288)$$

By Equation (288) we see that the minimum rank, minimum space matrix is the 2x2 identity matrix, that is

$$\underset{2x2}{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (289)$$

when the operator is expressed in a base and its dual base; when the operator is expressed completely in the $\langle \bar{f}$ base and the transpose the 2x2 matrix is

$$\underset{2x2}{M^{-1}} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \quad , \tag{290}$$

just as the coordinates of a vector look different in different bases, so also a matrix does.

We can also express the operator $\bar{\bar{P}}(1,2)$ in coordinates in the full $\langle d)\bar{f}$ space and the full $f^*(d\rangle$ base. Transposing Equation (287)

$$\bar{f}(2\rangle = E_1^T \, \bar{f}(d\rangle \quad . \tag{291}$$

The full-reciprocals are

$$\begin{aligned} \bar{f}^*(d\rangle &= \left(\bar{f}(d\rangle \text{ ⓘ } \langle d)\bar{f} \right)^{-1} \bar{f}(d\rangle \\ &= \underset{dxd}{M_{ff}^{-1}} \, \bar{f}(d\rangle \end{aligned} \tag{292}$$

or

$$\bar{f}(d\rangle = M_{ff} \, \bar{f}^*(d\rangle \quad . \tag{293}$$

Using Equation (293) in Equation (291)

$$\bar{f}(2\rangle = \underset{2xd}{E_1^T} \; \underset{(dxd)}{M_{fd}} \, \bar{f}^*(d\rangle \tag{294}$$

Using Equation (294) in Equation (285)

$$\bar{f}^*(1,2)(2\rangle = \underset{2x2}{M^{-1}} \; \underset{2xd}{E_1^T} \; \underset{dxd}{M_{ff}} \, \bar{f}^*(d\rangle \tag{295}$$

or Equation (295) in Equation (282) yields

$$\bar{\bar{P}}(1,2) = \left\langle d) \overleftarrow{f} \underset{dx2}{E_1}\ \underset{(2x2)}{M^{-1}}\ \underset{(2xd)}{E_1^T}\ \underset{dxd}{M_{ff}}\ \bar{f}^{*}(d \right\rangle \tag{296}$$

or the matrix of the projection operator of Equation (296) in the full space base and its dual is the dxd matrix

$$\underset{dx2}{E_1}\ \underset{(2x2)}{M^{-1}}\ \underset{(2xd)}{E_1^T}\ \underset{dxd}{M_{ff}} = \begin{bmatrix} \underset{2x2}{M^{-1}} & \underset{2c(1,2)}{0} \\ \underset{(d-2)x(2)}{0} & \underset{(d-2)(d-2)}{0} \end{bmatrix} \quad \underset{dxd}{M_{ff}}$$

or partitioning $\underset{dxd}{M_{ff}}$ such that sub-matrices are of compatible sizes.

$$E_1\ M^{-1}\ E_1^T\ M_{ff} = \begin{bmatrix} \underset{2x2}{I} & \underset{2x(d-2)}{0} \\ \underset{(d-2)x2}{0} & \underset{(d-2)(d-2)}{0} \end{bmatrix} \tag{297}$$

Since orthonormal bases are self dual that is

$$\bar{s}^{*}\rangle = \bar{s}\rangle \tag{298}$$

the nicest base to express the operator $\bar{\bar{P}}(1,2)$ is in the Schmidt orthonormal subspace base

$$\bar{\bar{P}}(1,2) = \left\langle 2) \bar{s}\ \bar{s}(2 \right\rangle = \bar{s}, \bar{s}_1 + \bar{s}_2\ \bar{s}_2, \tag{299}$$

even in the full Schmidt base one has

$$\bar{\bar{P}}(1,2) = \langle d)\bar{s} \begin{bmatrix} \underset{2x2}{I} & \underset{2x(d-2)}{0} \\ & \\ \underset{(d-2)x2}{0} & \underset{(d-2)(d-2)}{0} \end{bmatrix} \bar{s}(d\rangle \quad . \tag{300}$$

Returning now to Equation (264) and Equation (280) the third orthonormal vector is

$$\bar{s}_3 = \frac{\tilde{\tilde{f}}_3(1,2)}{\left[\bar{f}_3 \textcircled{i} \bar{\bar{P}}(1,2) \textcircled{i} \bar{f}_3 \right]^{\frac{1}{2}}} \tag{301}$$

Consider the vectors of Equation (226) and Equation (274) and the matrix of vectors not normalized

$$\left[\bar{f}_1, \tilde{f}_2(1), \tilde{f}_3(1,2) \right] = (\bar{g}_1, \bar{g}_2, \bar{g}_3) = \langle 3)\bar{g} \tag{302}$$

or

$$\langle 3)\bar{g} = \langle 3)\bar{f} \begin{bmatrix} 1 & \frac{-g_{12}}{g_{11}} & \frac{-g_{22}g_{13}+g_{12}g_{23}}{g_{11}g_{22}-g_{12}^2} \\ 0 & 1 & \frac{g_{12}g_{13}-g_{11}g_{23}}{g_{11}g_{22}-g_{12}^2} \\ 0 & 0 & 1 \end{bmatrix} \tag{303}$$

$$\langle\bar{g} = \langle\bar{f}\, B_g \tag{304}$$

which is called a <u>unit upper triangular matrix.</u>

The orthonormal Gram-Schmidt vectors can now be obtained from Equation (303) as

$$(\bar{s}_1,\bar{s}_2,\bar{s}_3) = \langle 2)\bar{g} \begin{bmatrix} \frac{1}{||\bar{g}_1||} & 0 & 0 \\ 0 & \frac{1}{||\bar{g}_2||} & 0 \\ 0 & 0 & \frac{1}{||\bar{g}_3||} \end{bmatrix} \tag{305}$$

The recursion on the projectors can be written in terms of the orthonormal vectors as

$$\bar{\bar{P}}(1) = \bar{s}_1\bar{s}_1$$

$$\bar{\bar{P}}(1,2) = \bar{s}_1\bar{s}_1+\bar{s}_2\bar{s}_2 = \sum_{j=1}^{2} \bar{s}_j\bar{s}_j$$

$$\vdots$$

$$\bar{\bar{P}}(1,2,\cdots k-1) = \bar{s}_1\bar{s}_1+\bar{s}_2\bar{s}_2 + \cdots + \bar{s}_{k-1}\bar{s}_{k-1} = \sum_{j=1}^{k-1} \bar{s}_j\bar{s}_j \tag{306}$$

$$\tilde{f}_2(1) = \bar{f}_2 - \bar{\bar{P}}(1) \textcircled{i} \bar{f}_2$$

$$\tilde{f}_3(1,2) = \bar{f}_3 - \bar{\bar{P}}(1,2) \textcircled{i} \bar{f}_3$$

$$\vdots$$

$$\tilde{f}_k(1,2, \cdots k-1) = \bar{f}_k - \bar{\bar{P}}(1,2, \cdots k-1) \textcircled{i} \bar{f}_k \tag{307}$$

In terms of summations

$$
\begin{aligned}
\bar{g}_2 &= \tilde{f}_2(1) = \bar{f}_2 - \bar{s}_1\bar{s}_1 \textcircled{i} \bar{f}_2 \\
\bar{g}_3 &= \tilde{f}_3(1,2) = \bar{f}_3 - \sum_{j=1}^{2} \bar{s}_j\bar{s}_j \textcircled{i} \bar{f}_3 \\
&\vdots \\
\bar{g}_k &= \tilde{f}_k(1,2 \ldots k-1) = \bar{f}_k - \sum_{j=1}^{k-1} \bar{s}_j\bar{s}_j \textcircled{i} \bar{f}_k \\
&\vdots \\
\bar{g}_d &= \tilde{f}_d(1,2, \cdots d-1) = \bar{f}_d - \sum_{j=1}^{d-1} \bar{s}_j\bar{s}_j \textcircled{i} \bar{f}_d
\end{aligned}
\tag{308}
$$

where k = 2, 3, ··· d

$$
\bar{s}_k = \frac{\tilde{f}_k(1,2 \cdots k-1)}{||\tilde{f}_k(1,2 \cdots k-1)||} = \frac{\bar{g}_k}{||\bar{g}_k||} \tag{309}
$$

and

$$
\bar{g}_1 = \bar{f}_1 \tag{310}
$$

and

$$
\bar{s}_1 = \frac{\bar{f}_1}{||\bar{f}_1||} \quad . \tag{311}
$$

Continuous Gram-Schmidt Polynomials (0,1).

Consider the set of base vectors (the monomial base)

$$(1,\ t,\ t^2) = \langle f \tag{312}$$

for a vector

$$x(t) = \langle f\ a \rangle = (1,\ t,\ t^2)\ a(3\rangle$$

with inner product on the bases defined as the Hilbert matrix

$$\underset{3x3}{M_{ff}} = f\rangle\ \text{ⓘ}\ \langle f = \int_0^1 f\rangle\langle f\ dt = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \tag{313}$$

By Equation (310) and Equation (311)

$$g_1 = s_1 = f_1 = 1 \tag{314}$$

since

$$s_1 = \frac{f_1}{(f_1\ \text{ⓘ}\ f_1)^{\frac{1}{2}}} = 1 = \langle 3)\, f\ e(3\rangle_1 \tag{315}$$

for

$$f_1\ \text{ⓘ}\ f_1 = {}_1\langle e\ f\rangle\ \text{ⓘ}\ \langle f\ e\rangle_1 = \tag{316}$$

$$= {}_1\langle 3)\, e\ \underset{3x3}{M_{ff}}\ e(3\rangle_1 = 1. \tag{317}$$

By Equation (308)

$$g_2 = f_2 - s_1 s_1\ \text{ⓘ}\ f_2 \tag{318}$$

and by Equation

$$s_1(i)f_2 = {}_1\Big\langle 3)e\ f \Big\rangle \textcircled{i} \Big\langle f \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{319}$$

$$= {}_1\Big\langle e \underset{3x3}{M_{ff}} e \Big\rangle_2 \tag{320}$$

$$= (1,0,0) \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \tfrac{1}{2} \tag{321}$$

Using Equation (321) in Equation (318)

$$g_2 = \Big\langle 3)f \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} 1/2 \right] = \Big\langle f \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} \tag{322}$$

and

$$s_2 = \frac{g_2}{(g_2 \textcircled{i} g_2)^{\frac{1}{2}}} \tag{323}$$

where

$$g_2 \textcircled{i} g_2 = (-\tfrac{1}{2},\ 1.\ 0) \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \tag{324}$$

$$g_2 \textcircled{i} g_2 = \frac{1}{12} \tag{325}$$

and

$$||g_2|| = \Big[g_2 \textcircled{i}\, g_2 \Big]^{1/2} = \frac{1}{2\sqrt{3}} \tag{326}$$

hence

$$s_2 = \Big\langle f \begin{bmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ 0 \end{bmatrix} = \Big\langle f\ b \Big\rangle_{2s} \tag{327}$$

The third orthogonal vector by Equation (308) is

$$g_3 = f_3 - s_1(s_1(i)f_3) - s_2(s_2 \textcircled{i}\, f_3) \tag{328}$$

where

$$\begin{aligned} s_1 \textcircled{i}\, f_3 &= \Big\langle e_1\ M_{ff}\ e_3 \Big\rangle \\ &= (1,0,0) \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1/3 \end{aligned} \tag{329}$$

and

$$\begin{aligned} s_2 \textcircled{i}\, f_3 &= {}_{2s}\Big\langle b\ M_{ff}\ e \Big\rangle_3 \\ &= (-\sqrt{3},\ 2\sqrt{3},\ 0) \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{3}} \end{aligned} \tag{330}$$

Using Equations (315),(327),(329) and (330) in Equation (328)

$$g_3 = \langle 2) f \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/3 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} \right] \tag{331}$$

$$g_3 = \langle 3) f \begin{bmatrix} 1/6 \\ -1 \\ 1 \end{bmatrix} \tag{332}$$

The norm of the g_3 vector is

$$||g_3|| = \left[(1/6, -1, 1) \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \begin{pmatrix} 1/6 \\ -1 \\ 1 \end{pmatrix} \right]^{\frac{1}{2}} \tag{333}$$

or

$$||g_3|| = \frac{1}{6\sqrt{5}} \tag{334}$$

Using Equation (334) in Equation (332)

$$s_3 = \langle f \begin{pmatrix} \sqrt{5} \\ -6\sqrt{5} \\ 6\sqrt{5} \end{pmatrix} \tag{335}$$

The $\langle g$ base connection is by Equations (314), (322) and (332)

$$(g_1, g_2, g_3) = (1, t, t^2)\begin{bmatrix} 1 & -1/2 & 1/6 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{336}$$

and the unit magnitude (orthonormal) vectors by Equations (327) and (335) is

$$(s_1, s_2, s_3) = (1, t, t^2)\begin{bmatrix} 1 & -\sqrt{3} & \sqrt{5} \\ 0 & 2\sqrt{3} & -6\sqrt{5} \\ 0 & 0 & 6\sqrt{5} \end{bmatrix} . \tag{337}$$

with respect to the inner-product on (0,1) interval.

TRIANGULAR FACTORS OF THE INVERSE OF A POSITIVE DEFINITE (FULL RANK) REAL MATRIX. This section will demonstrate how to obtain the inverse of a positive definite matrix having triangular factors and relate this to the Gram-Schmidt or orthogonal decomposition proceedures. By Equation (336) and Equation (337) we have

$$\langle \bar{g} = \langle \bar{f}\, B_g \tag{338}$$

and

$$\langle \bar{s} = \langle \bar{f}\, B_s \tag{339}$$

Transposing

$$\bar{g}\rangle = B_g^T\, \bar{f}\rangle \tag{340}$$

and

$$\bar{s}\rangle = B_s^T\, \bar{f}\rangle \tag{341}$$

Generating the metrics for the two orthogonal sets

$$\bar{g}\rangle \textcircled{i} \langle \bar{g} = B_g^T\, \bar{f}\rangle \textcircled{i} \langle \bar{f}\, B_g \tag{342}$$

and

$$\bar{s}\Big> \textcircled{i} \Big< \bar{s} = B_s^T\, \bar{f}\Big> \textcircled{i} \Big< \bar{f}\, B_s \tag{343}$$

or

$$M_{gg} = B_g^T\, M_{ff}\, B_g = D_g \tag{344}$$

$$M_{ss} = B_s^T\, M_{ff}\, B_s = I \tag{345}$$

or

$$B_g^{-T} D_g B_g^{-1} = M_{ff} = B_s^{-T} B_s^{-1} \tag{346}$$

or

$$\boxed{(B_g D_g^{-1} B_g^{T})^{-1} = M_{ff} = (B_s B_s^{T})^{-1}} \tag{347}$$

and inverting

$$\boxed{M_{ff}^{-1} = B_g D^{-1} B_g^{T} = B_s B_s^{T}} \tag{348}$$

Applying Equation (348) to the B_s matrix of Equation (337) or

$$\begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix} \tag{349}$$

Hence we see that we have an analytic method for the inverse of the classical dxd Hilbert matrix.

The dual polynomial base by Equation (237) is hence

$$f^{*}(3\rangle = \underset{3xd}{M_{ff}^{-1}}\, f\rangle \tag{350}$$

or transposing

$$\langle f^{*} = (1,t,t^{2}) \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix} . \tag{351}$$

THE INVERSE OF THE TRIANGULAR (FACTOR) MATRIX. The inverse of the upper triangular matrix B_s or B_g allows us to map back and forth between the bases. By Equation (339)

$$\langle \bar{s} = \langle \bar{f}\, B_s \tag{352}$$

and

$$\bar{s}\rangle = B_s^{T}\, \bar{f}\rangle \tag{353}$$

also

$$\langle \bar{s}\, B_s^{-1} = \langle \bar{f} \tag{354}$$

Since $\langle \bar{s}$ is orthonormal

$$\bar{s}\rangle \textcircled{i} \langle \bar{s} = I \tag{355}$$

operating on Equation (354)

$$B_s^{-1} = \bar{s}\rangle \textcircled{i} \langle \bar{f} \tag{356}$$

and using Equation (353) in Equation (356)

$$B_s^{-1} = B_s^{T}\ \tilde{f}\rangle \ \textcircled{i} \ \langle \bar{f} \tag{357}$$

or

$$\boxed{B_s^{-1} = B_s^{T}\ M_{ff}} \quad . \tag{358}$$

For the 3x3 case of Equation (337) and Equation (313)

$$B_s^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\sqrt{3} & 2\sqrt{3} & 0 \\ \sqrt{5} & -6\sqrt{5} & 6\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \tag{359}$$

or

$$B_s^{-1} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & \sqrt{3}/6 & \sqrt{3}/6 \\ 0 & 0 & \sqrt{5}/30 \end{bmatrix} \tag{360}$$

or

$$(1,\ t,\ t^2) = \langle s\ B_s^{-1} \tag{361}$$

Suppose we now have the coordinates of a vector in the $\langle f$ base, say $a(3\rangle$ is known, where

$$x(t) = (1,\ t,\ t^2)\ a(3\rangle$$

and we want the coordinates in the Schmidt base, then

$$x(t) = \langle s\ a^s\rangle = \langle f\ \ a\rangle \tag{362}$$

or

$$s\rangle \ \textcircled{i}\ x(t) = a^s\rangle = s\rangle \ \textcircled{i} \ \langle f\ \ a\rangle \tag{363}$$

or by Equation (356) in Equation (363)

$$a^{s}\rangle = B_{s}^{-1}\, a\rangle \quad . \tag{364}$$

CONTINUOUS MODIFIED LEGENDRE POLYNOMIALS [-1,1]. The definition of the inner-product as before but on the interval [-1,1]

$$-1 \leq t \leq 1$$

and the ensuing orthogonalization process generates a set of polynomials bearing the name Modified Legendre Polynomials. The matrix for the 3x3 case is

$$f\rangle \textcircled{i} \langle f = \int_{-1}^{1} \begin{pmatrix} 1 \\ t \\ t^{2} \end{pmatrix} (1,\ t,\ t^{2})dt \tag{365}$$

or

$$\underset{3x3}{M_{ff}} = \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix} \tag{366}$$

If one applys the equations for the general Gram-Schmidt procedure of Equation (308) for the metric of Equation (366).

By Equation (308)

$$g_{1} = f_{1} = (1,\ t,\ t^{2}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \langle f\ e\rangle_{1} \tag{367}$$

and normalizing

$$g_{1} \textcircled{i}\, g_{1} = {}_{1}\langle e \int_{-1}^{1} f\rangle\langle f\ dt\ e\rangle_{1} \tag{368}$$

$$= (1,0,0) \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2 \qquad (369)$$

and

$$s_1 = \langle f \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} \qquad (370)$$

The second vector is by Equation (308)

$$g_2 = f_2 - s_1 \, s_1 \textcircled{i} \, f_2 \qquad (371)$$

and

$$s_1 \textcircled{i} \, f_2 = (1/\sqrt{2}, 0, 0) \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \qquad (372)$$

hence

$$g_2 = f_2 = \langle f \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad (373)$$

and

$$s_2 = \frac{g_2}{||g_2||}$$

where

$$||g_2|| = \left({}_2\langle e \, M_{ff} \, e \rangle_2 \right)^{\frac{1}{2}}$$

$$= (0,1,0) \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^{\frac{1}{2}} = \left(\frac{2}{3} \right)^{\frac{1}{2}} \qquad (374)$$

or

$$s_2 = \langle f \begin{pmatrix} 0 \\ \sqrt{3}/\sqrt{2} \\ 0 \end{pmatrix} . \tag{375}$$

By Equation (308), the third vector is

$$g_3 = f_3 - s_1 s_1 \textcircled{i} f_3 - s_2 s_2 \textcircled{i} f_3 \tag{376}$$

where

$$s_1 \textcircled{i} f_3 = (1/\sqrt{2},\ 0,\ 0) \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{377}$$

$$= \frac{\sqrt{2}}{3}$$

and

$$s_2 \textcircled{i} f_3 = (0,\ \frac{\sqrt{3}}{\sqrt{2}},\ 0) \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \tag{378}$$

Using Equation (377) and Equation (378) in Equation (376)

$$g_3 = \langle f \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix} \frac{\sqrt{2}}{3} \right\} = \langle f \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} . \tag{379}$$

The norm of g_3 is

$$||g_3|| = \left\{ (-1/3,\ 0,\ 1) \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix} \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} \right\}^{\frac{1}{2}} \tag{380}$$

or

$$||g_3|| = \frac{2\sqrt{2}}{3\sqrt{5}} = \frac{2\sqrt{10}}{15} \tag{381}$$

and

$$s_3 = \langle f \begin{pmatrix} -1/2 \\ 0 \\ 3/2 \end{pmatrix} \frac{\sqrt{5}}{\sqrt{2}} \tag{382}$$

Packaging the $\langle g$ base

$$\langle \bar{g} = (1,t,t^2) \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \langle t\ B_g \tag{383}$$

and the orthonormial base is

$$\langle \bar{s} = (1,t,t^2) \begin{bmatrix} 1/\sqrt{2} & 0 & -\sqrt{5}/2\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} & 0 \\ 0 & 0 & 3\sqrt{5}/2\sqrt{2} \end{bmatrix} \tag{384}$$

The inverse of the metric M_{ff} is by Equation (348)

$$M_{ff}^{-1} = \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix}^{-1} = \begin{bmatrix} 9/4 & 0 & -15/4 \\ 0 & 3 & 0 \\ -15/4 & 0 & 3^2(5)/2^2 \end{bmatrix} \frac{1}{2} \quad . \tag{385}$$

and the metric-matrix for the $< g$ base is (386)

$$M_{gg} = g > \textcircled{i} < g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$M_{gg} = D_g = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 8/45 \end{bmatrix} \tag{387}$$

and its inverse

$$D_g^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 45/8 \end{bmatrix} \tag{388}$$

and

$$D_g^{-\frac{1}{2}} = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & \sqrt{3}/\sqrt{2} & 0 \\ 0 & 0 & 2\sqrt{2}/3\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & \sqrt{3}/\sqrt{2} & 0 \\ 0 & 0 & 2\sqrt{10}/15 \end{bmatrix}$$

We will now obtain the inverse of the base connection matrix B_g of Equation (383). Transposing Equation (383)

$$g\rangle = B_g^{T} f\rangle \tag{389}$$

and by Equation (384)

$$\langle f = \langle g \, B_g^{-1} \tag{390}$$

operate on Equation (390) with $g\rangle \textcircled{i}$

$$g\rangle \textcircled{i} \langle f = g\rangle \textcircled{i} \langle g \, B_g^{-1} \tag{391}$$

$$M_{ff} = D_g B_g^{-1} \tag{392}$$

Use Equation (389) in Equation (391)

$$B_g^{T} f\rangle \text{ (i) } \langle f = D_g B_g^{-1} \tag{393}$$

$$B_g^{T} M_{ff} = D_g B_g^{-1} \tag{394}$$

or

$$\boxed{B_g^{-1} = D_g^{-1} B_g^{T} M_{ff}} \tag{395}$$

and for the 3x3 case under consideration.

$$B_g^{-1} = \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{396}$$

GRAM-SCHMIDT PROCEEDURES IN DIFFERENT BASES. If the sequence of vectors of Equation (217) are known in a different background base that is for each f_k

$$\bar{f}_k = \langle \bar{b}\; f \rangle_k$$

then for the set we have

$$\langle d)\,\bar{f} = \langle d)\,\bar{b} \left[\; f\rangle_1 \; f\rangle_2 \; \cdots \; f\rangle_d \; \right] = \langle \bar{b}\; F \qquad (397)$$

where F is the dxd matrix of coordinates of the vectors in the $\langle \bar{b}$ base, whose metric matrix is

$$\bar{b}\rangle \; \textcircled{i} \; \langle \bar{b} = M_{bb} \quad . \qquad (398)$$

Thus by Equation (397) and Equation (354)

$$\langle \bar{f} = \langle \bar{b}\; F = \langle \bar{s}\; B_s^{-1} \qquad (399)$$

and

$$\langle \bar{s} = \langle \bar{f}\; B_s = \langle \bar{b}\; F\; B_s \qquad (400)$$

If we define the coordinates of the orthonormal Gram-Schmidt vectors in $\langle \bar{b}$ base as

$$S_b = F\; B_s \qquad (401)$$

then Equation (400) becomes

$$\langle \bar{s} = \langle \bar{f}\; B_s = \langle \bar{b}\; S_b \qquad (402)$$

Transposing

$$\bar{s}\rangle = B_s^{T}\, \bar{f}\rangle = S_b^{T}\, \bar{b}\rangle \tag{403}$$

and

$$\bar{s}\rangle \textcircled{i} \langle \bar{s} = \underset{dxd}{I} = B_s^{T}\, \bar{f}\rangle \textcircled{i} \langle \bar{f}\, B_s \tag{404}$$

$$= S_b^{T}\, \bar{b}\rangle \textcircled{i} \langle \bar{b}\, S_b \tag{405}$$

or

$$I = B_s^{T}\, M_{ff}\, B_s = S_b^{T}\, M_{bb}\, S_b \tag{406}$$

If the background base $\langle \bar{b}$ is also orthonormal, then clearly

$$I = S_b^{T} S_b \tag{407}$$

and

$$S_b^{T} = S_b^{-1} \tag{408}$$

and the matrix (of coordinates) S_b is said to be orthonormal. By Equation (401)

$$S_b B_s^{-1} = \underset{dxd}{F}\ , \tag{409}$$

Since B_s^{-1} is upper triangular, let

$$U = B_s^{-1} \tag{410}$$

hence

$$F = S_b U \tag{411}$$

and we see the "familiar decomposition of F into an orthonormal matrix and a triangular matrix. Transposing Equation (411)

$$F^T = U^T S_b{}^T \tag{412}$$

and

$$F^T F = U^T S_b{}^T S_b U \tag{413}$$

and if Equation (407) holds

$$F^T F = U^T U \tag{414}$$

Transposing Equation (393)

$$S_b{}^T = B_s{}^T F^T \tag{415}$$

and

$$S_b{}^T S_b = B_s{}^T F^T F\ B_s \tag{416}$$

By Equation (398)

$$S_b{}^{-T} S_b{}^{-1} = M_{bb} = (S_b S_b{}^T)^{-1} \tag{417}$$

and

$$B_s{}^{-T} B_s{}^{-1} = M_{ff} = (B_s B_s{}^T)^{-1} \tag{418}$$

or

$$S_b S_b{}^T = M_{bb}{}^{-1} \tag{419}$$

and

$$B_s B_s{}^T = M_{ff}{}^{-1} \quad . \tag{420}$$

The orthogonization proceedures for the polynomials (t, t, t^2) were preformed with respect to the $\langle \bar{f}$ sequence.

WEIGHTED GRAM-SCHMIDT. One can also perform the task with respect to the background base $\langle \bar{b}$. Let

$$\bar{g}_1 = \bar{f}_1 = \left\langle \bar{b}\ f \right\rangle_1 \tag{421}$$

and

$$\bar{s}_1 = \frac{\bar{g}_1}{||\bar{g}_1||} \tag{422}$$

where

$$||\bar{g}_1|| = \left({}_1\!\left\langle f\ M_{bb}\ f \right\rangle_1 \right)^{\frac{1}{2}} \tag{423}$$

and

$$\bar{s}_1 = \left\langle \bar{b}\ f \right\rangle_1 \frac{1}{\left({}_1\!\left\langle f\ M_{bb}\ f \right\rangle_1 \right)^{\frac{1}{2}}} \tag{424}$$

By Equation (308)

$$\bar{g}_2 = \bar{f}_2 - \bar{s}_1 \bar{s}_1 \textcircled{i} \bar{f}_2 \tag{425}$$

where

$$\bar{s}_1 \textcircled{i} \bar{f}_2 = {}_{1b}\!\left\langle s\ M_{bb}\ f \right\rangle_2 \tag{426}$$

$$= \frac{{}_1\!\left\langle f\ M_{bb}\ f \right\rangle_2}{{}_1\!\left\langle f\ M_{bb}\ f \right\rangle_2}$$

Using Equation (426) in Equation (425)

$$\bar{g}_2 = \Big\langle \bar{b} \left[\frac{f\Big\rangle_2 - f\Big\rangle_1 {}^{1}\Big\langle f M_{bb} f\Big\rangle_2}{{}_1\Big\langle f M_{bb} f\Big\rangle_1} \right] \tag{427}$$

In general the terms with respect to the metric of the inner-product are straight forward when applied to Equation (308).

DETERMINATION OF THE GRAM SCHMIDT FACTORS OF A MATRIX. By Equation (411) we see that the Gram-Schmidt orthogonalization can be regarded as being another way for decomposing a matrix with orthogonal columns and a triangular matrix or

$$\underset{dxd}{F} = \underset{(dxd)}{S} \underset{(dxd)}{U} \tag{428}$$

One must simply be careful of how inner-product is defined. Bierman in reference (85) introduces the notion of weighted-Gram-Schmidt factorization of the matrices, the weights are with respect to the metric of an inner-product.

MODIFIED GRAM-SCHMIDT. Rice in his paper "Experiments on Gram-Schmidt Orthogonalization" states that surprisingly the Gram-Schmidt and modified Gram-Schmidt show distinct differences in computational behavior. This is particularly remarkable since both methods perform basically the same operations, only in a different sequence. Indeed, ignoring computational errors, they produce the same set $\{\bar{s}_i\}$ with the same number of operations. He states that the modified-Gram-Schmidt method is more natural for machine computations since it economizes storage.

Consider the sequence Equation (216) and set the modified vector

$$\bar{m}_1 = \bar{f}_1 = \bar{g}_1 \tag{429}$$

and the unit magnitude modified vector

$$\bar{\mu}_1 = \frac{\bar{m}_1}{||\bar{m}_1||} = \bar{s}_1 \tag{430}$$

Designate the initial $\bar{m}$ sequence as

$$<\bar{m}^{(0)} = <\bar{f} = (\bar{f}_1, \bar{f}_2, \bar{f}_3 \cdots \bar{f}_d) \qquad (431)$$

Project all vectors orthogonally onto $\bar{f}_1$ (except $\bar{f}_1$) select the components perpendicular to $\bar{f}_1$, that is

$$\hat{f}_k = \bar{f}_1 \bar{f}_1^{*}(1) \textcircled{i} \bar{f}_k = \bar{s}_1 \bar{s}_1 \textcircled{i} \bar{f}_k \qquad (432)$$

for

$k = 2, 3, \cdots d$

$$\bar{f}_k = \bar{\mu}_1 \bar{\mu}_1 \textcircled{i} \bar{f}_k = \bar{\bar{P}}(1,1) \textcircled{i} \bar{f}_k \qquad (433)$$

and the complement component is

$$\tilde{f}_k(1) = (\bar{\bar{I}} - \bar{\bar{P}}(1,1)) \textcircled{i} \bar{f}_k \qquad (434)$$

Form the new base

$$^{1}<\bar{m} = (\bar{f}_1, \tilde{f}_2(1), \tilde{f}_3(1), \cdots \tilde{f}_k(1) \cdots \tilde{f}_d(1)) \qquad (435)$$

If we normalize the first two vectors we have

$$\bar{\mu}_1 = \bar{s}_1 \qquad (436)$$

and

$$\bar{\mu}_2 = \frac{\bar{m}_2^{(1)}}{||\bar{m}_2^{(1)}||} = \frac{\tilde{f}_2(1)}{||\tilde{f}_2(1)||} = \bar{s}_2 \qquad (437)$$

Now orthogonally project all vectors $\tilde{f}_k(1)$ for $k = 3,4, \cdots d$ onto $\tilde{f}_2(1)$ and select the components perpendicular to $\tilde{f}_2(1)$ that is

$$\hat{f}_k(1,2) = (\tilde{f}_2(1)\ \tilde{f}_2^{*}(1))\ \textcircled{i}\ \tilde{f}_k(1) \tag{438}$$

$$= (\bar{s}_2\bar{s}_2)\ \textcircled{i}\ \tilde{f}_k(1)$$

and the perpendicular component is

$$\tilde{f}_k(1,2) = \left[\bar{\bar{I}} - \frac{\tilde{f}_2(1)\ \tilde{f}_2(1)}{||\tilde{f}_2(1)||}\right] (i)\ \tilde{f}_k(1) \tag{439}$$

or

$$\bar{m}_k(1,2) = \tilde{f}_k(1,2) = (\bar{\bar{I}} - \bar{s}_2\bar{s}_2)\ (i)\ \tilde{f}_k(1) \tag{440}$$

using Equation (434) in Equation (440)

$$\tilde{f}_k(1,2) = (\bar{\bar{I}} - \bar{s}_2\bar{s}_2)\ \textcircled{i}\ (\bar{\bar{I}} - \bar{s}_1\bar{s}_1)\ \textcircled{i}\ \bar{f}_k \tag{441}$$

$$= \overset{\approx}{P}(2,2)\ \textcircled{i}\ \overset{\approx}{P}(1,1)\ \textcircled{i}\ \bar{f}_k \tag{442}$$

Note that the product of the two projectors

$$(\bar{\bar{I}} - \bar{s}_2\bar{s}_2)\ \textcircled{i}\ (\bar{\bar{I}}-\bar{s}_1\bar{s}_1)$$

$$= \bar{\bar{I}} - \bar{s}_1\bar{s}_1 - \bar{s}_2\bar{s}_2 = \overset{\approx}{P}(1,2) \tag{443}$$

where

$$\bar{s}_2\bar{s}_2 \textcircled{i} \bar{s}_1\bar{s}_1 = \bar{\bar{0}} \tag{444}$$

and the projector $\overset{\approx}{P}(1,2)$ of Equation (443) is the same projector as in Equation (281) and the third vector $\bar{m}_3$ is

$$\tilde{f}_3(1,2) = \bar{m}_3(1,2) = \bar{g}_3 = \overset{\approx}{P}(1,2) \textcircled{i} \bar{f}_3 \quad , \tag{445}$$

also the nomalized vector is

$$\bar{s}_3 = \bar{\mu}_3(1,2) \tag{446}$$

Form the set of base vectors

$${}^{2}\!\!\left\langle d\right)\bar{m} = (\bar{f}_1,\ \tilde{f}_2(1),\ \tilde{f}_3(1,2),\ \cdots\ \tilde{f}_k(1,2)\ \cdots\ \tilde{f}_d(1,2)) \tag{447}$$

Proceeding in the same manner project all vectors

$$\bar{m}_k(1,2) = \tilde{f}_k(1,2) \quad \text{for} \quad K = 4,5,6,\ \cdots\ d$$

onto $\tilde{f}_3(1,2)$ and select perpendicular components, the final $\bar{m}$ set of base vectors will be

$${}^{(d-1)}\!\!\left\langle \bar{m}\right. = (\bar{f}_1, \tilde{f}_2(1),\ \tilde{f}_3(1,2),\ \tilde{f}_4(1,2,3),\ \cdots\ \tilde{f}_d(1,2,\ \cdots\ d-1)) \tag{448}$$

and the orthonormal set will be

$$\left\langle d\right)\bar{\mu} = \left\langle d\right)\bar{s} \quad , \tag{449}$$

which is exactly the set for the conventional Gram-Schmidt proceedure. Note that the computational proceedures were different for this modified proceedure than for the conventional proceedures.

The MODIFIED GRAM-SCHMIDT recursive algorithms can be written as

$$\bar{\mu}_1 = \frac{\bar{f}_1}{||\bar{f}_1||}$$

$$\bar{m}_j(1) = \bar{f}_j - \bar{\mu}_1\bar{\mu}_1 \textcircled{i} \bar{f}_j \tag{450}$$

for

$$j = 2, 3, \cdots d$$

$$\bar{\mu}_k = \frac{\bar{m}_k(k-1)}{||\bar{m}_k(k-1)||} \qquad k = 2, \cdots d$$

$$\bar{m}_j(k) = \bar{m}_j(k-1) - \bar{\mu}_k\bar{\mu}_k \textcircled{i} \bar{m}_j(k-1)$$

for

$$j = k+1, \cdots d$$

where the abbreviated notation is

$$\bar{m}_j(k-1) \equiv \bar{m}_j\ (1,2,\cdots k-1)$$

TRIANGULAR DECOMPOSITIONS AND MATRIX INVERSION VIA CHOLESKY PROCEEDURE. There are many many chapters in numerical analysis texts and publications on triangular resolutions of arbitrary matrices. The method of Cholesky are generally reserved for the case of a symmetric matrix according to Fox. (30)

Let A be a real dxd matrix and suppose we wish to solve numerically the linear systems of equations

$$A\ x\Big\rangle = z\Big\rangle \tag{451}$$

The task is straight forward if we can factor A as

$$A = LU \tag{452}$$

where $L = (\ell_{ij})$ is lower triangular ($\ell_{ij}=0$ for $i < j$) and $U = (u_{ij})$ is upper triangular $u_{ij} = 0$ for $i > j$) for Equation (452) in Equation (451)

$$z\rangle = U\, x\rangle \tag{453}$$

$$z\rangle = Ly\rangle \tag{454}$$

where

$$y\rangle = U\, x\rangle \tag{455}$$

Although one must solve two systems

$$x\rangle = U^{-1}y\rangle \tag{456}$$

and

$$y\rangle = L^{-1}z\rangle \tag{457}$$

or

$$x\rangle = U^{-1}L^{-1}\, z\rangle \tag{458}$$

each of the two inverse factors has a very simple form. Thus the solution of Equation (451) via the triangular resolution requires the inverse of triangular matrices.

Theorem 1 is taken from Wendroff page 126 and stated without proof. (Previously stated on page (39)).

<u>THEOREM 1</u>. If A is symmetric and positive definite there exist an upper triangular matrix U such that

$$A = U^T U. \tag{459}$$

COROLLARY 1. If $\max|a_{ij}| \leq 1$, then $\max|u_{ij}| \leq 1$.

He states that for an arbitrary nonsingular matrix A the triangular decomposition may not exist. However, it is always possible to find a permutation matrix P so that

$$PA = LU \,, \qquad (460)$$

the permutation matrix P is one in which all elements are either 0 or 1, and every row and column contain exactly one element equal to 1. A simple proof of Equation (460) is given by Rose reference (77).

The Cholesky method is discussed in many of the numerical methods in papers and books and will not be developed in any detail here. An example of the applications will be presented for the discrete approximation case of fitting functions.

For a symmetric matrix A (3x3)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \qquad (461)$$

where

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} \frac{a_{11}}{u_{11}} & \frac{a_{12}}{u_{11}} & \frac{a_{13}}{u_{11}} \\ 0 & \frac{a_{22}-(u_{12})^2}{u_{22}} & \frac{a_{23}-u_{12}u_{13}}{u_{22}} \\ 0 & 0 & \frac{a_{33}-(u_{23})^2-(u_{13})^2}{u_{33}} \end{bmatrix} \qquad (462)$$

The general formula is

$$u_m \, u_{mn} = a_{mn} - \sum_{i=1}^{m-1} u_{im}u_{in} \qquad . \tag{463}$$

These relations are easily derived via partitioning. Partion U into its column space

$$\underset{dxd}{U} = \left[u\rangle_1 , \; u\rangle_2 , \; u\rangle_3 \; \cdots \; u\rangle_d \right]$$

and

$$U^T = \begin{bmatrix} {}^1\langle u \\ {}^2\langle u \\ \vdots \\ {}^d\langle u \end{bmatrix}$$

then

$$a_{ij} = {}_1\langle u \; u\rangle_j = \sum_{k=1}^{j} u_{ik} \, u_{kj}$$

giving

$$u_{ij} = (a_{ij} - \sum_{k=1}^{i-1} u_{ik}u_{jk}) \, 1/u_{jj} \tag{464}$$

for

$$i = 1, 2, \cdots j-1 \quad .$$

and

$$a_{ii} = \sum_{k=1}^{i} u_{ik}u_{ik}$$

giving

$$u_{ii} = a_{ii} - \sum_{k=1}^{i-1} u_{ik}^{2})^{\frac{1}{2}} \tag{465}$$

when one applies the algorithm of Equation (462) and Equation (463) to the (3x3) Hilbert matrix one obtains the factors of Equation (360) for the Gram polynomials on (0,1) interval, namely

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & \sqrt{3}/6 & 0 \\ 1/3 & \sqrt{3}/6 & \sqrt{5}/30 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & \sqrt{3}/6 & \sqrt{3}/6 \\ 0 & 0 & \sqrt{5}/30 \end{bmatrix} \tag{466}$$

If we apply the algorithms to the symmetric circulant matrix as an interesting example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} = U^T U \tag{467}$$

where

$$U^T U = \begin{bmatrix} 1 & 0 & 0 \\ 2 & i & 0 \\ 3 & 5_i & 3\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & i & 5_i \\ 0 & 0 & 3\sqrt{2} \end{bmatrix} . \tag{468}$$

where

$$i = \sqrt{-1}$$

thus we see that the factors of a real symmetric matrix can be imaginary.

GRAM-SCHMIDT OF THE "DISCREET METRIC MATRIX" FOR THE FITTING FUNCTIONS OF A DISCREET SPAN OF POINTS. For considerations of polynomials orthogonal over discreet point sets, the Gram-Schmidt proceedures developed in this section will be used later for Legendre, Laguerre and other polynomials. Consider the "discreet metric matrix" (also called the normal matrix in least squares discussions) of Equation (18)

$$\underset{dxk}{F^T}\,\underset{kxd}{F} = \underset{d(k)d}{M_{ff}} \qquad (469)$$

where the matrix size d(k)d indicates full rank factors with one side in k-space where

$$k \geq d.$$

The discreet vector-matrix equation by Equation (15) is

$$x(k\rangle = \underset{kxd}{F}\; a(d\rangle \qquad (470)$$

One can solve Equation (470) for the vector $a(d\rangle$ by computing the generalized inverse

$$a(d\rangle = \underset{dxk}{F^{*}}\; x(k\rangle \qquad (471)$$

where

$$\underset{dxk}{F^{*}} = \underset{dxd}{(F^TF)^{-1}}\;\underset{dxk}{F^T}$$

one can also obtain the generalized inverse as

$$\underset{dxk}{F^{*}} = \underset{dxk}{F^T}\;\underset{kxk}{(F\,F^T)^{*}} \qquad (473)$$

The Gram-Schmidt decomposition orthonormal of Equation (409) was for a full rank dxd matrix. If we partition the k row vectors of dimension d into its column space we have

$$F = \begin{bmatrix} \langle d)\, f(0) \\ \langle d)\, f(1) \\ \vdots \\ \langle d)\, f(k) \end{bmatrix} = \begin{bmatrix} f(k\rangle_1 , & f(k\rangle_2 & \cdots & f(k\rangle_d \end{bmatrix} \qquad (474)$$

or d linearly indepdent column vectors in the larger k space. If we now apply the Gram-Schmidt proceedures of Equation (308) to the column n-tuple vectors of Equation (474) we obtain

$$g(k\rangle_1 = f(k\rangle_1 \qquad (475)$$

$$s(k\rangle_1 = \frac{f(k\rangle_1}{{}^{1}\langle k)\, f\;\; f(k\rangle_1^{\frac{1}{2}}} \qquad (476)$$

$$g(k\rangle_2 = f(k\rangle_2 - \frac{f(k\rangle_1 \;\; {}^{1}\langle k)\, f\;\; f(k\rangle_2}{{}_{1}\langle k)\, f\;\; f(k\rangle_1} \qquad (477)$$

and the unit magnitude vector

$$s(k\rangle_2 = \frac{g(k\rangle_2}{\left({}_{2}\langle k)\, g\;\; g(k\rangle_2 \right)^{\frac{1}{2}}} \qquad (478)$$

etc.

The packaged results are

$$\underset{k\times d}{G} = \left[g(k\rangle_1, \; g(k\rangle_2 \; , \cdots \; g(k\rangle_d \right]$$

$$= \left[f(k\rangle_1 \cdots f(k\rangle_d \right] \begin{bmatrix} b^{1}_{\cdot 1} & b^{1}_{\cdot 2} \cdots & b^{1}_{\cdot d} \\ 0 & b^{2}_{\cdot 2} & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \cdot & b^{d}_{\cdot d} \end{bmatrix}_g \tag{479}$$

or

$$\underset{k\times d}{G} = \underset{k\times d}{F} \; \underset{d\times d}{B_g} \tag{480}$$

or

$$\underset{k\times d}{G} \; B_g^{-1} = \underset{k\times d}{F} \tag{481}$$

and for the orthonormal vectors

$$S \; B_s^{-1} = F \tag{482}$$

Transposing Equation (481) and Equation (482)

$$\underset{d\times k}{F^T} = B_g^{-T} G^T = \underset{d\times d}{G_s^{-T}} \; \underset{d\times k}{S^T} \tag{483}$$

Forming the full rank "inner-Grammian" we have

$$\underset{d(k)d}{F^TF} = B_g^{-T} D_g B_g^{-1} = B_s^{-T} B_s^{-1} \tag{484}$$

since

$$\underset{d(k)d}{S^TS} = \underset{dxd}{I} \tag{485}$$

and

$$\underset{dxd}{G^TG} = \underset{dxd}{D_g} \quad . \tag{486}$$

Rewritting Equation (484)

$$\underset{dxd}{F^TF} = (B_g D_g^{-1} B_g^T)^{-1} = (B_s B_s^T)^{-1} \tag{487}$$

By Equation (484) let the upper triangular matrix B_s^{-1}

$$U = B_s^{-1} \tag{488}$$

and

$$U^T = B_s^{-T} \tag{489}$$

and like-wise for the unit triangular matrix B_g^{-1}

$$B_g^{-1} = U_1 \tag{490}$$

Hence

$$\underset{d(k)d}{F^TF} = U_1^T D_g U_1 = U^T U \tag{491}$$

or upper-triangular factors in d space. Inverting Equation (487)

$$(F^TF)^{-1} = B_sB_s^T = B_gD_g^{-1}B_g^T \tag{492}$$

$$= U^{-1}U^{-T} = U_1D_gU_1^T \tag{493}$$

Hence by resolving Equation (491) into its triangular factors by the Cholesky algorithm of Equation (463)

$$F^TF = (g_{ij}) = U^TU$$

of that equation, one can solve Equation (471) for the unknown vector $a(d\rangle$.

The inverse of U^T for a 3x3 matrix is given by Bjerhammer on page 328 as

$$(U^T)^{-1} = \begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{u_{11}} & 0 & 0 \\ \frac{-u_{12}}{u_{11}u_{22}} & \frac{1}{u_{22}} & 0 \\ \frac{\frac{-1}{u_{11}} \; u_{13} \; \frac{u_{12}U_{23}}{u11u22}}{u_{33}} & \frac{-u_{23}}{u_{22}u_{33}} & \frac{1}{u_{33}} \end{bmatrix} \tag{494}$$

If we consider the solution of Equation (470) using Equation (472)

$$a(d\rangle = (F^TF)^{-1}F^T \; x(k\rangle \tag{495}$$

and the triangular factors of Equation (493)

$$a(d\rangle = (U^{-1}U^{-T}) \; F^T \; x(k\rangle \tag{496}$$

as the first method, then two other methods of computation will be pointed out.

SECOND METHOD. This method involves obtaining the Gram-Schmidt factors of F in k-space via Equation (482) and Equation (470) becomes

$$x(k\rangle = \underset{kxd}{S}\;\underset{dxd}{B_s^{-1}}\; a(d\rangle = F\; a(d\rangle \tag{497}$$

$$x(k\rangle = S\; a^b\; (d\rangle \tag{498}$$

where

$$a^b(d\rangle = B_s^{-1}\; a(d\rangle \tag{499}$$

or

$$\underset{dxk}{S^T}\; x(k\rangle = a^b(d\rangle = B_s^{-1}\; a(d\rangle \tag{500}$$

Multiply Equation (500) by B_s and

$$a(d\rangle = B_s S^T x(k\rangle \tag{501}$$

The solution of Equation (501) involves the computation of a triangular and an orthonormal matrix factor.

THIRD METHOD. One can also perform a Gram-Schmidt on the symmetric matrix

$$M = (F^T F) = S_m B_{sm}^{-1} \tag{502}$$

and

$$M^{-1} = B_{sm} S_m^T \tag{503}$$

or by Equation (495)

$$a(d\rangle = B_{sm} S_m^T\; F^T x(k\rangle \tag{504}$$

There are many papers on variations of Cholesky, square, root, Gram-Schmidt, etc., with numbers of computations counts, numerical stability etc. listed in the bibliography.

ALTERNATIVE, ORTHOGONAL PROCESS TO THE GRAM-SCHMIDT. There are many variations to the triangular-factorization for positive definite matrices for example see Schmidt (reference 79). Bierman (reference 85) and, Kailath (reference 42). Staib in his paper presents an alternative which he says if it is not original that it is not widely known. He reduces a positive definite matrix to triangular form by row operations and states that row multiplication can be freely used to avoid fractions which he says means that he can devise computer programs that will not suffer from round-off error. Row operations will not be used in this discussion but since his example centered around the 4x4 Legendre metric the 3x3 case of Equation (366) will be discussed

$$M_{ff} = \int_{-1}^{1} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} (1,\ t,\ t^2)dt) = \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix} \qquad (505)$$

If one has either the orthonormal base $\langle\bar{s}$ or the $\langle\bar{g}$ base which has a unit triangular matrix, a new orthogonal base can be obtained by an arbitrary diagonal matrix scaling or

$$\langle\bar{g}_a = \langle\bar{g}\ D_a \qquad (506)$$

and

$$\langle\bar{g} = \langle\bar{f}\ B_g$$

or

$$\langle g_a = \langle\bar{f}\ D_a B_g = \langle\bar{f}\ B_a \qquad (507)$$

If one replaces $g\rangle$ by $g\rangle_a$ and B_g by B_a for any alternate orthogonal base, the inverse is given by Equation (403)

$$B_a^{-1} = D_a^{-1} B_a^{T} M_{ff} \qquad (508)$$

and the inverse of the M_{ff} matrix is given by Equation (348)

$$M_{ff}^{-1} = B_a D_a^{-1} B_a^T \tag{509}$$

The transformation which Staib uses for the example is a triangular matrix of integers obtained from M_{ff} by row operations and is a matrix of integers

$$\langle \bar{g}_a = \langle \bar{f} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \langle \bar{f}\, B_a \tag{510}$$

The diagonal metric matrix is

$$D_g = B_a^T M_{ff} B_a = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 8/5 & 0 \\ 0 & 0 & 0 & 8/7 \end{bmatrix} \tag{511}$$

and the inverse is

$$D_g^{-1} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 \\ 0 & 0 & 5/8 & 0 \\ 0 & 0 & 0 & 7/8 \end{bmatrix} \tag{512}$$

The inverse by Equation (508) is

$$B_a^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} \end{bmatrix} \tag{513}$$

Alternative Gram-Schmidt (First Coordinate Unity)

We see by Eq (303) and Eq (308) that the orthogonalization procedure described by Eq (308) is quite standard in the linear vector space text books and generates a unit upper-triangular matrix. The coordinate of the higest subscript designated the base vector is taken to be unity, that is for the kth vector

$$\overline{g}_k = (\overline{f}_1, \overline{f}_2, \ldots \overline{f}_k) \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_k = 1 \end{bmatrix} \tag{514}$$

When the base vectors are the monomial base t we see that the polynomials are <u>monic</u> that is (slipping the index)

$$g_k = \lambda_o + \lambda_1 t + \lambda_2 t^2 + \ldots + 1t^k \tag{515}$$

the classical orthogonal polynomials reals correspond to the alternate Gram-Schmidt procedure described below. The 3x3 case will be applied to (o.1) interval to obtain the Legendre polynomials which agree with those presented by Milne ref. (60).

Consider now the sequence (with the index shifted to zero to conform with the polynomial degree) as

$$\langle d)f = (f_o, f_1, \ldots \overline{f}_{d-1}) \tag{516}$$

set

$$\overline{g}_o = \overline{f}_o \tag{517}$$

and by Fig (7) for the second vector

Figure (7) Two Dimensional Case

$$\overline{f}_o = \mu_1 \overline{f}_1 + \overline{g}_1 \tag{518}$$

or

$$\overline{g}_1 = (\overline{f}_o,\overline{f}_i)\begin{bmatrix} 1 & \\ -\mu_1 & 1 \end{bmatrix} = \langle \overline{f}\ \lambda(2\rangle_1 \tag{519}$$

Impose the constraint

$$\overline{f}_o \hat{\odot} \overline{g}_1 = 0 = \overline{f}_o \hat{\odot} (f_o,\overline{f}_1)\ \lambda(2\rangle_1 \tag{520}$$

or

$$\mu_1 = \frac{\overline{f}_o \odot \overline{f}_o}{\overline{f}_o \hat{\odot} f_1} \tag{521}$$

also for the third vector

$$\overline{g}_2 = (\overline{f}_o,\overline{f}_1,\overline{f}_2)\begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}_2 \tag{522}$$

$$\begin{bmatrix} f_o \\ \overline{f}_1 \end{bmatrix} \hat{\odot} \overline{g}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \overline{f}_o \\ f_1 \end{bmatrix} \hat{\odot} (\overline{f}_o,\overline{f}_1,\overline{f}_2)\ \lambda(3\rangle_2 \tag{523}$$

Define

$$\begin{bmatrix} \overline{f}_o \\ \overline{f}_1 \end{bmatrix} \hat{\odot} (\overline{f}_o,\overline{f}_1,\overline{f}_2) = \begin{bmatrix} f_{oo} & f_{o1} & f_{o2} \\ f_{1o} & f_{11} & f_{12} \end{bmatrix} \tag{524}$$

$$= [f(2\rangle_o, M_{12}] \tag{525}$$

Hence

$$- \begin{bmatrix} f_{oo} \\ f_{o1} \end{bmatrix} = M_{12}\, \mu(2\rangle_2 \tag{526}$$

or

$$\mu(2\rangle_2 = -M_{12}^{-1} \begin{bmatrix} f_{oo} \\ f_{o1} \end{bmatrix} \tag{527}$$

and for the kth vector

$$\bar{g}_{k-1} = (\bar{f}_o, \bar{f}_1, \ldots \bar{f}_{k-1}) \begin{bmatrix} \lambda_o = 1 \\ \lambda_2 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_{k-1} \end{bmatrix} \tag{528}$$

$$\begin{bmatrix} \bar{f}_o \\ \bar{f}_1 \\ \cdot \\ \cdot \\ \cdot \\ \bar{f}_{k-2} \end{bmatrix} @\bar{g}_{k-1} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = \begin{bmatrix} f_{oo} & f_{o1} & \cdots & f_{o,k-1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ f_{k-2,o} & \cdot & \cdot\ \cdot & f_{k-2,k-1} \end{bmatrix} \mu(k\rangle_{k-1} \tag{529}$$

or

$$0(k-1\rangle = \begin{bmatrix} m(k-1\rangle_o, & M_{(k-1)(k-1)} \end{bmatrix} \begin{bmatrix} 1 \\ \mu(k-1\rangle_{k-1} \end{bmatrix} \tag{530}$$

$$\mu(k-1\rangle = -M^{-1}\, m(k-1\rangle_o \tag{531}$$

The package of $\langle\bar{g}$ vectors are

$$\langle\bar{g} = \langle\bar{f} \begin{bmatrix} 1 & 1 & 1 & & 1 \\ 0 & \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1,d-1} \\ 0 & 0 & \lambda_{22} & \cdots & \lambda_{2,d-1} \\ \cdot & \cdot & 0 & & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \lambda_{d-1,d-1} \end{bmatrix} \qquad (532)$$

The orthonomal vectors obtained via this procedure will differ in signs from the procedure of Eq (305). This latter statement is obvious from the direction of the $\bar{g}$ vector. These two Gram-Schmidt procedures are used later to obtain some of the classical orthogonal polynomials.

TRANSFORMATION FROM UNIT UPPER TRIANGULAR TO UNIT UPPER ROW.

Consider an upper triangular matrix

$$U = \begin{bmatrix} u_{oo} & u_{o1} & u_{o2} \\ 0 & u_{11} & u_{12} \\ 0 & 0 & u_{22} \end{bmatrix} = [u\rangle_o, u\rangle_1, u\rangle_2] \qquad (533)$$

where

$$\begin{bmatrix} u_{oo} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_{oo} \qquad (534)$$

$$\begin{bmatrix} u_{o1} \\ u_{11} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{u_{o1}}{u_{11}} \\ 1 \\ 0 \end{bmatrix} u_{11} \qquad (535)$$

and

$$\begin{bmatrix} u_{o2} \\ u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} \frac{u_{o2}}{u_{22}} \\ \frac{u_{12}}{u_{22}} \\ 1 \end{bmatrix} u_{22} \qquad (536)$$

or

$$U = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u_{oo}, \begin{pmatrix} \frac{u_{o1}}{u_{11}} \\ 1 \\ 0 \end{pmatrix} u_{11}, \begin{pmatrix} \frac{u_{o2}}{u_{22}} \\ \frac{u_{12}}{u_{22}} \\ 1 \end{pmatrix} u_{22} \right] \qquad (537)$$

or

$$U = \begin{bmatrix} 1 & \frac{u_{o1}}{u_{11}} & \frac{u_{o2}}{u_{22}} \\ 0 & 1 & \frac{u_{12}}{u_{22}} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{oo} & 0 & 0 \\ 0 & u_{11} & 0 \\ 0 & 0 & u_{22} \end{bmatrix} \qquad (538)$$

or

$$U = U_1 D_1 \qquad (539)$$

In a similar manner

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{u_{11}}{u_{o1}} & \frac{u_{12}}{u_{o2}} \\ 0 & 0 & \frac{u_{22}}{u_{o2}} \end{bmatrix} \begin{bmatrix} u_{oo} & 0 & 0 \\ 0 & u_{o1} & 0 \\ 0 & 0 & u_{o2} \end{bmatrix} \qquad (540)$$

or

$$U = U_r D_r \tag{541}$$

By Eq (539) and Eq (541)

$$U_1 = U_r D_r D_1^{-1} = U_r T \tag{542}$$

where

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{u_{o1}}{u_{11}} & 0 \\ 0 & 0 & \frac{u_{o2}}{u_{22}} \end{bmatrix} \tag{543}$$

The Classical Continuous Gram Polynomials Via The Alternate G. S. Process.

The 4x4 case of the Gram ploynomials on the (0.1) interval are derived here using the unit first row upper triangularization procedure (G.S.). The monomial base $\langle t$ has the Hilbert matrix as metric, or

$$\int_o^1 t\rangle\langle t \; dt = H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix} \tag{544}$$

set

$$g_o = t_o = t^o = 1 \tag{545}$$

$$g_1 = (1,t)\begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

and by Eq (520)

$$\int_o^1 1 \; g_1 \; dt = 0 = \int_o^1 1 \; (1,t)dt \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} \tag{546}$$

or

$$0 = (1, \tfrac{1}{2})\begin{pmatrix}1\\ \lambda_1\end{pmatrix} \tag{547}$$

or

$$\lambda_1 = -2 \tag{548}$$

hence

$$g_1 = (1,t)\begin{pmatrix}1\\ -2\end{pmatrix} \tag{549}$$

The third vector is

$$\overline{g}_2 = (1,t,t^2)\begin{pmatrix}1\\ \lambda_1\\ \lambda_2\end{pmatrix}_2 \tag{550}$$

and by Eq (523)

$$\int_o^1 \begin{pmatrix}1\\ t\end{pmatrix}(1,t,t^2)dt = \begin{pmatrix}0\\ 0\end{pmatrix} = \begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4}\end{bmatrix} a(3\rangle \tag{551}$$

Clearly Eq (551) requires us to find a vector $a(3\rangle$ in the null-space of

$$\underset{2x3}{H}\, a(3\rangle = 0(2\rangle \tag{552}$$

One approach is to compute the orthogonal complement projector and its rank-one factor

$$\underset{3x3}{\tilde{P}} = I - \underset{3(2)3}{H^* H} = u(3\rangle_2 \overset{2}{\langle} 3)u^* \tag{553}$$

since P has rank one, the vector factor is always a solution. However we desire the first coordinate to be unity (since we have one free choice - 2 equations and 3 unknowns). The easiest approach appears to be via Eq (527) that is

$$\mu(2\rangle_2 = -\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \tag{554}$$

or

$$\mu(2\rangle_2 = -\begin{bmatrix} 18 & -24 \\ -24 & 36 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \tag{555}$$

or

$$\mu(2\rangle_2 = \begin{pmatrix} -6 \\ 6 \end{pmatrix} \quad . \tag{556}$$

Using Eq (556) in Eq (550)

$$g_2 = (1,t,t^2)\begin{pmatrix} 1 \\ -6 \\ 6 \end{pmatrix} \tag{557}$$

with the orthogonal constraint

$$\int_0^1 \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} g_3 \, dt = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix} \begin{pmatrix} 1 \\ \mu(3\rangle_3 \end{pmatrix} \tag{558}$$

or

$$\mu(3\rangle_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix} \tag{559}$$

The inverse of the 3x3 sub-Hilbert matrix is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{4.5^2.6} & -\frac{2}{3.4.5.6} & \frac{1}{3.4^2.5} \\ -\frac{2}{3.4.5.6} & \frac{4}{2.6.4^2} & -\frac{2}{2.3.4.5} \\ \frac{1}{3.4^2.5} & -\frac{2}{2.3.4.5} & \frac{1}{2.3^2.4} \end{bmatrix} \frac{1}{\det H} \qquad (560)$$

while

$$\det H = (\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}) \begin{bmatrix} \frac{1}{4.5^2.6} \\ -\frac{2}{3.4.5.6} \\ \frac{1}{3.4^2.5} \end{bmatrix} \qquad (561)$$

or

$$\det H = \frac{1}{2(5)(6)(8)(9)(10)} \qquad (562)$$

and

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix}^{-1} = \begin{bmatrix} 72 & -240 & 180 \\ -240 & 900 & -720 \\ 180 & -720 & 600 \end{bmatrix} \qquad (563)$$

or

$$\mu(3\rangle_3 = \begin{pmatrix} -12 \\ 30 \\ -20 \end{pmatrix} \qquad (564)$$

or

$$g_3 = (1, t, t^2, t^3) \begin{bmatrix} 1 \\ -12 \\ 30 \\ -20 \end{bmatrix} \qquad (565)$$

We have by Milne Ref (60) page (259)

$$\begin{bmatrix} a_o \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}_3 = \begin{bmatrix} (-1)^o & \binom{3}{o} & \binom{3+o}{o} \\ (-1)^1 & \binom{3}{1} & \binom{3+1}{1} \\ (-1)^2 & \binom{3}{2} & \binom{3+2}{2} \\ (-1)^3 & \binom{3}{3} & \binom{3+3}{3} \end{bmatrix} \qquad (566)$$

or for the mth vector

$$\begin{bmatrix} a_o \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_k \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{bmatrix}_m = \begin{bmatrix} (-1)^o & \binom{m}{o} & \binom{m+o}{o} \\ (-1)^1 & \binom{m}{1} & \binom{m+1}{1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (-1)^k & \binom{m}{k} & \binom{m+k}{k} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (-1)^m & \binom{m}{m} & \binom{2m}{m} \end{bmatrix} \qquad (567)$$

Clearly the Gram-Schmidt process would be quite difficult to derive a general term such as given by Eq (567). Milne in ref (60) has a scalar method for such a derivation. The G.S. insight is worthy of note, thus has been presented.

Section 2

GENERALIZATIONS OF SOME CONTINUOUS METRICS AND ORTHOGONAL POLYNOMIALS WITH RESPECT TO PARTICULAR WEIGHTS. The standard textbook on the subject of Orthogonal Polynomials is Szego (1939). The Erdelyi reference on page 153 states that with an interval (t_b, t_f) and a weight function w(t) which is non-negative there, we may associate the scalar product

$$(f_i, f_j) = \int_{t_b}^{t_f} w(t)\ f_i(t) f_j(t) dt \tag{1}$$

which is defined for all functions f(t) for which $w(t)^{\frac{1}{2}}$ f(t) is quadratically integrable in (t_b, t_f). More generally, a scalar product may be defined by a Stieltjes integral

$$(f_i, f_j) = \int_{t_b}^{t_f} f_i(t) f_j(t) d\sigma(t) \tag{2}$$

where σ(t) is a non-decreasing function called a distribution function.. If σ(t) is absolutely continuous Equation (2) reduces to Equation (1) with $w(t)=\sigma'(t)$. On the other hand if σ(t) is a jump function, that is constant except for jumps of the magnitude w_k at $t=t_k$ then Equation (2) reduces to a sum

$$(f_i, f_j) = \sum_k w_k f_i(t_k) f_j(t_k) \ . \tag{3}$$

which is the appropriate definition for functions of a discrete variable. The reference states further that the above definitions refer to real functions of a real variable(the case restricted to throughout most of this report).

It was shown in the previous section Equation (156) that the weighted inner-product or inner-product with respect to a given weight function can be considered as a base change.

Certainly when one defines a norm on a vector, and then an inner-product on two vectors, nothing of interest ensues if one does not look at the inner-product on a sequence of base vectors for the full space or a subspace, for example when one is given the coordinates of the two vectors in a base we have

$$(x,\ y) = \bar{x}\textcircled{i}\bar{y} = \left\langle x\ M\ y \right\rangle \qquad (4)$$

where the matrix M is the metric.

The classical orthogonal polynomials, the interval and the weight function are given by Rice on page 36 in the following table.

CLASSICAL ORTHOGONAL POLYNOMIALS

INTERVAL	WEIGHT FUNCTION	NAME
$(-1,\ 1)$	$w(t) = 1$	LEGENDRE
$(-1,\ 1)$	$w(t) = (1-t)^{\lambda-\frac{1}{2}}$	GEGENBAUER
$(-1,\ 1)$	$w(t) = (1-t)^{\alpha}(1+t)^{\beta}$	JACOBI
$(-\infty,\ \infty)$	$w(t) = e^{-t^2}$	HERMITE
$(\ 0,\ \infty)$	$w(t) = t^{d}e^{-t}$	GENERALIZED LAGUERRE

TABLE (1) RICE

Davis in his book on page 168 states that the following special selections of (t_b, t_f) and $w(t)$ have been studied extensively, and the resulting orthonormal polynomials constitute the "classical" orthogonal polynomials.

SOME CLASSICAL ORTHOGONAL POLYNOMIALS

INTERVAL	WEIGHT FUNCTION	NAME
$(-1,\ 1)$	$w(t) = 1$	LEGENDRE
$(-1,\ 1)$	$w(t) = (1-t^2)^{-\frac{1}{2}}$	TSCHEBYSCHEFF POLYNOMIALS (of the first kind)
$(-1,\ 1)$	$w(t) = (1-t^2)^{\frac{1}{2}}$	TSCHEBYSCHEFF POLYNOMIALS (of the second kind)
$(-1,\ 1)$	$w(t) = (1-t)^{\alpha}(1+t)^{\beta}$ $\alpha,\beta > -1$	JACOBI POLYNOMIALS
$(\ 0,\ \infty)$	$w(t) = t^{d}e^{-t}$ $d > -1$	LAGUERRE POLYNOMIALS
$(-\infty,\ \infty)$	$w(t) = e^{-t^2}$	HERMITE POLYNOMIALS

TABLE (2) DAVIS

Rice states that the Legendre and Gegenbauer polynomials are special cases of the Jacobi polynomials. He states further that the classical orthogonal polynomials are not the only systems of polynomials orthogonal on the interval (0,1). For any weight function w(t) there corresponds a system of orthogonal polynomials (obviously one can always do a Gram-Schmidt (procedure). On page 6 of Rice we find the statement that the L_p - norms may be generalized by the introduction of a weight function w(t). He states that while we may conceivably use any integrable function w(t) ··· we normally consider only those weight functions for which

$$\int_0^1 w(t)dt = 1 \tag{5}$$

$$w(t) > 0 \quad , \quad 0 \leq t \leq 1$$

However for the purpose of this report and in the derivations of exponentially weighted least-squares filters via Laguerre polynomials the weight function is

$$\int_0^\infty w(t)dt = \int_0^\infty e^{-at}dt = \frac{1}{a} \quad . \tag{6}$$

Hildebrand on page 282 states that the weighting function

$$w(t) = (1-t)^\alpha (1+t)^\beta \tag{7}$$
$$\alpha > -1, \beta > -1$$

over (-1, 1) reduces to the Legendre case when $\alpha = \beta = 0$ and to the Chebyshev case when $\alpha = \beta = -\frac{1}{2}$. Note that Tschebyscheff and Chebyshev refer to the same person.

Hildebrand on page 282 is a little more general in what he calls the generalized Laguerre Polynomials (also he says frequently called the Sonine polynomials) and uses the weight

$$w(t) = t^d e^{-at} (d > -1, a > 0) \tag{8}$$

Rainville on page 213 refers to the simple Laguerre polynomials, which correspond to d=0 and a=1 in Equation (8) of Hildebrands weight.

There has been extensive applications of Laguerre polynomials to linear and non-linear systems analysis, both in the time and frequency domain. Morrison in his paper "Smoothing and Extrapolation of Continuous Time Series Using Laguerre Polynomials" defines the norm

$$||f||=\left(\int_0^\infty f^2(t)e^{-at}dt\right)^{\frac{1}{2}} \tag{9}$$

and the inner product of two functions f(t) and g(t) as

$$(f,g) = \int_0^\infty f(t)g(t)e^{-at}dt \quad . \tag{10}$$

The weight of Equation (10) is the one primarly used in this paper and its discrete analog in a later section.

The standard approach to the various types of polynomials is via the route of generating functions, a very tedious area of mathematics; however, they will not be discussed in this report.

Before proceeding to a simple derivation via Gram-Schmidt of the Laguerre polynomials, some terminology, jargon, and concepts associated with classical approximation theory and modern vector-space theory will be discussed. Even though Davis' book is quite vector space oriented, for the state-space orientation of this paper some of the terminology will be aliased in parenthesis. On page 169 Davis presents a definition:

DEFINITION: Let $s_1, s_2, \cdots$, be a finite or infinite sequence of orthonormal elements (the orthonormal condition implies linear independence and for clarification and applications herein the elements are to be taken from a vector space and form a base for the space or a subspace). Let x(t) be an arbitrary element (vector). The series

$$\sum_{n=1}^{\infty} (x,s_n)s_n$$

is the Fourier Series for x(t). (The representation of the vector x(t) in the particular orthonormal base $\langle s)$. If the sequence is finite we use a finite sum. The constants (x,s_n) are known as the Fourier Coefficients of x(t) (the constants (x,s_n) are the coordinates of the vector x(t) in the orthonormal base $\langle s)$. One frequently writes

$$x(t) \sim \sum_{n=1}^{\infty} (x,s_n)s_n \tag{11}$$

to indicate that the right-hand sum is associated in a formal way with the left-hand side. (The vector x(t) can always be written as

$$x(t) = \sum_{n=1}^{\infty} (x,s_n)s_n + x_r(t) \tag{12}$$

where the residual or error vector is $x_r(t)$). The relation between an element and its Fourier Series has been the object of vast investigation and theories.

Davis states further that we may write

$$x(t) \sim \sum_{n=0}^{\infty} \quad (\text{Projection of } x \text{ on } s_n)$$

and hence the Fourier series of an element (vector) is merely the sum of the projections of the element on a system of orthonormal elements (base vectors). On page 162 Davis presents the equation, projection of

$$x_1 \text{ on } x_2 = \frac{(x_1,x_2\ )}{(x_2,x_2\ )}\ x_2 \quad , \tag{13}$$

which he states serves to define projection in the abstract case. (See previous section for concrete projections with polynomials over the continuous reals).

B. CONTINUOUS LAGUERRE POLYNOMIALS. Two types of Laguerre polynomials will be used in this report corresponding to the weight of Equation (6) namely

$$w(t) = e^{-at} \tag{14}$$

and for a=1

$$w(t) = e^{-t} \quad . \tag{15}$$

Not only the Laguerre polynomials with respect to the two weights above but also two modified Laguerre polynomial bases will be derived. The Modified Laguerre set will be derived via a Grahm-Schmidt procedure on the base

$$e^{-\frac{a}{2}t} \langle t \quad .$$

These are in analogy to the Modified Legendre discussed and derived in a later section and related to the Classical Legendre Polynomials.

The base change approach will be used for all orthogonal polynomials starting with the monomial base and in later sections their velocities etc., will be derived. The previous section derived the 3x3 Gram and Modified Legendre polynomials on the intervals (0,1) and (-1,1). If we look at the weights for the Modified Legendre case, we see it is equal to one. If we look at the inner-product interval for the Modified Laguerre case we see $(0,\infty)$ and this span for the monomial base infinite entries in the metric matrix without the

$$e^{-\frac{a}{2}t}$$

weight.

We designate the monomial base

$$\langle d)t = (t^0, t^1, t^2, t^3, \cdots t^{d-1}) = \langle t \tag{16}$$

that is the subscript on the first base element of the sequence will correspond to the power of t, and

$$t\rangle\langle t = \left[t^i t^j \right] = \left[i,j^{\text{th}} \text{ element} \right] \tag{17}$$

$$i,j = 0,\ 1,\ 2,\ \cdots\ d-1\ .$$

If we want the first base element subscript 1, etc., as before

$$(f_1, f_2, f_3, \cdots f_d) = \langle d)\, f \qquad (18)$$

and we want

$$\langle d)\, f = \langle t \qquad (19)$$

then

$$f\rangle\langle f = \left[f_i, f_j\right] = \left[t^{i-1}, t^{j-1}\right] \qquad (20)$$

$$i, j = 1,\ 2,\ \cdots\ d$$

and the indefinite integral is

$$\int t^{i-1} t^{j-1} dt = \int t^{i+j-2} dt \qquad (21)$$

$$= \frac{t^{i+j-1}}{i+j-1} \qquad (22)$$

$$i,\ j = 1,\ 2,\ 3,\ \cdots\ d$$

and the i^{th} row and j^{th} column of the entry of the metric for this indexing is

$$\int_0^\infty f\rangle\langle f = \left[\frac{t^{i+j-1}}{i+j-1}\right]_0^\infty \qquad (23)$$

Returning to the indexing of Equation (16) where the index runs

$$i = 0,\ 1,\ 2,\ 3,\ \cdots\ d-1$$

$$\int t^i t^j dt = \int t^{i+j} dt = \frac{t^{i+j+1}}{i+j+1} \tag{24}$$

as compared with Equation (23).

If we make a time-varying base change on $\langle t$

$$\langle t \; D(t) = \langle t \; I \; e^{\frac{-a}{2} t} = \langle t_e \tag{25}$$

Then the metric-matrix for the this new base is

$$M_{tt_e} = \int_0^\infty t\rangle_e \langle t_e \; dt = \int_0^\infty t\rangle\langle t \; e^{-at} \; dt \tag{26}$$

the i-jth element of the integral matrix is

$$t^{i+j} \; e^{-at}$$

and by Pierces Handbook of Integrals on page 63

$$\int_0^\infty t^{i+j} \; e^{-at} \; dt = \frac{(i+j)!}{a^{i+j+1}} \tag{27}$$

applying Equation (27) to the metric matrix of Equation (26)

$$\int_0^\infty t\rangle\langle t \; e^{-at} \; dt = \left[\frac{(i+j)!}{a^{i+j+1}} \right] = \left[g_{ij} \right]_{tt_e} \tag{28}$$

or in open-form

$$M_{tt_e} = \begin{bmatrix} \frac{0!}{a} & \frac{1!}{a^2} & \frac{2!}{a^3} & \frac{3!}{a^4} & \cdots & \frac{(d-1)!}{a^d} \\ \frac{1!}{a^2} & \frac{2!}{a^3} & \frac{3!}{a^4} & \frac{4!}{a^5} & \cdots & \cdot \\ \frac{2!}{a^3} & \frac{3!}{a^4} & \frac{4!}{a^5} & \frac{5!}{a^6} & & \cdot \\ \frac{3!}{a^4} & \frac{4!}{a^5} & \frac{5!}{a^6} & & & \cdot \\ \vdots & & & & & \\ \frac{(d-1)!}{a^d} & & \cdots & & & \frac{(2(d-1))!}{a^{2d-1}} \end{bmatrix} \tag{29}$$

We now have the inner-product of the base elements and will run a Gram-Schmidt proceedure on the first four elements

$$\langle f_e = \langle 4) t_e = (1, t, t^2, t^3)\ e^{-a}t \quad . \tag{30}$$

We have as before (index starting at 1, 2, ...)

$$g_{e1} = f_{e1} = (f_{e1}, f_{e2}, f_{e3}, f_{e4}) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{31}$$

and the orthonormal vector is

$$s_{e_1} = \frac{g_{e1}}{||g_{e_1}||} \tag{32}$$

where

$$||g_{e1}||^2 = \int_0^\infty g^2{}_{e1} dt = \int_0^\infty e^{-at} dt = \frac{1}{a} \qquad (33)$$

or

$$||g_{e1}|| = \frac{1}{\sqrt{a}} \quad . \qquad (34)$$

The second orthogonal vector by Equation (308) of section (1) is

$$g_{e2} = f_{e2} - s_{e1}\, s_{e1}, \textcircled{i} f_{e2} \qquad (35)$$

and is

$$g_{e2} = \langle f_e \begin{pmatrix} -\frac{1}{a} \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad (36)$$

The second orthonormal vector

$$s_{2e} = \frac{g_{e2}}{||g_{e2}||} \qquad (37)$$

and

$$||g_{e2}|| = (-\frac{1}{a} , 1, 0, 0)\, M_{ff_e} \begin{pmatrix} -\frac{1}{a} \\ 1 \\ 0 \\ 0 \end{pmatrix}^{\frac{1}{2}} = \frac{1}{\sqrt{a^3}} \qquad (38)$$

and

$$s_{e2} = \langle f_e \begin{pmatrix} -1 \\ a \\ 0 \\ 0 \end{pmatrix} \sqrt{a} \qquad (39)$$

Proceeding through the process one obtains for the first four

$$\langle 4) g_e = \langle 4) f_e \begin{bmatrix} 1 & -\frac{1}{a} & \frac{2}{a^2} & -\frac{3!}{a^3} \\ 0 & 1 & -\frac{4}{a} & \frac{18}{a^2} \\ 0 & 0 & 1 & -\frac{9}{a} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (40)$$

or

$$\langle 4) g_e = \langle 4) f_e \underset{4x4}{B_{ge}} \qquad (41)$$

The first 3 orthonormal vectors are from the computations

$$\langle 3) s_e = \langle 3) f_e \begin{bmatrix} 1 & -1 & 1 \\ 0 & a & -2a \\ 0 & 0 & \frac{a^2}{2} \end{bmatrix} \sqrt{a} \qquad (42)$$

If we shift our index back to zero that is index starting at (0, 1, 2, ··· d-1)

$$\langle g = (g_0, g_1, g_2, \cdots g_{d-1}) = (t^0, t^1, t^2, \cdots t^{d-1})_e B_{ge} \qquad (43)$$

and

$$\langle t_e = (t^0, t^1, t^2, \cdots t^{d-1}) \, e^{\frac{-a}{2} t} \quad . \qquad (44)$$

By Equation (25) in Equation (43)

$$\langle g = \langle t \, B_{ge} \, e^{\frac{-a}{2} t} \qquad (45)$$

The Modified Laguerre Equations are defined by Equation (45) and Equation (46)

$$\langle \ell_{m_f} = \langle \ell_m \, e^{-at}/2 \quad . \tag{47}$$

The n^{th} Classical Laguerre polynomials (for n=0, 1, 2, ··· d-1) (non-normalized) are given by (Morrison) reference (61)

$$\boxed{\ell_n^{(t)} = n! \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(a_t)^j}{j!}} \tag{48}$$

Expressing the summation of Equation (48) as inner-products of row and column vectors we have

$$\ell_0 = 1$$

$$\ell_1 = \left(\frac{(at)^0}{0!} , \frac{(at)^1}{1!} \right) \begin{bmatrix} (-1)^0 \binom{1}{0} \\ \\ (-1)^1 \binom{1}{1} \end{bmatrix} 1!$$

$$\ell_2 = \left(\frac{(at)^0}{0!} , \frac{(at)^1}{1!} , \frac{(at)^2}{2!} \right) \begin{bmatrix} (-1)^0 \binom{2}{0} \\ (-1)^1 \binom{2}{1} \\ (-1)^2 \binom{2}{2} \end{bmatrix} 2!$$

$$\ell_3 = \left(\frac{(at)^0}{0!} , \frac{(at)^1}{1!} , \frac{(at)^2}{2!} , \frac{(at)^3}{3!} \right) \begin{bmatrix} (-1)^0 \binom{3}{0} \\ (-1)^1 \binom{3}{1} \\ (-1)^2 \binom{3}{2} \\ (-1)^3 \binom{3}{3} \end{bmatrix} 3!$$

$$\ell_n = \left(\frac{(at)^0}{0!} , \frac{(at)^1}{1!} , \frac{(at)^2}{2!} \dots \frac{(at)^n}{n!} \right) \begin{bmatrix} (-1)^0 \binom{n}{0} \\ (-1)^1 \binom{n}{1} \\ (-1)^2 \binom{n}{2} \\ \vdots \\ (-1)^n \binom{n}{n} \end{bmatrix} n!$$

$$\ell_{d-1} = \left(\frac{(at)^0}{0!} , \frac{(at)^1}{1!} \dots \frac{(at)^{d-1}}{(d-1)!} \right) \begin{bmatrix} (-1)^0 \binom{d-1}{0} \\ (-1)^1 \binom{d-1}{1} \\ \vdots \\ (-1)^{d-1} \binom{d-1}{d-1} \end{bmatrix} (d-1)!$$

Packaging the previous equations as a row of base vectors

$$(\ell_0,\ell_1,\ell_2 \cdots \ell_{d-1}) = (1,t,t^2 \dots t^n \dots t^{d-1}) \begin{bmatrix} a^0 & & & & \\ & a^1 & & & \\ & & a^2 & & \\ & & & \ddots & \\ & & & & a^{d-1} \end{bmatrix}$$

$$x \begin{bmatrix} \frac{1}{0!} & & & & & \\ & \frac{1}{1!} & & & 0 & \\ & & \frac{1}{2!} & & & \\ & & & \frac{1}{3!} & & \\ & 0 & & & \ddots & \\ & & & & & \frac{1}{(d-1)!} \end{bmatrix}$$

$$\begin{bmatrix} (-1)^0\binom{0}{0} & (-1)^0\binom{0}{1} & (-1)^0\binom{2}{0} & \dots & (-1)^0\binom{n}{0} & \dots & (-1)^0\binom{d-1}{0} \\ 0 & (-1)^1\binom{1}{1} & (-1)^1\binom{2}{1} & \dots & (-1)^1\binom{n}{1} & & (-1)^1\binom{d-1}{1} \\ 0 & 0 & (-1)^2\binom{2}{2} & & \cdot & & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & (-1)^b\binom{n}{n} & & \cdot \\ & & & & 0 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & (-1)^{d-1}\binom{d-1}{d-1} \end{bmatrix}$$

$$x \begin{bmatrix} 0! & & & & & \\ 0 & 1! & & & 0 & \\ & & 2! & & & \\ 0 & & & 3! & & \\ & & & & \ddots & \\ & & & & & (d-1)! \end{bmatrix} \tag{50}$$

Define the following matrices

$$D(a^n) = \begin{bmatrix} a^0 & & & & & & \\ & a^1 & & & & & \\ & & a^2 & & & & \\ & & & \ddots & & & \\ & & & & a^n & & \\ & & & & & \ddots & \\ & & & & & & a^{d-1} \end{bmatrix} \tag{51}$$

$$\mathbb{F} = \begin{bmatrix} 0! & & & & & \\ & 1! & & & & \\ & & 2! & & & \\ & & & 3! & & \\ & & & & \ddots & \\ & & & & & (d-1)! \end{bmatrix} \tag{52}$$

and the matrix of modified binomial coefficients by Equation (90) in Appendix A is the Rutishauser matrix

$$R = \begin{bmatrix} (-1)^0\binom{0}{0} & (-1)^0\binom{1}{0} & (-1)^0\binom{2}{0} & \cdots & (-1)^0\binom{d-1}{0} \\ 0 & (-1)^1\binom{1}{1} & (-1)^1\binom{2}{1} & \cdots & (-1)^1\binom{d-1}{1} \\ & & (-1)^2\binom{2}{2} & & \cdot \\ & & & & \cdot \\ & & & & \cdot \\ & & & & (-1)^{d-1}\binom{d-1}{d-1} \end{bmatrix} \tag{53}$$

hence Equation (50) can be written in factored form

$$\boxed{\langle \ell = \langle t \; D(a^n) \; \mathrm{F}\!\!\!\mid^{-1} \; R \; \mathrm{F}\!\!\!\mid} \tag{54}$$

which is the matrix representation of the Classical Laguerre Polynomials and are orthogonal on $(0,\infty)$ with respect to the weight e^{-at} or by the base change

$$\langle \ell_f = \langle t \; D(a^n) \; \mathrm{F}\!\!\!\mid^{-1} \; R \; \mathrm{F}\!\!\!\mid \; e^{\frac{-at}{2}} \tag{55}$$

$$= \langle t \; T \; e^{\frac{-at}{2}} \tag{56}$$

we have the Laguerre functions of Equation (48) which Gillis and Bolgiano (for a=1) in separate papers refer to as forming an orthogonal set complete in $L^2(0,\infty)$.

If we form the metric of Equation (56) as

$$\int_0^\infty \ell\rangle_f \langle\ell_f \, dt = M_{\ell\ell_f} \quad . \tag{57}$$

and use the $\ell\ell f$ subscript,no ambiguity can occur. Using Equation (56) in Equation (59)

$$M_{\ell\ell_f} = T^T \left[\int_0^\infty t\rangle \langle t \; e^{-at} \right] T \tag{58}$$

or

$$M_{\ell\ell_f} = T^T M_{tt_e} T \tag{59}$$

The vectors of Equation (56) (not normalized) and their magnitudes (thus the metric matrix) is a diagonal matrix whose entries are the squares of the norms, that is

$$M_{\ell\ell_f} = \begin{bmatrix} ||g_{00}||^2 & & & \\ & ||g_{11}||^2 & & \\ & & \ddots & \\ & & & ||g_{d-1}||^2 \end{bmatrix} = \left[g_{nn}\right] \tag{60}$$

The squares of the magnitudes of the Laguerre polynomials are given by Hildebrand page 276 as

$$\boxed{g_{nn} \equiv \int_0^\infty e^{-at} \, \ell_n^2(t) \, dt = \frac{1}{a} \; (n!)^2} \tag{61}$$

where g_{nn} are the elements of the metric matrix Equation (60) and where $\ell_n(t)$ is given by Equation (48), hence

$$\boxed{M_{\ell\ell_f} = \frac{1}{a}\mathbb{F}^2} = \frac{1}{a}\begin{bmatrix} (0!)^2 & & & & \\ & (1!)^2 & & & \\ & & (2!)^2 & & \\ & & & \ddots & \\ & & & & (d-1)!^{\,2} \end{bmatrix} \tag{62}$$

The normalizing matrix to obtain the unit magnitude vectors $\langle s$ is the inverse square root of Equation (62) and (64) or

$$\langle s = \langle \ell_u = \langle \ell \; M_{\ell\ell_f}^{-\frac{1}{2}} \quad . \tag{63}$$

and by Equation (62)

$$\sqrt{a}\,\mathbb{F}^{-1} = M_{\ell\ell_f}^{-\frac{1}{2}} \tag{64}$$

hence using Equation (64) in Equation (63) and Equation (54)

$$\langle \ell_u = \langle t \; D(a^n)\; \mathbb{F}^{-1} R \; \mathbb{F}\mathbb{F}^{-1} \sqrt{a} \tag{65}$$

or the <u>Classical Orthonormal Laguerre</u> polynomials are given by

$$\boxed{\langle \ell_u = \langle t \; D(a^n)\; \mathbb{F}^{-1} R \sqrt{a}} \tag{66}$$

The individual elements are given by Equation (66) and Equation (48) as

$$\boxed{(\ell_u(t))_n = \sqrt{a}\sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(at)^j}{j!}} \tag{67}$$

which is the n^{th} orthonormal Laguerre polynomial for n=0, 1, 2 ··· d-1 and the metric is

$$\int_0^\infty \ell_u \rangle \langle \ell_u \, e^{-at} dt = I \qquad (68)$$

If we now compare the Classical Laguerre polynomials with the modified Laguerre polynomials for the first four of Equation (50) and Equation (40) respectively we have

$$\langle \ell = \langle t \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -a & -4a & -18a \\ 0 & 0 & a^2 & 9a^2 \\ 0 & 0 & 0 & -a^3 \end{bmatrix} \qquad (69)$$

and for the modifieds

$$\langle g = \langle \ell_m = \langle t \begin{bmatrix} 1 & -1/a & 2/a^2 & -6/a^3 \\ 0 & 1 & -4/a & 18/a^2 \\ 0 & 0 & 1 & -9/a \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (70)$$

Clearly the two matrices differ by signs and a and a^{-1} powers. Notice also that the Classical Laguerre Polynomials do not have unity entries on the main diagonal, hence are not in the monic polynomial form as are the modified Laguerre polynomials derived via Gram-Schmidt. The monic constraint on functions given by say

$$\langle b = \langle f \, U \qquad (71)$$

where U is upper triangular we have

$$b_1 = \angle f \begin{pmatrix} u_{11} \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

or

$$\frac{b_1}{u_{11}} = b_{1m} = \angle f \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$b_2 = \angle f \begin{pmatrix} u_1 \\ u_2 \\ 0 \\ 0 \\ \vdots \end{pmatrix}_2$$

$$\begin{gathered} \frac{b_2}{u_{22}} = b_{2m} = \angle f \begin{pmatrix} \frac{u_{12}}{u_{22}} \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \\ \cdot \\ \cdot \\ \cdot \\ \frac{b_n}{u_{nn}} = b_{nm} = \angle f \begin{pmatrix} \frac{u_{1n}}{u_{nn}} \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \end{gathered} \tag{72}$$

packaging Equation (72)

$$\langle b_m = \langle b \begin{pmatrix} u_{11} & & & \\ & u_{22} & & \\ & & \ddots & \\ & & & u_{nn} \end{pmatrix}^{-1} \tag{73}$$

Hence applying Equation (73) to the elements of Equation (69) and Equation (70)

$$\ell_0 = \langle t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = g_0 = \langle t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{74}$$

$$\ell_1 = \langle t \begin{pmatrix} 1 \\ -a \\ 0 \\ 0 \end{pmatrix} \tag{75}$$

$$\frac{\ell_1}{-a} = \ell_{1m} = \langle t \begin{pmatrix} -1/a \\ 1 \\ 0 \\ 0 \end{pmatrix} = g_1 \tag{76}$$

$$\ell_2 = \langle t \begin{pmatrix} 2 \\ -4a \\ a^2 \\ 0 \end{pmatrix} \tag{77}$$

$$\frac{\ell_2}{a^2} = \ell_{2m} = \langle t \begin{pmatrix} {}^{2}/_{a^2} \\ -{}^{4}/_{a} \\ 1 \\ 0 \end{pmatrix} = g_2 \tag{78}$$

$$\ell_3 = \langle t \begin{pmatrix} 6 \\ -18a \\ 9a \\ -a^3 \end{pmatrix} \tag{79}$$

$$\frac{\ell_3}{-a^3} = \ell_{3m} = \langle t \begin{pmatrix} -{}^{6}/_{a^3} \\ +{}^{18}/_{a^2} \\ -{}^{9}/_{a} \\ 1 \end{pmatrix} = g_3 \tag{80}$$

or matrix-wise

$$\langle \ell_m = \langle 4) g = \langle \ell \begin{pmatrix} 1 & & & \\ & -{}^{1}/_{a} & & \\ & & {}^{1}/_{a^2} & \\ & & & -{}^{1}/_{a^3} \end{pmatrix} \tag{81}$$

Using Equation (53) in Equation (77) of Appendix A

$$\langle \ell_m = \langle \ell \; I(-1) \; D^{-1}(a^n) \qquad (82)$$

Using Equation (54) in Equation (82)

$$\langle \ell_m = \langle t \; D(a^n) \; \#^{-1} \; R \; \# \; I(-1) \; D^{-1}(a^n) \qquad (83)$$

Using the commutitive property of diagonal matrices

$$\langle \ell_m = \langle t \; D(a^n) \; \#^{-1} \; R \; I(-1) \; \# \; D^{-1}(a^n) \qquad (84)$$

The alternating sign binomial matrix of Equation (77) Appendix A is

$$R \; I(-1) = B(-1) \qquad (85)$$

hence using Equation (85) in Equation (84)

$$\boxed{\langle \ell_m = \langle t \; D(a^n) \; \#^{-1} \; B(-1) \; \# \; D^{-1}(a^n)} \qquad (86)$$

For the special case a=1, the first six Laguerre polynomials by Equation (60) are the same as those given by Hildebrand on page 275 of reference (37) for the non-orthonormal case

$$\langle \ell = \langle t \begin{bmatrix} 1 & 1 & 2 & 6 & 24 & 120 \\ 0 & -1 & -4 & -18 & -96 & -600 \\ 0 & 0 & 1 & 9 & 72 & 600 \\ 0 & 0 & 0 & -1 & -16 & -200 \\ 0 & 0 & 0 & 0 & 1 & 25 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad (87)$$

If we compare Hildebrands' with Equation (86) for the first five polynomials with a=1

$$\langle \ell_m = \langle t \begin{bmatrix} 1 & -1 & 2 & -6 & 24 \\ 0 & 1 & -4 & 18 & -96 \\ 0 & 0 & 1 & -9 & 72 \\ 0 & 0 & 0 & 1 & -16 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (88)$$

we see the disagreement in signs only. Equation (87) becomes by Equation (54)

$$\boxed{\begin{array}{c} \langle \ell = \langle t\ \#^{-1}\ R\ \# \\ a = 1 \end{array}} \qquad (89)$$

and the modified polynomials and

$$\boxed{\langle \ell_m = \langle t\ \#^{-1}\ B(-1)\ \#} \qquad (90)$$

The <u>modified Laguerre Functions</u> are given by Equation (86) and Equation (45)

$$\langle \ell_{m_f} = \langle \ell_m\ e^{-a t/2} \qquad (91)$$

Using Equation (82) in Equation (91)

$$\langle \ell_{m_f} = \langle \ell\ I(-1)\ D^{-1}(a^n)\ e^{-a t/2} \qquad (92)$$

$$\langle \ell_{m_f} = \langle \ell_f \; I(-1) \; D^{-1}(a^n) \qquad (93)$$

The modified metric becomes

$$\int_0^{\infty} \ell\rangle_{m_f} \langle \ell_{m_f} \, dt = D^{-1}(a^n) \; I(-1) \; M_{\ell\ell_f} \; I(-1) \; D^{-1}(a^n) \qquad (94)$$

By Equation (62) in Equation (94)

$$M_{\ell\ell m_f} = D^{-1}(a^n) \; I(-1) \; \frac{\text{Ħ}^2}{a} I(-1) \; D^{-1}(a^n) \qquad (95)$$

or

$$M_{\ell\ell m_f} = \begin{bmatrix} \frac{(0!)^2}{a} & & & & & & & \\ & \frac{(1!)^2}{a^3} & & & & & & \\ & & \frac{(2!)^2}{a^5} & & & & & \\ & & & \frac{(3!)^2}{a^7} & & & & \\ & & & & \ddots & & & \\ & & & & & \frac{(j!)^2}{a^{2j+1}} & & \\ & & & & & & \ddots & \\ & & & & & & & \frac{(d-1)!^{\;2}}{a^{2d-1}} \end{bmatrix} \qquad (96)$$

or factoring out the a power terms

$$\underset{d\times d}{M_{\ell\ell m_f}} = \begin{bmatrix} {}^1/_a & & & & & & \\ & {}^1/_{a^3} & & & & & \\ & & {}^1/_{a^5} & & & & \\ & & & \ddots & & & \\ & & & & {}^1/_{a^{2j+1}} & & \\ & & & & & \ddots & \\ & & & & & & {}^1/_{a^{2d-1}} \end{bmatrix} \begin{bmatrix} 0! & & & & & & \\ & 1! & & & & & \\ & & 2! & & & & \\ & & & \ddots & & & \\ & & & & j! & & \\ & & & & & \ddots & \\ & & & & & & (d-1)! \end{bmatrix}^2 \tag{97}$$

Define the inverse odd powers of a matrix as

$$D^{-1}(a^{2n+1}) = \begin{bmatrix} {}^{1}/_{a} & & & \\ & {}^{1}/_{a^3} & & \\ & & {}^{1}/_{a^5} & \\ & & & \ddots \end{bmatrix} \tag{98}$$

Hence Equation (98) in Equation (97)

$$\boxed{M_{\ell\ell m_f} = D^{-1}(a^{2n+1})\ \mathbb{F}^2} \tag{99}$$

which is the metric-matrix for the Modified Laguerre Functions, each element of which is given as

$$\boxed{M_{\ell\ell m_f} = \begin{bmatrix} g_{nn} \end{bmatrix} = \begin{bmatrix} \frac{(n!)^2}{a^{2n+1}} \end{bmatrix}} \tag{100}$$

for

$$n = 0,\ 1,\ 2,\ \cdots\ d-1$$

The orthonormal Modified Laguerre polynomials are given as

$$\langle \ell_{m_u} = \langle \ell_m\ M^{-\frac{1}{2}}_{\ell\ell m_f} \tag{101}$$

The square root of the matrix of Equation (100) is

$$M^{\frac{1}{2}}_{\ell\ell m_f} = \begin{bmatrix} \frac{n!}{a^{n+\frac{1}{2}}} \end{bmatrix} \tag{102}$$

and the inverse of the diagonal matrix of Equation (102) is

$$M_{\ell\ell m_f}^{-\frac{1}{2}} = \left[\frac{a^{n+\frac{1}{2}}}{n!} \right] \tag{103}$$

or in terms of factors

$$M_{\ell\ell m_f}^{-\frac{1}{2}} = D(a^{n+\frac{1}{2}})\ \mathbb{F}^{-1} \tag{104}$$

Using Equation (104) in Equation (101)

$$\langle \ell_{m_u} = \langle \ell_m\ D(a^{n+\frac{1}{2}})\ \mathbb{F}^{-1} \tag{105}$$

Using Equation (86) in Equation (105)

$$\langle \ell_{m_u} = \langle t\ D(a^n)\ \mathbb{F}^{-1}\ B(-1)\ \mathbb{F}\ D^{-1}(a^n) D(a^{n+\frac{1}{2}})\ \mathbb{F}^{-1} \tag{106}$$

observe for the first 4x4

$$\underset{4x4}{M_{\ell\ell m_f}^{-\frac{1}{2}}} = \underset{4x4}{D(a^{n+\frac{1}{2}})}\ \underset{4x4}{\mathbb{F}^{-1}} = \begin{pmatrix} a^{1/2} & & & \\ & a^{3/2} & & \\ & & a^{5/2} & \\ & & & a^{7/2} \end{pmatrix} \begin{pmatrix} 1/0! & & & \\ & 1/1! & & \\ & & 1/2! & \\ & & & 1/3! \end{pmatrix}$$

$$M_{\ell\ell m_f}^{-\frac{1}{2}} = \begin{pmatrix} a^{1/2} & & & \\ & aa^{1/2} & & \\ & & \frac{a^2 a^{1/2}}{2!} & \\ & & & \frac{a^3 a^{1/2}}{3!} \end{pmatrix} \tag{107}$$

$$M_{\ell\ell m_f}^{-\frac{1}{2}} = \begin{pmatrix} 1 & & & \\ & a & & \\ & & \frac{a^2}{2!} & \\ & & & \frac{a^3}{3!} \end{pmatrix} \sqrt{a} \tag{108}$$

The first four Modified Laguerre polynomials are given by Equation (70)

$$\langle \ell_m = \langle t \begin{bmatrix} 1 & -1/_a & 2/_{a^2} & -6/_{a^3} \\ 0 & 1 & -4/_a & 18/_{a^2} \\ 0 & 0 & 1 & -9/_a \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{109}$$

Using Equation (109) in Equation (105)

$$\langle t)\ell_{m_u} = \langle t \begin{bmatrix} 1 & -1/_a & 2/_{a^2} & -6/_{a^3} \\ 0 & 1 & -4/_a & 18/_{a^2} \\ 0 & 0 & 1 & -9/_a \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^2/_{2!} & 0 \\ 0 & 0 & 0 & a^3/_{3!} \end{bmatrix} \sqrt{a} \tag{110}$$

or

$$\Big\langle \ell_{m_u} = \Big\langle t \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & a & -2a & 3a \\ 0 & 0 & a^2/_2 & -3/_2 a^2 \\ 0 & 0 & 0 & a^3/_{3!} \end{bmatrix} \sqrt{a} = \Big\langle t \; B_s \qquad (111)$$

which agrees with the first three normalized polynomials obtained via Gram-Schmidt in Equation (42).

The first four <u>orthonormal Laguerre polynomials</u> obtained from Equation (66) are

$$\Big\langle \ell_u = \Big\langle t \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -a & -2a & -3a \\ 0 & 0 & a^2/_2 & 3/_2 a^2 \\ 0 & 0 & 0 & -a^3/_{3!} \end{bmatrix} \sqrt{a} \qquad (112)$$

which differ only in the signs hence the connection is

$$\Big\langle \ell_{u_m} = \Big\langle \ell_u \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad (113)$$

or

$$\Big\langle s = \Big\langle \ell_{u_m} = \Big\langle \ell_u \; I(-1) = \Big\langle t_e \; B_s \quad . \qquad (114)$$

By Equation (66) in Equation (114)

$$\langle s = \langle t \; D(a^n) \; \mathbb{F}^{-1} \; R \; I(-1) \; \sqrt{a} \tag{115}$$

and by Equation (77) in Appendix A

$$\langle \ell_{u_m} = \langle s = \langle t \; D(a^n) \; \mathbb{F}^{-1} \; B(-1) \; \sqrt{a} \tag{116}$$

The orthonormal modified Laguerre functions are

$$\langle \ell_{um_f} = \langle t_e \; D(a^n) \; \mathbb{F}^{-1} \; B(-1) \; \sqrt{a} \tag{117}$$

$$= \langle t_e \; B_s \tag{118}$$

The matrix B_s is

$$B_s = D(a^n) \; \mathbb{F}^{-1} \; B(-1) \; \sqrt{a} \tag{119}$$

and the inverse is easily obtained

$$B_s^{-1} = B \; \mathbb{F} \; D^{-1}(a^n) \; \frac{1}{\sqrt{a}} \tag{120}$$

and for the 3x3 case of Equation (42) the inverse is

$$\underset{3x3}{B_s^{-1}} = \frac{1}{\sqrt{a}} \begin{pmatrix} 1/_a & 1/_{a^2} & 2!/_{a^3} \\ 0 & 1/_{a^2} & 4/_{a^3} \\ 0 & 0 & 2/_{a^3} \end{pmatrix} \tag{121}$$

By Equation (359) of the previous section we could also compute the inverse connection matrix as

$$B_s^{-1} = B_s^T M_{tt_e} \quad . \tag{122}$$

The inverse of the metric-matrix M_{tt_e} can be obtained via Equation (348) of section (1) for the 3x3 case of Equation (29) is

$$\begin{bmatrix} 1/_a & 1/_{a^2} & 2!/_{a^3} \\ 1/_{a^2} & 2!/_{a^3} & 3!/_{a^4} \\ 2!/_{a^3} & 3!/_{a^4} & 4!/_{a^5} \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -3a & a^2/2 \\ -3a & 5a^2 & -a^3 \\ a^2/_2 & -a^3 & a^4/_4 \end{bmatrix} \tag{123}$$

obtained from

$$M_{tt_e}^{-1} = B_s B_s^T \quad , \tag{124}$$

where the general B_s term is given by Equation (119).

In conclusion for the Laguerre and Modified Laguerre polynomials we have by Equation (54) and Equation (83)

$$\Big< \ell = \Big< t \; D(a^n) \; \#^{-1} \; R \; \# \tag{125}$$

$$\Big< \ell_m = \Big< t \; D(a^n) \; \#^{-1} \; B(-1) \; \# \; D^{-1}(a^n) \tag{126}$$

and for the Laguerre functions and Modified Laguerre functions by Equation (55) and Equation (47)

$$\Big< \ell_f = \Big< t_e \; D(a^n) \; \#^{-1} \; R \; \# \tag{127}$$

$$\Big< \ell_{m_f} = \Big< t_e \; D(a^n) \; \#^{-1} \; B(-1) \; \# \; D^{-1}(a^n). \tag{128}$$

For the orthonormal Laguerre polynomials and modified polynomials by Equation (66) and Equation (109)

$$\langle \ell_u = \langle t \; D(a^n) \; \mathbb{F}^{-1} \; R \; \sqrt{a} \tag{129}$$

$$\langle \ell_{u_m} = \langle t \; D(a^n) \; \mathbb{F}^{-1} \; B(-1) \; \sqrt{a} \tag{130}$$

with the corresponding change to the e subscript on $\langle t$ of Equation (129) and Equation (130) for the orthonormal functions.

The inverse transformations are easily obtained since by Appendix A the Rutishauser matrix R is its own inverse that is

$$R^{-1} = R \tag{131}$$

and

$$B(-1)^{-1} = B \quad . \tag{132}$$

C. CONTINUOUS POISSON POLYNOMIALS. Morrison states in reference (61) that the Laguerre polynomials give rise, in a very natural way, to a real-true spectrum, analyzer. He says that another virtue of the Laguerre polynomials is that their Laplace transforms are given by very simple formulas.

Bolgiano in his paper "Relationship of Poisson Transform to Laguerre Expansions" reference, states that ··· one obtains the Laguerre function used by Steiglitz in his paper " The Equivalence of Digital and Analog Signal Processing" reference (84) to represent positive time signals for the digital simulation of analog signal processing. He says Steiglitz's representation offers the convenience of an orthogonal representation for signals. However, for systems it does not offer as direct a representation as the Poisson transform, which ··· permits obtaining the unit pulse response for a discrete system from the unit impulse response of the analog system it simulates in the same manner as discrete-time signals are derived from analog signals. From the above statements of the two authors it is felt that the frequency domain study of digital and analog (discrete and continuous) filters can be enhanced by knowledge of both types of polynomials.

If we attempt to apply some state-space techniques to the results of Bolgianos paper where he treats the Laguerre orthonormal polynomials for $a=1$ by Equation (129)

$$\langle t \, \mathbb{F}^{-1} R = \langle \ell_u \tag{133}$$

Define the Poisson polynomials by

$$\langle p(t) = \langle t \, \mathbb{F}^{-1} \tag{134}$$

and like-wise the Poisson Functions by

$$\langle p(t)_f = \langle t_e \, \mathbb{F}^{-1} = \langle t \, \mathbb{F}^{-1} \, e^{-t/2} \tag{135}$$

Using Equation (134) in Equation (133)

$$\langle \ell_u = \langle p \, R \tag{136}$$

and multiplying Equation (136) by $e^{-t/2}$

$$\langle \ell_{u_f} = \langle p_f \, R \tag{137}$$

The inverse base change is now very simple, by Equation (131) in Equation (137)

$$\langle p_f = \langle \ell_{u_f} \, R \tag{138}$$

The metric matrix for the Poisson Functions is

$$M_{pp} = \int_0^\infty p(t)\rangle_f \, {}_f\langle p \, dt = R^T \left. \int_0^\infty \ell \rangle_{u_f} \langle \ell_{u_f} \, dt \right| R \tag{139}$$

or

$$M_{pp} = R^T R \quad . \tag{140}$$

The 5x5 case yields Equation (95) Appendix A

$$R^T R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 60 \end{bmatrix} \tag{141}$$

Thus we see that the Poisson polynomials are "oblique" that is "non-orthogonal".

If one has a finite vector function f(t) given by

$$f(t) = \langle \ell_{uf} \; a \rangle \tag{142}$$

and makes a base change given by Equation (36)

$$f(t) = \langle p_f(t) \; R \; a \rangle \tag{143}$$

$$= \langle p(t) \; a^P \rangle \tag{144}$$

we see that the connection between the coordinates is

$$a^P \rangle = R \; a \rangle \tag{145}$$

and by Equation (131)

$$a \rangle = R \; a^P \rangle \tag{146}$$

Thus Bolgiano states that the elements of these two sequences have been shown by Bédard to be related to each other by the symmetrical equations

$$(a_n)^p = \sum_{j=0}^{n} (-1)^j \binom{n}{j} a_j \tag{147}$$

and

$$(a_n) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (a_j)^p \tag{148}$$

which is an obvious deduction from Equation (145) and Equation (146). No further applications in this section are planned for the Poisson functions.

Another earlier paper in 1955 by Ule "Weighted Least-Squares Smoothing Filters" applies Laguerre polynomials. In an earlier paper 1954 W. Kautz "Transient Synthesis in The Time Domain" applies Laguerre polynomials. Indeed one can apply these polynomials with Kalman filters, where one for example, assures accelerations constant over a span, hence implying second degree polynomials. If we form the dyadic product of the Poisson Polynomials given by Equation (136) we have

$$P(t)\Big\rangle\Big\langle P(t) = \mathbb{F}^{-1} t\Big\rangle\Big\langle t\, \mathbb{F}^{-1} \tag{149}$$

and for the 4x4 case we have,

$$\begin{bmatrix} 1/_{0!} & 0 & 0 & 0 \\ 0 & 1/_{1!} & 0 & 0 \\ 0 & 0 & 1/_{2!} & 0 \\ 0 & 0 & 0 & 1/_{3!} \end{bmatrix} \begin{bmatrix} 1 & t & t^2 & t^3 \\ t & t^2 & t^3 & t^4 \\ t^2 & t^3 & t^4 & t^5 \\ t^3 & t^4 & t^5 & t^6 \end{bmatrix} \begin{bmatrix} 1/_{0!} & 0 & 0 & 0 \\ 0 & 1/_{1!} & 0 & 0 \\ 0 & 0 & 1/_{2!} & 0 \\ 0 & 0 & 0 & 1/_{3!} \end{bmatrix} \tag{150}$$

$$= \begin{bmatrix} {}^{1}/_{0!} & {}^{t}/_{0!} & {}^{t^2}/_{0!} & {}^{t^3}/_{0!} \\ {}^{t}/_{1!} & {}^{t^2}/_{1!} & {}^{t^3}/_{1!} & {}^{t^4}/_{1!} \\ {}^{t^2}/_{2!} & {}^{t^3}/_{2!} & {}^{t^4}/_{2!} & {}^{t^5}/_{2!} \\ {}^{t^3}/_{3!} & {}^{t^4}/_{3!} & {}^{t^5}/_{3!} & {}^{t^6}/_{3!} \end{bmatrix} \begin{bmatrix} {}^{1}/_{0!} & 0 & 0 & 0 \\ 0 & {}^{1}/_{1!} & 0 & 0 \\ 0 & 0 & {}^{1}/_{2!} & 0 \\ 0 & 0 & 0 & {}^{1}/_{3!} \end{bmatrix} \tag{151}$$

$$= \begin{bmatrix} {}^{1}/_{0!0!} & {}^{t}/_{0!1!} & {}^{t^2}/_{0!2!} & {}^{t^3}/_{0!3!} \\ {}^{t}/_{0!1!} & {}^{t^2}/_{1!1!} & {}^{t^3}/_{1!2!} & {}^{t^4}/_{1!3!} \\ {}^{t^2}/_{2!0!} & {}^{t^3}/_{2!1!} & {}^{t^4}/_{2!2!} & {}^{t^5}/_{2!3!} \\ {}^{t^3}/_{3!0!} & {}^{t^4}/_{3!1!} & {}^{t^5}/_{3!2!} & {}^{t^6}/_{3!3!} \end{bmatrix} \tag{152}$$

Clearly we could evaluate this matrix on the interval (0,1),(-1,1),(t_b,t_f) etc., and obtain finite elements.

D. CONTINUOUS EXPONENTIAL POLYNOMIALS. Exponential fitting functions are of interest to systems engineers, economists, statisticis, numerical analysis or wherever polynomials of the form can be used

$$x(t) = \langle d) f(t) \ a(d \rangle \tag{153}$$

where

$$(f_0, f_1, f_2, \cdots f_{d-1}) = (e^{-\lambda_0 t}, e^{-\lambda_1 t}, \dots e^{-\lambda_{d-1} t}) \tag{154}$$

Using the integration formula

$$\int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda} \tag{155}$$

From the dyadic product of Equation (154) for the 3x3 case and integral

$$\int_0^\infty f\rangle\langle f \, dt = \int_0^\infty \begin{pmatrix} e^{-\lambda_0 t} \\ e^{-\lambda_1 t} \\ e^{-\lambda_2 t} \end{pmatrix} (e^{-\lambda_0 t}, e^{-\lambda_1 t}, e^{-\lambda_2 t}) dt \tag{156}$$

$$\int_0^\infty \begin{bmatrix} e^{-2\lambda_0 t} & e^{-(\lambda_0+\lambda_1)t} & e^{-(\lambda_0+\lambda_2)t} \\ e^{-(\lambda_1+\lambda_0)t} & e^{-2\lambda_1 t} & e^{-(\lambda_1+\lambda_2)t} \\ e^{-(\lambda_2+\lambda_0)t} & e^{-(\lambda_2+\lambda_1)t} & e^{-2\lambda_2 t} \end{bmatrix} dt \tag{157}$$

Using Equation (155) in Equation (157)

$$\int_0^\infty f\rangle\langle f \, dt = \begin{bmatrix} \frac{1}{2\lambda_0} & \frac{1}{\lambda_0+\lambda_1} & \frac{1}{\lambda_0+\lambda_2} \\ \frac{1}{\lambda_1+\lambda_0} & \frac{1}{2\lambda_1} & \frac{1}{\lambda_1+\lambda_2} \\ \frac{1}{\lambda_2+\lambda_0} & \frac{1}{\lambda_1+\lambda_2} & \frac{1}{2\lambda_2} \end{bmatrix} \tag{158}$$

For the special case

$$\lambda_n = (n+1)\lambda_0 \tag{159}$$

for

$$n = 0,\ 1,\ 2,\ \cdots\ d-1$$

we have

$$\langle f = (e^{-\lambda_0 t},\ e^{-2\lambda_0 t},\ e^{-3\lambda_0 t}\ \ldots) \tag{160}$$

and Equation (160) in Equation (158) yields for the special case $\lambda_0 = 1$

$$M_{ee} = \left[\begin{matrix} -\dfrac{e^{-2t}}{2} & -\dfrac{e^{-3t}}{3} & -\dfrac{e^{-4t}}{4} \\ \dfrac{e^{-3t}}{3} & \dfrac{e^{-4t}}{4} & \dfrac{e^{-5t}}{5} \\ -\dfrac{e^{-4t}}{4} & -\dfrac{e^{-5t}}{5} & -\dfrac{e^{-6t}}{6} \end{matrix}\right]_{0}^{\infty} \tag{161}$$

$$= \left[\begin{matrix} 1/_2 & 1/_3 & 1/_4 \\ 1/_3 & 1/_4 & 1/_5 \\ 1/_4 & 1/_5 & 1/_6 \end{matrix}\right] = M_{ee} \tag{162}$$

Wendroff on page 120 of his book reference (89) calls the matrix

$$H_n = \begin{bmatrix} 1/_2 & 1/_3 & 1/_4 & 1/_5 & \cdots & 1/_{n+1} \\ 1/_3 & 1/_4 & 1/_5 & & & \\ 1/_4 & 1/_5 & & \ddots & & \\ 1/_5 & & & & & \\ \vdots & & & & & \\ 1/_{n+1} & & & & & 1/_{2n} \end{bmatrix} \qquad (163)$$

a Hilbert matrix of order n and says it is ill-conditioned also.

We find Fox in his book on page 75 gives the inverse of the 4x4 case of Equation (163) as

$$H^{-1}_{4x4} = \begin{bmatrix} 200 & -1200 & 2100 & -1120 \\ -1200 & 8100 & -15120 & 8400 \\ 2100 & -15120 & 29400 & -16800 \\ -1120 & 8400 & -16800 & 9800 \end{bmatrix} \qquad (164)$$

exactly. Paul Brock uses exponential polynomials in his paper "A Relation Between Exponential and Polynomial Methods For the Numerical Solution of Ordinary Differential Equations" reference (15).

E. <u>CONTINUOUS GRAM-POLYNOMIALS.</u> The Gram-Schmidt proceedure for the interval (0.1) was derived in the previous section. The general Hilbert matrix is

$$M_{tt}(0,1) = \int_0^1 t\rangle\langle t \; dt \qquad (165)$$

$$= \left[(i+j+1)^{-1} \right] \tag{166}$$

for

$$0 \leq i, \quad j \leq d-1$$

or in open form

$$\underset{d\times d}{M} = \begin{bmatrix} 1 & {}^{1}/_{2} & {}^{1}/_{3} & {}^{1}/_{4} & \cdots & {}^{1}/_{d} \\ {}^{1}/_{2} & {}^{1}/_{3} & {}^{1}/_{4} & & & \\ {}^{1}/_{3} & {}^{1}/_{4} & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ {}^{1}/_{d} & & . \quad . \quad . & . \quad . & & {}^{1}/_{2d-1} \end{bmatrix} \tag{167}$$

The Gram-Schmidt proceedure applied to the monomial base on the interval (0,1) is given by Equation (337) of Section (1) for the 3x3 case as

$$\langle g = \langle t \begin{bmatrix} 1 & -{}^{1}/_{2} & {}^{1}/_{6} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{168}$$

and for the orthonormal case by Equation (338) Section (1) as

$$\langle s = \langle t \begin{bmatrix} 1 & -\sqrt{3} & \sqrt{5} \\ 0 & 2\sqrt{3} & -6\sqrt{5} \\ 0 & 0 & 6\sqrt{5} \end{bmatrix} . \tag{169}$$

The polynomials of Equation (168) will be designated as the Modified Gram Polynomials as in the Laguerre case because of the sign differences and diagonal terms to be discussed next.

Milne in his book reference (60) on page 257 gives the general term for what I will call the Classical Gram Polynomials, the n^{th} element is

$$\ell_n(t) = \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{n+j}{j} t^j \tag{170}$$

for

$$n = 0,\ 1,\ 2,\ 3,\ \cdots\ d-1$$

By Equation (170) we have

$$\ell_0 = \langle d)\, t \begin{pmatrix} 1 & & \bigcirc & \\ & 0 & & \\ \bigcirc & & \ddots & \\ & & & 0 \end{pmatrix} \begin{pmatrix} (-1)^0 \binom{0}{0} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{171}$$

$$\ell_1 = \langle d)\, t \begin{pmatrix} \binom{1}{0} & & & \\ & \binom{2}{1} & \bigcirc & \\ & & 0 & \\ \bigcirc & & & \ddots \\ & & & & 0 \end{pmatrix} \begin{pmatrix} (-1)^0 \binom{1}{0} \\ (-1)^1 \binom{1}{1} \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{172}$$

$$\ell_2 = \langle t \begin{pmatrix} \binom{2}{0} & & & & \\ & \binom{3}{1} & & \bigcirc & \\ & & \binom{4}{2} & & \\ & \bigcirc & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} (-1)^0 \binom{2}{0} \\ (-1)^1 \binom{2}{1} \\ (-1)^2 \binom{2}{2} \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{173}$$

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$$\ell_n = \langle d)t \begin{pmatrix} \binom{n}{0} & & & \\ & \binom{n+1}{1} & \bigcirc & \\ & & \binom{n+2}{2} & \\ \bigcirc & & & \ddots \\ & & & & \binom{2n}{n} \end{pmatrix} \begin{pmatrix} (-1)^0 \binom{n}{0} \\ (-1)^1 \binom{n}{1} \\ (-1)^2 \binom{n}{2} \\ \vdots \\ (-1)^j \binom{n}{j} \\ \vdots \\ (-1)^n \binom{n}{n} \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{174}$$

and for the d^{th} vector $n = d-1$

$$\ell_{d-1} = \langle t \begin{bmatrix} \binom{d-1}{0} & & & & & \\ & \binom{d}{1} & & & & \\ & & \binom{d+1}{2} & & & \\ & & & \ddots & & \\ & & & & & \binom{2d-2}{d-1} \end{bmatrix} \begin{bmatrix} (-1)^{0}\binom{d-1}{0} \\ (-1)^{1}\binom{d-1}{1} \\ \vdots \\ (-1)^{d-1}\binom{d-1}{d-1} \end{bmatrix} \tag{175}$$

The first nine Gram polynomials are given by Milne reference (60) on page 260 as

$$\langle \ell = \langle t \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & -2 & -6 & -12 & -20 & -30 & -42 & -56 & -72 \\ & & 6 & 30 & 90 & 210 & 420 & 756 & 1260 \\ & & & -20 & -140 & -560 & -1680 & -4200 & -9240 \\ & & & & 70 & 630 & 3150 & 11550 & 34650 \\ & & & & & -252 & -2772 & -16632 & -72072 \\ & & & & & & 924 & 12102 & 84084 \\ & & & & & & & -3432 & -51480 \\ & & & & & & & & 12870 \end{bmatrix} \tag{176}$$

If we compare the first three of these we see as before that the signs differ as well as the main diagonals of the Classical Gram polynomials are not unity. Normalizing these diagonal elements by Equation (73) we obtain

$$\langle g = \langle t \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/6 \end{bmatrix} \tag{177}$$

$$\langle g = \langle t \begin{bmatrix} 1 & -1/2 & 1/6 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{178}$$

which is an agreement with Equation (168) for the <u>modified Gram</u> polynomials obtained via the Gram-Schmidt procedure.

The metric-matrix for the classical Gram polynomials is given by Milne on page 261 of reference (60) as

$$\int_0^1 \ell_n^2(t)\,dt = \frac{1}{2_{n+1}} \tag{179}$$

for

$$n = 0,\ 1,\ 2,\ \cdots\ d-1$$

and for the dxd case

$$M_{\ell\ell} = \int_0^1 \ell\rangle\langle\ell \; dt = \begin{bmatrix} 1 & & & & & \\ & 1/_3 & & & & \\ & & 1/_5 & & & \\ & & & 1/_7 & & \\ & & & & \ddots & \\ & & & & & \frac{1}{2d-1} \end{bmatrix} \tag{180}$$

or element wise

$$\boxed{M_{\ell\ell} = \left[\frac{1}{2n+1}\right]} \tag{181}$$

The metric-matrix of Equation (180) will be referred to as the Gram-polynomial metric-matrix. The orthonormal vectors are obtained by

$$\langle\ell_u = \langle\ell \; M_{\ell\ell}^{-\frac{1}{2}} \tag{182}$$

or

$$\langle\ell_u = \langle\ell \begin{bmatrix} 1 & & & & \\ & \sqrt{3} & & & \\ & & \sqrt{5} & & \\ & & & \ddots & \\ & & & & \sqrt{2d-1} \end{bmatrix} \tag{183}$$

which are the Classical orthonormal Gram polynomials.

Returning to Equation (171) we have

$$\ell_0 = \langle t \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \ell_{0m} \tag{184}$$

By Equation (172) normalizing the second coordinate

$$\frac{\ell}{\binom{2}{1}(-1)} = \ell_{1m} = \langle t \begin{pmatrix} \frac{1}{\binom{2}{1}(-1)} \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{185}$$

the n^{th} term by Equation (174) is

$$\frac{\ell_n}{\binom{2n}{n}(-1)^n} = \ell_{nn} \tag{186}$$

and the d^{th} vector is

$$\frac{\ell_{d-1}}{\binom{2d-2}{d-1}(-1)^{d-1}} = \ell_{(d-1)m} \tag{187}$$

The connection matrix is

$$\langle \ell_m = \langle \ell \left[\binom{2n}{n} \right]^{-1} I(-1) \tag{188}$$

where

$$L = \left[\binom{2n}{n}\right] = \begin{bmatrix} \binom{0}{0} & & & & & & & \\ & \binom{2}{1} & & & & & & \\ & & \binom{4}{2} & & & & & \\ & & & \binom{6}{3} & & & & \\ & & & & \ddots & & & \\ & & & & & \binom{2n}{n} & & \\ & & & & & & \ddots & \\ & & & & & & & \binom{2d-2}{d-1} \end{bmatrix} \tag{189}$$

and by Equation (34) in Appendix A

$$\binom{2n}{n} = \frac{(2n)!}{n!(2n-n)!} \tag{190}$$

or

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \tag{191}$$

The metric-matrix of the modified Gram polynomial is by Equation (188)

$$\int_0^1 \ell\rangle_m \langle\ell_m \, dt = I(-1)L^T M_{\ell\ell} L\, I(-1) \qquad (192)$$

and by Equation (181) and (187)

$$M_{\ell\ell_m} = I(-1) \left[\binom{2n}{n}\right] \left[\frac{1}{2n+1}\right] \left[\binom{2n}{n}\right] I(-1) \qquad (193)$$

for

$$n = 0,\ 1,\ 2,\ \cdots\ d-1$$

a diagonal matrix.

Using Equation (191) in the inner matrix product of Equation (193) all diagonal matrices

$$\left[\binom{2n}{n}\right]\left[\frac{1}{2n+1}\right]\left[\binom{2n}{n}\right] = \left[\frac{1}{(2n+1)} \quad \frac{(2n!)^2}{(n!)^4}\right] \qquad (194)$$

and the metric matrix for the <u>modified Gram</u> polynomials is

$$\boxed{M_{\ell\ell_m} = \text{dig}\left[\frac{1}{(2n+1)}\frac{(2n!)^2}{(n!)^4}\right]} \qquad (195)$$

since I(-1) enters as a square with respect to diagonal matrices in Equation (193).

One can now obtain the inverse of the Hilbert matrix by Equation (348) of section (1)

$$M_{tt}^{-1}\ (0,1) = H_{i_{\ell\ell}}^{-1} = B_s B_s^T \qquad (196)$$

where the matrix B_s connects the orthonormal Gram polynomials to the monomial base, that is

$$\langle \ell_u = \langle t\ B_s = \ \ell\ \ M_{\ell\ell}^{-1/2} \tag{197}$$

where $\langle \ell$ is given by Equation (176) $M_{\ell\ell}^{-1/2}$ is obtained via Equation (181).

The details will not be carried through here.

D. CONTINUIOUS LEGENDRE POLYNOMIALS. The continuious modified Legendre polynomials on the interval (-1, 1) were obtained via a Gram-Schmidt process for the 3x3 case in the previous section (1) Equation (383). This section presents some relations for the general case on the interval (-1,1), and a latter section derives the state-space relations for an arbitrary interval and a re-scaled time axis.

The general metric for (-1,1) is

$$\int_{-1}^{1} f\rangle\langle f\ dt = \int_{-1}^{0} f\rangle\langle f\ dt + \int_{0}^{1} f\rangle\langle f\ dt \tag{198}$$

and

$$\int_{-1}^{0} f\rangle\langle f\ dt = -\int_{0}^{-1} f\rangle\langle f\ dt \tag{199}$$

For the monomial base

$$\langle f = \langle t \tag{200}$$

we have

$$\int_0^{-1} t\rangle\langle t \, dt = \left[\begin{array}{cccccc} t & \frac{t^2}{2} & \frac{t^3}{3} & \frac{t^4}{4} & \cdots & \frac{t^d}{d} \\ \frac{t^2}{2} & & & & & \\ \vdots & & & & & \\ \frac{t^d}{d} & & & & & \frac{t^{2d-1}}{2d-1} \end{array}\right]_0^{-1} \tag{201}$$

(202)

$$\int_0^{-1} t\rangle\langle t \, dt \quad \left[\begin{array}{cccccc} -1 & \frac{1}{2} & \frac{-1}{3} & \frac{1}{4} & \cdots & \frac{(-1)^d}{d} \\ \frac{1}{2} & \frac{-1}{3} & \frac{1}{4} & & & \\ \frac{1}{4} & & & & & \\ \vdots & & & & & \\ \frac{(-1)^d}{d} & & & & & \frac{(-1)^{2d-1}}{2d-1} \end{array}\right]$$

Using Equation (202) and Equation (167) in Equation (198)

$$\int_{-1}^{1} t\rangle\langle t \, dt = \begin{bmatrix} 2 & 0 & \frac{2}{3} & 0 & \frac{2}{5} & \cdots \\ 0 & \frac{2}{3} & 0 & \frac{2}{5} & & \\ \frac{2}{3} & 0 & \frac{2}{5} & 0 & & \\ 0 & \frac{2}{5} & 0 & \frac{2}{7} & & \\ \frac{2}{5} & & & & & \\ 0 & & & & & \\ \vdots & & & & & \end{bmatrix} \qquad (203)$$

The 3x3 modified Legendre polynomial case is given in the previous section by Equation (384). Applying the Gram-Schmidt formula of Equation (308) of that same section, one finds

$$g_3 = \langle d)t \begin{pmatrix} 0 \\ -\frac{3}{5} \\ 0 \\ 1 \end{pmatrix} \qquad (204)$$

One can continue the orthogonization process, the first nine modified Lagendre polynomials are given by Barker in Reference (9) as (The modified terminology is due to Barker).

$$\langle g = \langle t \begin{bmatrix} 1 & 0 & \frac{-1}{3} & 0 & \frac{3}{35} & 0 & \frac{-5}{231} & 0 & \frac{7}{1287} \\ 0 & 1 & 0 & \frac{-3}{5} & 0 & \frac{5}{21} & 0 & \frac{-35}{429} & 0 \\ 0 & 0 & 1 & 0 & \frac{-6}{7} & 0 & \frac{5}{11} & 0 & \frac{-28}{143} \\ 0 & 0 & 0 & 1 & 0 & \frac{-10}{9} & 0 & \frac{105}{143} & 0 \\ & & & & 1 & 0 & \frac{-15}{11} & 0 & \frac{14}{13} \\ & & & & & 1 & 0 & \frac{-21}{13} & 0 \\ & & & & & & 1 & 0 & \frac{-28}{15} \\ & & & & & & & 1 & 0 \\ & & & & & & & & 1 \end{bmatrix} \qquad (205)$$

Note the alternation of zeros in the coordinates and the corresponding alternation of zeros in the metric on the interval of Equation (203). The first three modified Legendre polynomials are given by Davis on page (168) of reference and agree with those derived here via the Gram-Schmidt approach. Davis calls them Legendre polynomials even though he uses the Gram-Schmidt proceedure as done in this report.

The metric for the first four vectors of Equation (205) is

$$M_{gg} = \int_{-1}^{1} g \rangle\!\langle g \, dt = \begin{bmatrix} 2 & & & \\ & \frac{2}{3} & & \\ & & \frac{8}{45} & \\ & & & \frac{8}{175} \end{bmatrix} \qquad (206)$$

The first three orthonormal Modified Legendre polynomials are given by Equation (384) of section (1) as

$$\langle s = \langle t \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-\sqrt{5}}{2\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{3\sqrt{5}}{\sqrt{2}} \end{bmatrix} \qquad (207)$$

Erdelyi on page 180 of reference (29) and Rice on page (68) of reference both given the following general formula for the n^{th} Legendre polynomial,

$$\boxed{\ell_n(t) = 2^{-n} \sum_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \binom{n}{j} \binom{2(n-j)}{n} t^{n-2j}} \qquad (208)$$

$n = 0, 1, 2, \cdots d\text{-}1$

where $\left[\frac{n}{2}\right]$ means the greatest integer less than or equal to $\frac{n}{2}$ depending on whether n is odd or even.

If we evaluate the formula given by Equation (208) for the first seven polynomials we obtain the first seven "Classical" Legendre polynomials.

$$\langle 7)\ell(t) = \langle t \begin{bmatrix} 1 & 0 & \frac{-1}{2} & 0 & \frac{3}{8} & 0 & \frac{-5}{16} \\ 0 & 1 & 0 & \frac{-3}{2} & 0 & \frac{15}{8} & 0 \\ 0 & 0 & \frac{3}{2} & 0 & \frac{-15}{4} & 0 & \frac{105}{16} \\ 0 & 0 & 0 & \frac{5}{2} & 0 & \frac{-35}{4} & 0 \\ 0 & 0 & & & \frac{35}{8} & 0 & \frac{-315}{16} \\ 0 & 0 & & & & \frac{63}{8} & 0 \\ 0 & 0 & & & & & \frac{231}{16} \end{bmatrix} \qquad (209)$$

We see that the main diagonal terms are not unity, hence the polynomials are not the result of a Gram-Schmidt as applied in the previous section. Rainville on page (160) of reference gives exactly the polynomials of Equation (208). Also Hildebrand on page 273 of reference gives the first six polynomials which agree precisely with the first six of Equation (209).

Hildebrand on page 273 describes the n^{th} Legendre polynomial in the usual way as

$$\ell_n(t) = \frac{1}{2^n} \frac{1}{n!} \frac{d^n}{dt^n} (t^2-1)^n \tag{210}$$

which is the Rodrigues' formula.

Luenberger on page 61 of reference (54) relates the polynomials applied on the (-1,1) interval via the Gram-Schmidt orthonormal proceedure to the well known Legendre polynomials satisfying Equation (210) and Equation (208) as

$$(\ell_u)_n = \frac{\sqrt{2n+1}}{2} \ell_n(t) \tag{211}$$

for

$$n = 0, 1, 2, \cdots d-1$$

and package-wise we have

$$\langle \ell_u = \langle \ell \begin{bmatrix} \frac{1}{\sqrt{2}} & & & & & & & & \\ & \sqrt{\frac{3}{2}} & & & & & & & \\ & & \sqrt{\frac{5}{2}} & & & & & & \\ & & & \sqrt{\frac{7}{2}} & & & & & \\ & & & & \ddots & & & & \\ & & & & & \sqrt{\frac{2n+1}{d}} & & & \\ & & & & & & \ddots & & \\ & & & & & & & & \sqrt{\frac{2d-1}{2}} \end{bmatrix}$$

which connects the Classical orthogonal to the orthonormal polynomials.

Luenberger on page 61 of reference (54) states that the polynomials obtained on the (-1,1) interval via the orthonormal proceedure are related to the well known Legendre polynomials satisfying Equation (210) via

$$(\ell_u)_n = \sqrt{\frac{2n+1}{2}} \quad \ell_n(t) \ . \tag{212}$$

Luenberger is correct in his statement for this case since we see by Equation (209) the main-diagonal non-unity numbers are all positive, hence the alternating signs which occurred in the Laguerre and Gram polynomials are not present for the Legendre polynomials.

Rice also gives on page 47 of reference (75), the elements of the diagonal metric-matrix for the Classical Legendre polynomials as

$$\boxed{\int_{-1}^{1} \ell_n^{\,2}(t)dt = \frac{2}{2n+1}} \qquad (213)$$

the metric-matrix (diagonal) is

$$\boxed{M_{\ell\ell}(-1,1) = \int_{-1}^{1} t \rangle\!\langle t \; dt = \text{dig} \left[\frac{2}{2n+1}\right]} \qquad (214)$$

In open form the Classical Legendre polynomial metric-matrix is for the dxd case

$$\underset{dxd}{M_{\ell\ell}(-1,1)} = \begin{bmatrix} 2 & & & & & & & & \\ & \frac{2}{3} & & & & & & & \\ & & \frac{2}{5} & & & & & & \\ & & & \frac{2}{7} & & & & & \\ & & & & \ddots & & & & \\ & & & & & \frac{2}{2n+1} & & & \\ & & & & & & \ddots & & \\ & & & & & & & \frac{2}{2d-1} \end{bmatrix} \qquad (215)$$

Clearly the normalizing matrix to obtain the orthonormal Classical Legendre polynomials is

$$\langle \ell_u(t) = \langle \ell(t)\ M_{\ell\ell}^{-1/2} = \langle \ell \ \mathrm{dig} \ \left[\frac{2}{2n+1}\right]^{-1/2} \tag{216}$$

or

$$\langle \ell_u(t) = \langle \ell \ \mathrm{dig} \ \left(\sqrt{\frac{2n+1}{2}}\right) . \tag{217}$$

which are the elements of Equation (211).

The connection between the Classical (non-orthogonal) polynomials of Equation (209) and the modified Legendre polynomials of Equation (205) derived via the Gram-Schmidt proceedure of this report can be obtained. By Equation (73) we have for the 4x4 case

$$\langle \ell_m = \langle \ell \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \frac{2}{3} & \\ & & & \frac{2}{5} \end{bmatrix} \tag{218}$$

or

$$\langle \ell_m = \langle t \begin{bmatrix} 1 & 0 & -1/_2 & 0 \\ 0 & 1 & 0 & -3/_2 \\ 0 & 0 & 3/_2 & 0 \\ 0 & 0 & 0 & 5/_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2/_3 & 0 \\ 0 & 0 & 0 & 2/_5 \end{bmatrix} \qquad (219)$$

which yields

$$\langle \ell_m = \langle t \begin{bmatrix} 1 & 0 & -1/_3 & 0 \\ 0 & 1 & 0 & -3/_5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (220)$$

which agrees with Equation (205).

If we put the polynomials in Equation (208) in monic form we obtain the connection matrix between the modified Legendre polynomials and the Classical Legendre polynomials

$$\langle \ell_m = \langle \ell \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & \binom{4}{2}^{-1} 2^2 & & & & & & \\ & & & \binom{6}{3}^{-1} 2^3 & & & & & \\ & & & & \binom{8}{4}^{-1} 2^4 & & & & \\ & & & & & \ddots & & & \\ & & & & & & \binom{2n}{n}^{-1} 2^n & & \\ & & & & & & & \ddots & \\ & & & & & & & & \binom{2d-2}{d-1}^{-1} 2^{d-1} \end{bmatrix} \qquad (221)$$

or

$$\langle\ell_m = \langle\ell \ \mathrm{dig}\left[\binom{2n}{n}^{-1} 2^{n}\right] \qquad (222)$$

where dig (   ) designates the diagonal matrix of Equation (221).

We can now obtain the metric-matrix for the modified Legendre polynomials via Equation (222)

$$\int_{-1}^{1} \ell\rangle_m \langle\ell_m \ dt = \mathrm{dig}\left[\binom{2n}{n}^{-1} 2^{n}\right] M_{\ell\ell}\ \mathrm{dig}\left[\binom{2n}{n}^{-1} 2^{n}\right] \qquad (223)$$

or by Equation (215)

$$M_{\ell\ell_m} = \mathrm{dig}\left[\frac{2^{2n+1}}{(2n+1)} \binom{2n}{n}^{-2}\right] \quad . \qquad (224)$$

With the transformations derived above one can relate Legendre polynomials to the Laguerre polynomials etc.

Section 3

TRANSLATION OF ORIGIN AND SCALING ON THE TIME AXIS AND INDUCED TRANSFORMATION METRIC MATRICES AND GRAM-SCHMIDT VECTOR. The continuous metric matrices and the Gram-Schmidt obtained orthogonal polynomials of the previous section were all with respect to inner-products on classical intervals. This section will extend the classical intervals (0,1) (-1,1), $(0,t_2)$etc., to arbitrary intervals on the independent variable axis (time) and derive a number of the corresponding polynomials orthogonal on the arbitrary continuous intervals which actually occur in optimal estimation and approximation theory in practice. The necessary induced transformations for the monomial base metrics on these arbitrary intervals and the corresponding orthogonal polynomials with respect to the arbitrary metrics will be defined in terms of new upper triangular connection matrices for Gram, Legendre, and Laguerre polynomials over thse intervals.

Monomial Base Metric Matrix for Arbitrary Intervals. For the purposes of this section we will assume that the classical time axis variable is τ that ranges over the classical intervals and we have a linear relation

$$\tau = b_0 + B_1 \; t = \langle tb\rangle \qquad (1)$$

where b_0 is the translation to the new origin and b_1 is the scale - factor.

Consider the metric matrix

$$\int_{t_1}^{t_2} t\rangle\langle t \; dt = M_{tt}(t_1 t_2) \qquad (2)$$

Case I - Transformation Metric $(0,1) \rightarrow (0,t_2)$. The first case considered is the interval

$$(\tau_1,\tau_2) = (0,1) \qquad (3)$$

for which the metric is the standard, Hilbert matrix given by Equation

$$\int_0^1 \tau\rangle\langle\tau \; dt = M(0,1) = H_{i11} \qquad (4)$$

and we want the metric for the inner product defined on the variable t for the arbitrary interval

$$(t_1 t_2) = (0,t_2) \tag{5}$$

By Equation (1)

$$\tau_1 = 0 = (1,0)\, b\rangle \tag{6}$$

and

$$\tau_2 = 1 = (1,t_2)\, b\rangle \tag{7}$$

Packaging Equation (6) and Equation (7)

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & t_2 \end{bmatrix} b\rangle \tag{8}$$

or

$$b\rangle = \frac{1}{t_2} \begin{bmatrix} t_2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{t_2} \tag{9}$$

Using Equation (9) in Equation (1) we find

$$\tau = \frac{t}{t_2} = \frac{t}{w_g} \tag{10}$$

where the interval width is

$$\int_0^{t_2} 1\, dt = t_2 = w_g \tag{11}$$

hence

$$t = w_g \tau = t_2 \tau \tag{12}$$

and

$$dt = t_2 d\tau \tag{13}$$

Packaging the powers of Equation (12)

$$\langle t = \langle \tau D(t_2^n) \tag{14}$$

where the diagonal matrix

$$D(t_2^n) - \begin{bmatrix} 1 & & & & & \\ & t_2 & & & & \\ & & t_2^2 & & & \\ & & & t_2^3 & & \\ & & & & \ddots & \\ & & & & & t_2^{d-1} \end{bmatrix} \tag{15}$$

Transposing Equation (14) and forming the metric

$$\int_0^{t_2} t \rangle \langle t \, dt = D(w_g) \int_0^1 \tau \rangle \langle \tau \; dt \; D(w_g) t_2 \tag{16}$$

or

$$\boxed{M_{tt}(0,t_2) = t_2 D(t_2^n) \; M_{tt}(0,1) D(t_2^n)} \tag{17}$$

Case II = Transformation Metric (-1,1) → $(-t_2,t_2)$. Consider next the case for the interval

$$(\tau_1,\tau_2) = (-1,1) \tag{18}$$

and we want the metric for the case of symmetry about 0

$$(t_1,t_2) - (-t_2,t_2) \tag{19}$$

as shown in Figure (1)

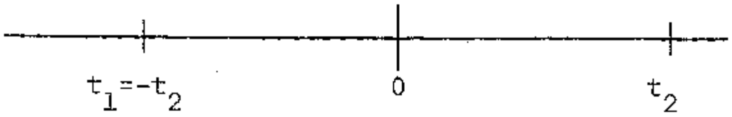


FIGURE (1)
SYMMETRIC SPAN ABOUT ORIGIN

Using the limits of Equation (18) and (19) in Equation (1)

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -t_2 \\ 1 & t_2 \end{bmatrix} b\rangle \tag{20}$$

Inverting

$$b\rangle = \begin{bmatrix} t_2 & t_2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \frac{1}{2t_2} \tag{21}$$

$$b\rangle = \begin{bmatrix} t_2 & t_2 \\ -1 & 1 \end{bmatrix} \frac{1}{2t_2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{t_2} \tag{22}$$

Using Equation (22) in Equation (1)

$$\tau = \frac{t}{t_2} \tag{23}$$

$$\int_{-t_2}^{t_2} 1dt = 2t_2 = w_\ell \tag{24}$$

Hence in terms of the interval width (Legendre interval width w_ℓ)

$$\tau = \frac{t_2}{w_\ell} \tag{25}$$

or

$$t = \frac{w_\ell}{2}\tau = t_2\tau \qquad (26)$$

and

$$dt = t_2 \, d\tau \qquad (27)$$

and packagewise

$$\langle t = \langle \tau \begin{bmatrix} 1 & & & & & \\ & t_2 & & & & \\ & & t_2^2 & & & \\ & & & t_2^3 & & \\ & & & & \ddots & \\ & & & & & t_2^{d-1} \end{bmatrix} \qquad (28)$$

or

$$\boxed{\langle t = \langle \tau D(t_2^n)} \qquad (29)$$

Note that Equation (29) is the same transformation as Equation (14), however the metrics are different, since

$$\int_{-t_2}^{t_2} t\rangle\langle t \, dt - t_2 D(t_2^n) \int_{-1}^{1} t\rangle\langle t \, dt \, D(t_2^n) \qquad (30)$$

or

$$\boxed{M(-t_2,t_2) = t_2 D(t_2^n) M(-1,1) D(t_2^n)} \qquad (31)$$

since the metric M(-1,1) has zeros and even powers as given by Equation (145) of Section (4).

Case III - Transformation Metric $(0,1) \rightarrow (t_1,t_2)$. Consider next a translation to a new origin at t, as shown in Figure (2)

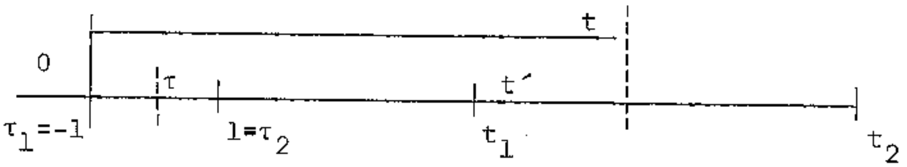


FIGURE (2)
TRANSLATION TO t ORIGIN

where the interval

$$(\tau_1,\tau_2) = (0,1) \tag{32}$$

is translated and scaled to

$$(t_1,t_2) = (t_1,t_2) \tag{33}$$

Using the integration end points of Equation (32) and Equation (33) in Equation (1)

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \end{bmatrix} b\rangle \tag{34}$$

and inverting

$$b\rangle = \frac{1}{t_2-t_1} \begin{bmatrix} t_2 & -t_1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{35}$$

or

$$b\rangle = \frac{1}{t_2-t_1} \begin{bmatrix} -t_1 \\ 1 \end{bmatrix} \tag{36}$$

Using Equation (36) in Equation (1)

$$\tau = \frac{-t_1}{t_2-t_1} + \frac{t}{t_2-t_1} \tag{37}$$

and solving for t

$$t = t_1 + (t_2-t_1)\tau \tag{38}$$

$$dt = (t_2-t_1)d\tau \tag{39}$$

The width of the integration interval is

$$\int_{t_1}^{t_2} 1dt = t_2-t_1 = w_\ell \tag{40}$$

By Figure (2) set

$$t' = w_\ell \tau \tag{41}$$

hence

$$t = t_1 + t' \tag{42}$$

and

$$dt = dt' = w_\ell d\tau \tag{43}$$

If t_1 is different from zero normalize Equation (42) as non-dimensional time

$$\frac{t}{t_1} = \frac{t'}{t_1} + 1 \tag{44}$$

or

$$x = x' + 1 \tag{45}$$

Using Equation (22) of Appendix A

$$\langle x = \langle x' \; B \tag{46}$$

where B is the binomial matrix. Also

$$\langle x = \langle t \begin{bmatrix} 1 & & & & \\ & \frac{1}{t_1} & & & \\ & & \frac{1}{t_1^2} & & \\ & & & \ddots & \\ & & & & \frac{1}{t_1^{d-1}} \end{bmatrix} \tag{47}$$

and by Equation (41)

$$\langle x' = \langle \tau \begin{bmatrix} 1 & & & \\ & \frac{1}{t_1} & & \\ & & \ddots & \\ & & & \frac{1}{t_1^{d-1}} \end{bmatrix} \tag{48}$$

Using Equation (47) and Equation (48) in Equation (46)

$$\begin{aligned} \langle t &= \langle \tau \; D^{-1}(t_1^n) \; B \; D(t_1^n) \\ &= \langle \tau \; T_{\tau t} \end{aligned} \tag{49}$$

Transposing Equation (49)

$$t\rangle = D(t_1^n) \; B^T \; D^{-1}(t_1^n) \; \tau\rangle = T^T_{\tau t} \; \tau\rangle \tag{50}$$

and forming the metric-matrix

$$\int_{t_1}^{t_2} t\rangle \langle t \; dt = w_d T_{\tau t}^T \left[\int_0^1 \tau\rangle \langle \tau \; dt \right] T_{\tau t} \tag{51}$$

or

$$M_{tt}(t_1,t_2) = w_\ell T^T \; M_{\tau\tau}(0,1) \; T \tag{52}$$

where

$$T_{\tau t} = D^{-1}(t_1^n) \; BD(t_1^n) \tag{53}$$

Using Equation (53) in Equation (52)

$$M_{tt}(t_1,t_2) = (t_2-t_1) \; D(T_1^n) \; B^T D^{-1}(t_1^n) \; H_{i\ell\ell} D^{-1}(t_1^n) \; BD(t_1^n) \tag{54}$$

As example of the validity and the appearance of the results of Equation (54) consider the 3χ3 case which the left hand side of Equation (54) is

$$M_{tt}(t_1 t_2) = \int_{t_1}^{t_2} t\rangle \langle t \; dt = \begin{bmatrix} t_2-t_1 & \frac{t_2^2-t_1^2}{2} & \frac{t_2^3-t_1^3}{3} \\ \frac{t_2^2-t_1^2}{2} & \frac{t_2^3-t_1^3}{3} & \frac{t_2^4-t_1^4}{4} \\ \frac{t_2^3-t_1^3}{3} & \frac{t_2^4-t_1^4}{4} & \frac{t_2^5-t_1^5}{5} \end{bmatrix} \tag{55}$$

The three center matrices of the right hand side of Equation (54) are, where w is given by Equation (40),

$$w_g \; D^{-1}(t_1^n) \; H_{i\ell\ell} \; D^{-1}(tn) = \begin{bmatrix} w & \frac{w^2}{2t_1} & \frac{w^3}{3t_1^2} \\ \frac{w^2}{2t_1} & \frac{w^3}{3t_1^2} & \frac{w^4}{4t_1^3} \\ \frac{w^3}{3t_1^2} & \frac{w^4}{4t_1^4} & \frac{w^5}{5t_1^5} \end{bmatrix} \tag{56}$$

and Equation (56) used in Equation (54) is

$$M_{tt}(t_1,t_2) =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ t_1 & t_1 & 0 \\ t_1^2 & 2t_1^2 & t_1^2 \end{bmatrix} \begin{bmatrix} w & \frac{w^2}{2t_1} & \frac{w^3}{3t_1^3} \\ \frac{w^2}{2t_1} & \frac{w^3}{3t_1^2} & \frac{w^4}{4t_1^3} \\ \frac{w^3}{3t_1^2} & \frac{w^4}{4t_1^4} & \frac{w^5}{5t_1^5} \end{bmatrix} \begin{bmatrix} 1 & t_1 & t_1^2 \\ 0 & t_1 & 2t_1^2 \\ 0 & 0 & t_1^2 \end{bmatrix} \tag{57}$$

$$= w \begin{bmatrix} 1 & \frac{w}{2t_1} & \frac{w^2}{3t_1} \\ \frac{2t_1+w}{2} & \frac{w}{2} + \frac{w^2}{3t_1} & \frac{w^2}{3t_1} + \frac{w^3}{4t_1^2} \\ \frac{t_1^2+t_1w+w^2}{4} & wt_1+w^2 \; \frac{2}{3} + \frac{w^3}{4t_1} & \frac{w^2}{3} + \frac{w^3}{2t_1} + \frac{w^5}{5t_1^2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & t_1 & t_1^2 \\ 0 & t_1 & 2t_1^2 \\ 0 & 0 & t_1^2 \end{bmatrix} \tag{58}$$

If one multiplies out the terms of Equation (53) and uses the relations

$$\begin{aligned} t_2^2 - t_1^2 &= (t_2-t_1)(t_2+t_1) \\ t_2^3 - t_1^3 &= (t_2-t_1)(t_2^2+t_2t_1+t_1^2) \\ (t_2^4-t_1^4) &= (t_2-t_1)(t_2^3+t_2^2t_1+t_2t_1^2+t_1^3) \\ (t_2^5-t_1^5) &= (t_2-t_1)(t_2^4+t_2^3t_1+t_2^2t_1^2 - t_2t_1^2+t_1^4) \end{aligned} \tag{59}$$

or in general for x and y

$$x^n - y^n = (x-y)(1,y,y^2,\ldots y^{n-1}) \begin{bmatrix} & & & 1 \\ 0 & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & 0 \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{n-1} \end{bmatrix} \tag{60}$$

or

$$x^n - y^n = x \left\langle n)y \; L \; x(n \right\rangle - y \left\langle y \; L \; x(n \right\rangle \tag{61}$$

where the linear convolution matrix L is

$$L = \begin{bmatrix} 0 & & & 1 \\ & & & 1 \\ & & 1 & \\ 0 & \cdot^{\cdot^{\cdot}} & & \\ & & 0 & \\ 1 & 0 & & \end{bmatrix} \qquad (62)$$

It is shown in Equation (117) of Section (4) that

$$M_{tt}(t_1 t_2) = t_2\ D(t_2^n)\ H_{i\ell\ell}\ D(t_1^n) \qquad (63)$$

$$-t_1\ D(t_1^n)\ H_{i\ell\ell}\ D(t_1^n)$$

If t_1=0 Equation (63) reduces to

$$M_{tt}(0,t_2) = t_2\ D(t_2^n)\ H_{i\ell\ell}\ D(t_2^n) \qquad (64)$$

Which agrees with Equation (17).

Case IV - Transformation Metric $(-1,1) \rightarrow (-t_2,t_c,t_2)$. Consider next the translated and scaled symmetric interval corresponding to (-1,1) the Legendre case about a translated origin at t_c the center point of the symmetric interval as shown in Figure (3).

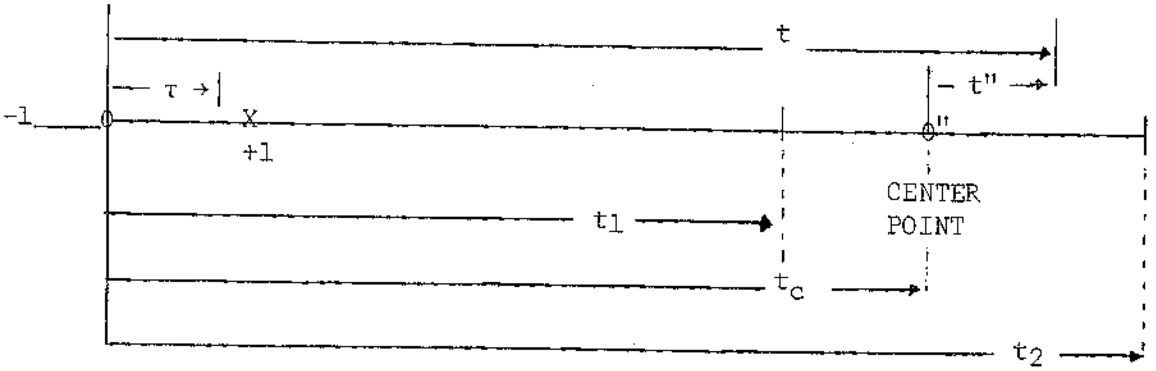


FIGURE (3)
SYMMETRIC SPAN TRANSLATED TO t_c

The variables are

$$t = t_c + t'' \tag{65}$$

By Equation (1) the τ variables end points with midpoint at zero is

$$(\tau_1, \tau_c, \tau_2) = (-1, 0, 1) \tag{66}$$

Consider next the translated and rescaled end points, the span length is

$$\int_{t_1}^{t_2} dt = t_2 - t_1 = \omega_\ell \tag{67}$$

and the midpoint is translated to

$$t_c = t_1 + \frac{t_2 - t_1}{2} \tag{68}$$

or

$$t_c = \frac{t_1 + t_2}{2} \tag{69}$$

The end points in Equation (1) yield

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \end{bmatrix} b> \tag{70}$$

or solving for $b>$

$$b> = \begin{bmatrix} \dfrac{(t_1 + t_2)}{(t_2 - t_1)} \\ \\ \dfrac{2}{(t_2 - t_1)} \end{bmatrix} \tag{71}$$

Using Equation (71) in Equation (1)

$$\tau(t) = -\frac{(t_2+t_1)}{(t_2-t_1)} + \frac{2}{(t_2-t_1)} t \tag{72}$$

and solving for t

$$t = \frac{(t_2+t_1)}{2} + \frac{(t_2-t_1)}{2} \tau \tag{73}$$

or by Equation (64)

$$t = t_c + t'' \tag{74}$$

where

$$t'' = \frac{(t_2-t_1)}{2} \tau = \frac{\omega_\ell}{2} \tau \tag{75}$$

The differentials are

$$dt = \frac{(t_2-t_1)}{2} d\tau \tag{76}$$

$$dt = dt'' \tag{77}$$

We see by Equation (73) if $t_1 = -t_2$ then Equation 68) reduces to Equation (26).

If we normalize Equation Equation (74)

$$\frac{t}{t_c} = \frac{t''}{t_c} + 1 \tag{78}$$

or

$$x = x'' + 1 \tag{79}$$

which is of the form of Equation (45) we have

$$<x = <x''B \tag{80}$$

and

$$<x'' = <t'' \begin{bmatrix} 1 & & & \\ & \frac{1}{t_c} & & \\ & & \frac{1}{t_c^2} & \\ & & & \ddots \end{bmatrix} \tag{81}$$

and by Equation (75)

$$<t'' = <\tau \begin{bmatrix} 1 & & & \\ & \left[\frac{\omega_\ell}{2}\right]^2 & & \\ & & \ddots & \\ & & & \left[\frac{\omega_\ell}{2}\right]^{d-1} \end{bmatrix} \tag{82}$$

also we have

$$<x = <t \begin{bmatrix} 1 & & & & \\ & \frac{1}{t_c} & & & \\ & & \frac{1}{t_c^2} & & \\ & & & \ddots & \\ & & & & \frac{1}{t_c^{d-1}} \end{bmatrix} \tag{83}$$

Using Equations (81), (82), and (83) in Equation (80)

$$\langle t D\left[\left(\frac{1}{tc}\right)^n\right] = \langle \tau D\left[\left(\frac{W_\ell}{2}\right)^n\right] \; D\left[\left(\frac{1}{tc}\right)\right] B \tag{84}$$

or

$$\langle t = \langle \tau D\left[\left(\frac{W_\ell}{2}\right)^n\right] \; D\left[\left(\frac{1}{tc}\right)^n\right] \; BD\left[\left(\frac{1}{tc}\right)^n\right]^{-1} \tag{85}$$

or

$$\langle t = \langle \tau \; T_{\tau t} \tag{86}$$

Transposing and forming the metric via use of Equation (76)

$$\int_{t_1}^{t_2} t\rangle\langle t \, dt = T_{\tau t}^{T} \int_{-1}^{1} \tau\rangle\langle \tau d\tau \;\; T_{\tau t} \, \frac{(t_2 - t_1)}{2} \tag{87}$$

By Equations (74) and (140) in the Appendix A we can also express

$$\langle t = \langle t'' \; T_u(tc) \tag{88}$$

and by Equation (70) in Equation (88)

$$\langle t = \langle \tau D\left[\left(\frac{t_2 - t_1}{2}\right)^n\right] \;\; T_u(tc) \tag{89}$$

$$\langle t = \langle \tau \; T_{\tau t} \tag{90}$$

By Equations (89) and (88)

$$T_u(tc) = D\left[\left(\frac{1}{tc}\right)^n\right] BD\left[\left(\frac{1}{tc}\right)^n\right]^{-1} \tag{91}$$

Note that there are three different metrics corresponding to the intervals of Figure (4)

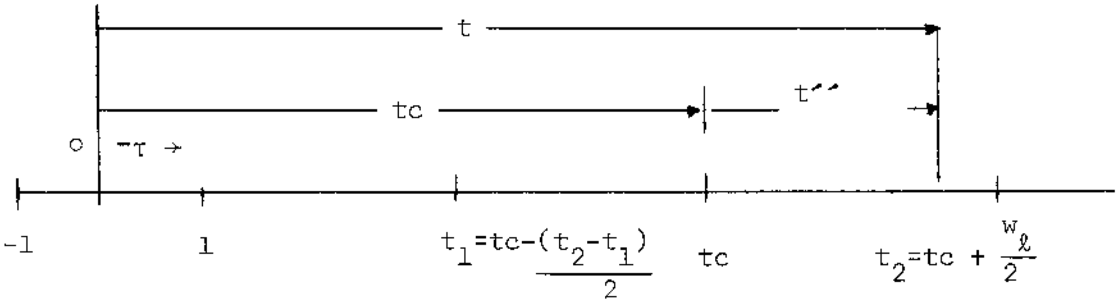


FIGURE (4)
TRANSLATED SYMMETRIC SPAN

The transformation relating the metrics are

$$\langle t = \langle t'' \ T_u(tc) \tag{92}$$

and

$$\langle t'' = \langle \tau D \ \frac{w_\ell}{2} \tag{93}$$

and

$$\int_{t_1}^{t_2} t\rangle \langle t \ dt = T_u^T(tc) \int_{\frac{-w_\ell}{2}}^{\frac{w_\ell}{2}} t''\rangle \langle t'' dt'' \ T_u(tc) \tag{94}$$

and

$$\int_{\frac{-w_\ell}{2}}^{\frac{w_\ell}{2}} t''\rangle \langle t'' dt'' = D\left(\frac{w_\ell}{2}\right) \int_{-1}^{1} \tau\rangle \ \langle \tau d\tau \ D\left(\frac{w_\ell}{2}\right) \frac{w_\ell}{2} \tag{95}$$

Case V - Transformation Metric Due to Sign Change. Consider next a variable change in sign on the interval (0,1) by Equation (1)

$$\tau = b_o + b_1 t \tag{96}$$

we want

$$(\tau_1, \tau_2) = (0,1) \tag{97}$$

and

$$(t_1, t_2) = (0,-1) \tag{98}$$

or

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} b\rangle \tag{99}$$

or

$$b\rangle = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{100}$$

or

$$\tau = -t \tag{101}$$

as shown in Figure (5)

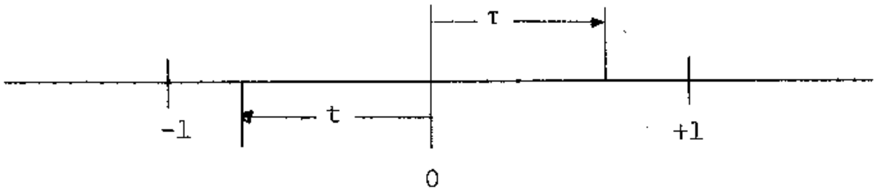


FIGURE (5)

AXIS REVERSAL

and

$$d\tau = (-1)dt \tag{102}$$

packagewise

$$\langle t = \langle \tau \; I(-1) \tag{103}$$

The metric is

$$\int_{o}^{-1} t\rangle\langle t \; dt = I(-1) \int_{o}^{1} (-1)\tau\rangle\langle \tau \; d\tau \; I(-1) \tag{104}$$

For the 3x3 case is for example

$$\int_{o}^{-1} t(3\rangle\langle 3)t \; dt = \begin{bmatrix} -1 & \frac{1}{2} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} \\ -\frac{1}{3} & \frac{1}{4} & -\frac{1}{5} \end{bmatrix} \tag{105}$$

The right hand side of Equation (104) is

$$-1\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4}\\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix} =$$

$$-1\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{2} & \frac{1}{3}\\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{4}\\ \frac{1}{3} & -\frac{1}{4} & \frac{1}{5}\end{bmatrix} \qquad (106)$$

$$=(-1)\begin{bmatrix}1 & -\frac{1}{2} & \frac{1}{3}\\ -\frac{1}{2} & \frac{1}{3} & -\frac{1}{4}\\ \frac{1}{3} & -\frac{1}{4} & \frac{1}{5}\end{bmatrix} = \begin{bmatrix}-1 & \frac{1}{2} & -\frac{1}{3}\\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{4}\\ -\frac{1}{3} & \frac{1}{4} & -\frac{1}{5}\end{bmatrix} \qquad (107)$$

which agrees with Equation (105).

Case VI - Transformation to New Origin with Sign Reversal $t=t_f-t'''$. The next case corresponds to Figure (6) where the origin is translated to the front of the time span of length t_f-t_b and the sign of the new variable is reversed as shown in Figure (6)

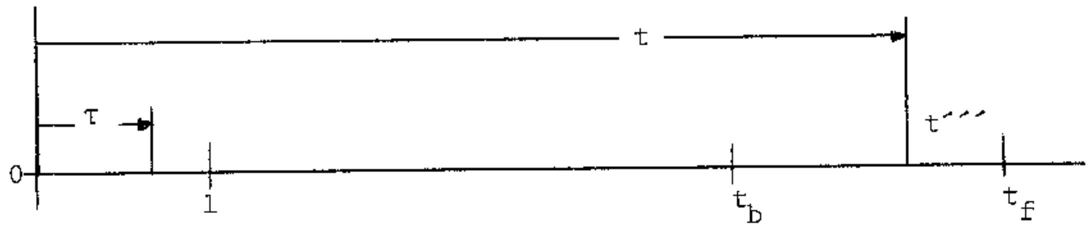


FIGURE (6)
ORIGIN TRANSLATION, SCALING AND AXIS REVERSAL

By Equation (1)

$$\tau = b_o + b_1 t$$

at the τ end points

$$(\tau_1, \tau_2) = (0,1)$$

and at the t end points

$$(t_1, t_2) = (t_f, t_b) \tag{108}$$

or

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & t_f \\ 1 & t_b \end{pmatrix} \begin{pmatrix} b_o \\ b_1 \end{pmatrix} \tag{109}$$

or

$$b\rangle = \begin{bmatrix} \dfrac{t_f}{t_f - t_b} \\ \dfrac{-1}{t_f - t_b} \end{bmatrix} \tag{110}$$

and

$$\tau = \frac{t_f}{t_f - t_b} - \frac{1}{(t_f - t_b)} t \tag{111}$$

and solving for t

$$t = t_f - (t_f - t_b)\tau \tag{112}$$

Define

$$t''' = (t_f - t_b)\tau = w_g \tau \tag{113}$$

hence

$$t = t_f - t''' \tag{114}$$

and differential-wise

$$dt = -dt''' = (t_f - t_b)d\tau \tag{115}$$

The end-points on the t''' variable are

$$\begin{pmatrix} t_1''' \\ t_2''' \end{pmatrix} = \begin{pmatrix} 0 \\ t_f - t_b \end{pmatrix} = \begin{pmatrix} 0 \\ w_g \end{pmatrix} \tag{116}$$

Package-of-powers Equation (113) yields

$$\langle t''' = \langle \tau \; D(w_g) \tag{117}$$

also by Equation (114)

$$t''' = t_f - t \tag{118}$$

or nomalizing

$$\frac{t'''}{t_f} = 1 - \frac{t}{t_f} \tag{119}$$

and packaging into a row vector

$$\begin{aligned} \langle t''' &= \langle t D^{-1}(t_f) R D(t_f) \\ &= \langle t \;\; T_{tt'''} \end{aligned} \tag{120}$$

The metric-matrix for $\langle t'''$ is by Equation (120), Equation (116) and Equation (115)

$$\int_0^{w_g} t'''\rangle \langle t''' dt''' = -T^T_{tt'''} \int_{t_f}^{t_b} t\rangle \langle t \; dt \; T_{tt'''} \tag{121}$$

also by Equation (117) and Equation (115)

$$\int_0^{w_g} t''' \quad t''' dt''' = w_g D(w_g) \int_0^1 \tau\rangle \langle \tau \; d\tau \; D(w_g) \tag{122}$$

or

$$M_{t''',t'''}(0,w_g) = w_g D(w_g) H_{i11} D(w_g) \tag{123}$$

also by Equation (118)

$$t''' = -(t - t_f) \tag{124}$$

and by Equation (120) Section (A)

$$\langle t''' = \langle t\ T_u(-t_f)I(-1) \tag{125}$$

and another expression for Equation (120) is obtained by equating the matrices

$$T_{t,t'''} = D^{-1}(t_f)RD(t_f) = T_u(-t_f)I(-1) \tag{126}$$

As examples of the way the results look let the t''' end points be

$$(t_1''',t_2''') = (0,1) \tag{127}$$

and

$$t_f-t_b = w_g = 1 \tag{128}$$

and

$$t_f = 1 \tag{129}$$

and by Equation (129) in the diagonal matrix of Equation (126)

$$D(t_f) = I \tag{130}$$

hence Equation (126) becomes

$$T_{t,t'''} = R = T_u(-1)I(-1) \tag{131}$$

Where by Equation (33) Appendix A

$$T_u(-1) = B(-1) = B^{-1} \tag{132}$$

and the connection matrix is

$$\langle t''' = \langle tR \tag{133}$$

The metric of Equation (121) using integration limits of Equation (108) for t becomes

$$\int_{o}^{1} t'''\rangle\langle t'''dt''' = R^T\int_{1}^{o}(-1)\ t\rangle\langle t\ dt\ R \tag{134}$$

or

$$H_{ill} = R^TH_{ill}R \tag{135}$$

a novel result which says that the Hilbert matrix is congruent to itself via the Rutishauser matrix.

also

$$H_{ill}^{-1} = R^T H_{ill}^{-1} R \qquad (136)$$

For example the 3x3 case is for the matrix H

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad (137)$$

<u>Case VII - Transformation Metric (0,1) (-1,1)</u>. This case will show the application of the methods to the two classical intervals. Consider the transformation

$$\tau = b_o + b_1 t$$

where

$$(\tau_1, \tau_2) = (0,1) \qquad (138)$$

and

$$(t_1, t_2) = (-1,1) \qquad (139)$$

or

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} b\rangle \qquad (140)$$

and solving for b

$$b = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (141)$$

or

$$\tau = \frac{1}{2} + \frac{t}{2} \qquad (142)$$

and

$$t = -1 + 2\tau \qquad (143)$$

and differential-wise

$$dt = 2\ d\tau \qquad (144)$$

The package of powers of Equation (143) is

$$\langle t = \langle \tau \; D(2^n) \; B^{-1} \tag{145}$$

and transposing and forming the metric (B is binomial matrix)

$$\int_{-1}^{1} t\rangle \; \langle t \; dt = B^{-T} D(2^n) \int_{0}^{1} (2)\tau\rangle \langle \tau \; d\tau \; D(2^n) B^{-1} \tag{146}$$

or

$$M_{t,t}(-1,1) = B^{-T} D(2^n) \; H_{ill} D(2^n) B^{-1} 2 \tag{147}$$

For the 3x3 case one has

$$\begin{bmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2^2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \text{x}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2^2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \tag{148}$$

Multiplying the matrices on the right yields the equivalence relation.

The interesting aspect of the above case is made apparent when one attempts the transformation (0,1) to (o,-1) simply by changing the limits of integration

$$\int_{o}^{-1} \tau\rangle \; \langle \tau \; d\tau = \left[\begin{matrix} \tau & \frac{\tau^2}{2} & \frac{\tau^3}{3} \\ \frac{\tau^2}{2} & \frac{\tau^3}{3} & \frac{\tau^4}{4} \\ \frac{\tau^3}{3} & \frac{\tau^4}{4} & \frac{\tau^5}{5} \end{matrix} \right]_{0}^{-1} \tag{149}$$

$$= \begin{bmatrix} -1 & \frac{1}{2} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} \\ -\frac{1}{3} & \frac{1}{4} & -\frac{1}{5} \end{bmatrix}$$

which does not qualify as a metric matrix since the norms on the main diagonals have negative signs. This is avoided when one changes variables thus

$$\tau = -t \tag{150}$$

and

$$d\tau = -dt \tag{151}$$

and by Equation (150)

$$\langle t = \langle \tau \; I(-1) \tag{152}$$

and

$$\int_0^{-1} t \rangle \langle t \; dt = (-1)I(-1) \int_0^{1} \tau \rangle \langle \tau \; d\tau \; I(-1) \tag{153}$$

$$\int_0^{-1} t \rangle \; \langle t \; (-dt) = I(-1) \int_0^{1} \tau \rangle \langle \tau \; d\tau \; I(-1) \tag{154}$$

or

$$M_{tt}(0,-1) = \int_0^{-1} t \quad t(-dt) \tag{155}$$

and

$$M_{t,t}(0,-1) = I(-1) \; M_{\tau\tau}(0,1) \; I \; (-1) \tag{156}$$

B. <u>TRANSFORMATIONS ON GRAM-SCHMIDT VECTORS FROM CLASSICAL INTERVALS TO ARBITRARY INTERVALS.</u>

<u>Case 1 - Transformation Gram-Schmidt Vectors from (0,1) Interval to (-1,1).</u> The orthonormal Gram-Schmidt vectors were derived in Section (1) Equation (337) for the (0,1) interval as

$$\langle s(0,1) = \langle \tau \; B_s(0,1) \tag{157}$$

where the coordinate matrix is for 3x3 case

$$B_s(0,1) = \begin{bmatrix} 1 & -\sqrt{3} & \sqrt{5} \\ 0 & 2\sqrt{3} & -6\sqrt{5} \\ 0 & 0 & 6\sqrt{5} \end{bmatrix} \qquad (158)$$

We seek to derive the orthonormal Gram-Schmidt vectors previously obtained on the interval (-1,1) called the modified Legendre polynomials via transformation matrices rather than proceeding through the Gram-Schmidt construction equations given by Equation (384) of Section (1). Let

$$\langle s(-1,1) = \langle t\ B_s(-1,1) \qquad (159)$$

where $B_s(-1,1)$ is unknown and where the connection of Equation (145) is used

$$\langle t = \langle \tau\ D(2^n)B^{-1} = \langle \tau\ T_{\tau t} \qquad (160)$$

Using Equation (160) in Equation (159)

$$\langle s(-1,1) = \langle \tau D(2^n)B^{-1}B_s(-1,1) \qquad (161)$$

Transposing

$$s(-1,1)\rangle = B_s^{\ T}(-1,1)\ B^{-T}\ D(2^n)\tau\rangle \qquad (162)$$

Forming the O,N metric (constant)

$$\int_{-1}^{1} s(-1,1)\rangle\ \langle s(-1,1)dt = I = M_{ss}(-1,1) \qquad (163)$$

or by Equation (144)

$$I = B_s^T(-1,1)B^{-T}D(2^n)\int_0^1 (2)\ \tau\ \ \tau d\tau D(2^n)B^{-1}B_s(-1,1) \qquad (164)$$

or

$$I = 2B_s^T(-1,1)B^{-T}D(2^n)H_{i11}D(2^n)B^{-1}B_s(-1,1) \qquad (165)$$

By Equation (348) of Section (1) the triangular factors of the Hilbert matrix are

$$M_{\tau\tau}(0,1) = H_{i11} = [B_s(0,1)B_s^T(0,1)]^{-1} \qquad (166)$$

Modifying Equation (165)

$$B_s^T(-1,1)\ B_s^{-1}(-1,1) = [B_s(-1,1)B_s^T(-1,1)]^{-1}$$

$$=2B^{-T}D(2^n)[B_s(0,1)B_s^T(0,1)]^{-1}D(2^n)B^{-1} \qquad (167)$$

Inverting Equation (167)

$$B_s(-1,1)B_s^T(-1,1) = \frac{1}{2} BD^{-1}(2^n)B_s(0,1)B_s^T(0,1)D^{-1}(2^n)B^T \tag{168}$$

Equating factors of the metric matrix

$$B_s(-1,1) = \frac{1}{\sqrt{2}} BD^{-1}(2^n)\ B_s(0,1) \tag{169}$$

and for the 3x3 case using Equation (158)

$$\underset{3x3}{B_s(-1,1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2^2} \end{bmatrix} \begin{bmatrix} 1 & -\sqrt{3} & \sqrt{5} \\ 0 & 2\sqrt{3} & -6\sqrt{5} \\ 0 & 0 & 6\sqrt{5} \end{bmatrix} \tag{170}$$

or

$$B_s(-1,1) = \begin{bmatrix} 1 & 0 & -\frac{\sqrt{5}}{2} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \frac{3\sqrt{5}}{2} \end{bmatrix} \frac{1}{\sqrt{2}} \tag{171}$$

which agrees with Equation (384) of Section (1). One can easily obtain the general expressions for the dxd matrices. Using Equation (160) in Equation (159)

$$\langle s(-1,1) = \langle \tau D(2^n)B^{-1}B_s(-1,1) \tag{172}$$

Using Equation (169) in Equation (172)

$$\langle s(-1,1) = \langle \tau\ D(2^n)B^{-1}BD^{-1}(2^n)\ B_s(0,1)\frac{1}{\sqrt{2}} \tag{173}$$

$$\langle s(-1,1) = \langle \tau\ B_s(o,1)\frac{1}{\sqrt{2}} \tag{174}$$

which relates the O.N. Schmidt vectors to the old base.

C. EXPONENTIAL BASE METRICS. The modified Laguarre polynomial metrics with translated origins and time axis sign change are presented below.

The interval $(0,\infty)$ will be referred to as the classical interval and by Equation (20) in Appendix D

$$\int_0^\infty \tau\rangle\langle\tau e^{-a\tau}d\tau = \int_0^\infty[\tau^{i+j}e^{-a\tau}d\tau] \tag{175}$$

$$M_{\tau\tau}(0,\infty) = \left[\frac{(i + j)\ !}{a^{i + j + 1}}\right] \tag{176}$$

for i,j=0,1,2,...d-1

<u>Case I - The Variable Change.</u>

$$\tau = -t \tag{177}$$

and

$$d\tau = -dt \tag{178}$$

and

$$e^{-a\tau} = e^{at} \tag{179}$$

yields for the powers

$$\langle\tau = \langle tI(-1) \tag{180}$$

The end points for the new variable are

$$(t_1, t_2) = (0, -\infty) \tag{181}$$

and the metric is

$$\int_0^{\infty} \tau\rangle\langle\tau\ e^{-a\tau} d\tau = I(-1)\int_0^{-\infty} t\rangle\langle t\ e^{at}(-dt)I(-1) \tag{182}$$

By Equation (23) in Appendix B

$$\int_0^{-\infty} e^{at} t\rangle\langle t\ dt = \left[\frac{(i + j)\ !\ (-1)^{i+j+1}}{a^{i+j+1}}\right] \tag{183}$$

$$= \begin{bmatrix} -\frac{1}{a} & -\frac{1}{a^2} & \frac{-2!}{a^3} \\ \frac{1}{a^2} & \frac{-2}{a^3} & \frac{3!}{a^4} \\ \frac{-2!}{a^3} & \frac{3!}{a^4} & \frac{-4!}{a^5} \end{bmatrix} \tag{184}$$

and we see as in Equation (149) that the diagonal elements have negative signs, hence do not serve as a good metric. By Equation (182) define the metric

$$M_{tt}(0,-\infty) = \int_0^{-\infty} t \rangle \langle te^{at}(-dt) = -\int_0^{\infty} t \rangle \langle t\ e^{at}dt \tag{185}$$

and using Equation (184) in Equation (185)

$$M_{tt}(0,-\infty) = \begin{bmatrix} \frac{1}{a} & \frac{-1}{a^2} & \frac{2!}{a^3} \\ \frac{-1}{a^2} & \frac{2!}{a^3} & \frac{-3!}{a^4} \\ \frac{2!}{a^3} & \frac{-3!}{a^4} & \frac{4!}{a^5} \end{bmatrix} \tag{186}$$

and the metrics are connected via Equation (186) in Equation (182)

$$M_{\tau\tau}(0,\infty) = I(-1)\ M_{tt}(0,-\infty)I(-1) \tag{187}$$

and

$$M_{tt}(0,-\infty) = I(-1)\ M_{\tau\tau}(0,\infty)I(-1) \tag{188}$$

and the inverse metrics are related

$$M^{-1}_{tt}(0,-\infty) = I(-1)M^{-1}_{\tau\tau}(0,\infty)I(-1) \tag{189}$$

The inverse metric on $(0,\infty)$ is given by Equation (123) Section (2) as

$$M^{-1}(0,\infty) = a \begin{bmatrix} 3 & -3a & \frac{a^2}{2} \\ -3a & 5a^2 & -a^3 \\ \frac{a^2}{2} & -a^3 & \frac{a^4}{4} \end{bmatrix} \tag{190}$$

and the inverse of Equation (190)

$$M^{-1}_{tt}(0,-\infty) = a \begin{bmatrix} 3 & 3a & \frac{a^2}{2} \\ 3a & 5a^2 & a^3 \\ \frac{a^2}{2} & a^3 & \frac{a^4}{4} \end{bmatrix} \tag{191}$$

Case II - If we translate the origin to a new origin t_1 0 where

$$\tau = t_1 + t \tag{192}$$

then the metric for the variable τ on the interval $(0,\infty)$ can be written as

$$\int_0^\tau \tau\rangle_e \langle\tau_e \, d\tau = \int_0^{t_1} \tau\rangle_e \langle\tau_e \, d\tau + \int_{t_1}^\infty \tau\rangle_e \langle\tau_e \, d\tau \tag{193}$$

The first term on the right is rather messy, however the second integral is interesting. For the 3x3 case we obtain

$$M_{\tau\tau}(t_1,\infty) = \int_{t_1}^\infty \tau\rangle \langle\tau \, e^{-a\tau} \, d\tau \tag{194}$$

$$= \frac{e^{-at_1}}{a} \begin{bmatrix} 1 & (t_1+\frac{1}{a}) & (t_1^2+\frac{2t_1}{a}+\frac{1}{a^2}) \\ (t_1+\frac{1}{a}) & (t_1^2+\frac{2t_1}{a}+\frac{1}{a^2}) & (t_1^3+\frac{3t_1^2}{a}+\frac{6t_1}{a^2}+\frac{3}{a^3}) \\ (t_1^2+\frac{2t_1}{a}+\frac{1}{a^2}) & (t_1^3+\frac{3t_1^2}{a}+\frac{6t_1}{a^2}+\frac{3}{a^3}) & (t_1^4+\frac{4t_1^3}{a}+\frac{12t_1^2}{a^2}+\frac{24t_1}{a^3}+\frac{12}{a^4} \end{bmatrix} \tag{195}$$

D. <u>TRANSFORMATION OF THE GRAM-SCHMIDT VECTORS FROM $(0,\infty\ )$ $(0,-\infty\ )$ INTERVALS</u>. The Gram-Schmidt proceedure was applied to derive the orthonormal modified Laguarre functions of Equation (42) Section (2) for the 3x3 case as

$$\langle s(0,\ ) = \langle\tau_e B_s(0,\infty) \tag{196}$$

where

$$B_s(0,\infty) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & a & -2a \\ 0 & 0 & \frac{a^2}{2} \end{bmatrix} \tag{197}$$

The O. N. Gram-Schmidt vectors on $[0,-\infty]$ is given by

$$\langle s(0,-\infty) = \langle t_e B_s(0,-\infty) \tag{198}$$

where $B_s(0,-\infty)$ is unknown.

Transpose Equation (198) and form the metric

$$\int_0^{-\infty} s(0,-\infty)\rangle \langle s(0,-\infty)(-dt) = B_s^T(0,-\infty)\int_0^{-\infty} t\rangle_e \langle t_e(-dt)B_s(0,-\infty) \tag{199}$$

or

$$I = B_s^T(0,-\infty)\; M_{t_e t_e}(0,-\infty)\; B_s(0,-\infty) \tag{200}$$

Using Equation (188) in Equation (200)

$$I = B_s^T(0,-\infty)I(-1)M_{\tau_e\tau_e}(0,-\infty)I(-1)B_s(0,-\infty) \tag{201}$$

and the triangular factors of Equation (417) Section (1)

$$M_{\tau_e\tau_e}(0,\infty) = [B_s(0,\infty)B_s^T(0,\infty)]^{-1} \tag{202}$$

and as before

$$[B_s(0,-\infty)B_s^T(0,-\infty)]^{-1} = I(-1)[B_s(0,\infty)B_s^T(0,\infty)]^{-1}I(-1) \tag{203}$$

or

$$B_s(0,-\infty)B_s^T(0,-\infty) = I(-1)B_s(0,\infty)B_s^T(0,\infty)I(-1) \tag{204}$$

and if one attempts as a solution

$$B_s(0,\infty) \overset{?}{=} I(-1)B_s(0,\infty) \tag{205}$$

and for the 3x3 case we have by Equation (197) in Equation (205)

$$B_s(0,-\infty) \overset{?}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & a & -2a \\ 0 & 0 & \frac{a^2}{2} \end{bmatrix} \sqrt{a} \tag{206}$$

or

$$B_s(0,-\infty) \overset{?}{=} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -a & 2a \\ 0 & 0 & \frac{a^2}{2} \end{bmatrix} \tag{207}$$

If we perform the O. N. Gram-Schmidt process using the metric of Equation (186)

$$g(0,-\infty) = \langle t \begin{bmatrix} 1 & \frac{1}{a} & \frac{2}{a^2} \\ 0 & 1 & \frac{2^2}{a} \\ 0 & 0 & 1 \end{bmatrix} \tag{208}$$

and the O.N. vectors are

$$s(0,-\infty) = \langle t \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & 2a \\ 0 & 0 & \frac{a^2}{2} \end{bmatrix} \tag{209}$$

Equation (209) is seen to differ in sign from Equation (207), however if we use the solution of Equation (24) in Appendix C that is

$$B_s(0,-\infty) = I(-1)B_s(0,\infty)I(-1)$$

we obtain the correct matrix

$$B_s(0,-\infty) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2a & 2a \\ 0 & 0 & \frac{a^2}{2} \end{bmatrix} \sqrt{a} \tag{210}$$

Section 4

POLYNOMIAL VECTORS, BASES, BASE CHANGES, POLYNOMIAL DERIVATIVES, BASE DERIVATES AND BASE INTEGRALS. It is well known from elementary algebra texts (see for example Halmos Ref [36]) that the set of polynomials of degree d-1 form a vector space of degree d, for example

$$x(t)=a_0+a_1\ t\ldots a_{d-1}\ t^{d-1} \tag{1}$$

is a vector. The coordinates are $a_0, a_1 \ldots a_{d-1}$ and the base elements are the powers of t. If we separate the coordinates from the base elements

$$x(t)=(1,t,t^2\ \cdots\ t^{d-1})\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{d-1} \end{bmatrix} \tag{2}$$

or

$$x(t)=\langle t\ a\rangle \tag{3}$$

where the column of field elements is

$$a\ (d\rangle = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{d-1} \end{pmatrix} \tag{4}$$

and the row of base vectors is

$$\langle t = (t^0, t^1, t^2\ \ldots\ t^{d-1}) \tag{5}$$

Equation (5) is a row of d base vectors, that is the i^{th} base vector is the polynomial t^i. The d base vectors are linearly independent etc. Any full rank invertible dxd matrix B maps the $\langle t$ base to a new base, for example

$$\langle t \underset{dxd}{B} = \langle d)\ x_b \tag{6}$$

or

$$\langle t = \langle x_b\ B^{-1} = \langle x_b\ M_{x_{bt}} \tag{7}$$

Some examples of base changes commonly occurring in classical mathematics are given below.

Linear Scaling and Translation on the Time Axis.

$$\tau = b_{11} + b_{12}\ t$$

or simplifying the notation (8)

$$\tau = b_0 + b_1 t$$

or nomalizing the coefficient of t in Equation (8)

$$\tau = b_1\ (\beta + t) \tag{9}$$

where

$$\beta = b_0/b_1 \tag{10}$$

By Equation (59-A), the binomial term of Equation (8) yields

$$(\tau^0, \tau, \tau^2, \cdots\ \tau^{d-1}) = \langle \tau = \langle t\ D(b_1)T_u(\beta) \tag{11}$$

where

$$D(b_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \\ 0 & b_1 & 0 & 0 & & \cdot \\ 0 & 0 & b_1^2 & 0 & & \cdot \\ & & & b_1^3 & & \cdot \\ & & & & \ddots & b_1^{d-1} \end{bmatrix}$$

and

$$T_u(\beta) = \begin{bmatrix} 1 & \beta & \beta^2 & \beta^3 & \beta^4 & \beta^5 & \cdots \\ 0 & 1 & 2\beta & 3\beta^2 & 4\beta^3 & 5\beta^4 & \cdots \\ 0 & 0 & 1 & 3\beta & 6\beta^2 & 10\beta^3 & \cdots \\ 0 & 0 & 0 & 1 & 4\beta & 10\beta^2 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 5\beta & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & & & & & \vdots & \cdots \end{bmatrix} \qquad (12)$$

By Equation (12) we see that the bases are connected by a diagonal matrix D times an upper triangular matrix $T_u(\beta)$ containing the binomial coefficients or

$$\langle\tau = \langle t \; M_{t\tau} \qquad (13)$$

where

$$M_{t\tau} = D(b_1) \; T_u(\beta) \qquad (14)$$

The inverse base change is

$$\langle t = \langle\tau \; T_u^{-1} \; (\beta) \; D^{-1}(b_1) \qquad (15)$$

The diagonal matrix inversion is simple, the upper triangular matrix inversion is more complicated. However the inverse can be constructed easily, for by Equation (9)

$$t = \frac{1}{b_1}(\tau - b_0) \tag{16}$$

and by Equation (60-A)

$$\langle t = \langle \tau \begin{bmatrix} 1 & & & & \\ & b_1^{-1} & & 0 & \\ & & b^{-1} & & \\ & 0 & & \ddots & \\ & & & & b^{(d-1)} \end{bmatrix} x$$

$$\begin{bmatrix} 1 & -b_0 & b_0^2 & -b_0^3 & b_0^4 & -b_0^5 \\ 0 & 1 & -2b_0 & 3b_0^2 & -4b_0^3 & 5b_0^4 \\ 0 & 0 & 1 & -3b_0 & 6b_0^2 & -10b_0^3 \\ 0 & 0 & 0 & 1 & -4b_0 & 10b_0^2 \\ 0 & 0 & 0 & 0 & 1 & -5b_0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

Consider the derivative of Equation (2)

$$\dot{x}(t) = \left(\frac{d \langle t}{dt} \right) a\rangle + \langle t \; \frac{da\rangle}{dt} \tag{18}$$

For many studies the vector $a\rangle$ is a constant and

$$\frac{da\rangle}{dt} = 0\rangle \tag{19}$$

For filtering example of varying parameters see paper by Luenberger [54]. The derivatives of the base polynomials in the same base is

$$\frac{d}{dt}\,(1,t,t^2,t^3,\ \cdots t^{d-1}) = (0,1,2t,3t^2,\ \cdots\ (d-1)t^{d-2}) \tag{20}$$

or

$$\frac{d}{dt}\,\langle t = \langle t\ V \tag{21}$$

where the dxd velocity matrix is

$$\underset{dxd}{V} = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 2 & 0 & & & & . \\ 0 & 0 & 0 & 3 & & & & . \\ . & & & 0 & & & & . \\ . & & 0 & & & & & \\ . & & & & & & & (d-1) \\ 0 & & . & . & . & & & 0 \end{bmatrix} \tag{22}$$

The higher derivatives are

$$\begin{aligned} \ddot{x} &= \langle t\ V^2\ \ a\rangle \\ \dddot{x} &= \langle t\ V^3\ \ a\rangle \\ &\vdots \\ x^{(d-1)} &= \langle t\ V^{d-1}\ \ a\rangle \\ x^{(d)} &= \langle t\ V^d\ \ a\rangle = 0 \end{aligned} \tag{23}$$

The matrix of Equation (22) is singular, or the polynomial derivatives are linearly dependent. From the above we see that

$$\frac{d}{dt} \langle t = \langle t \; V \qquad (24)$$

$$\frac{d^2}{dt^2} \langle t = \langle t \; V^2$$

$$\vdots$$

$$\frac{d^m}{dt^m} \langle t = \langle t \; V^m$$

$$\vdots$$

$$\frac{d^d}{dt^d} \langle t = \langle t \; V^d = \langle t \; [0]$$

and the singular velocity-matrix V is said to be nilpotent, index d, that is

$$V^d = 0 \qquad (25)$$

Note the interesting dyadic decomposition of V as a sum of rank one dyads

$$V = 1 \; E_{12} + 2 \; E_{23} + 3 \; E_{34} + \cdots + (d-1) \; E_{(d-1)d} \qquad (26)$$

where each rank-one dyad is

$$E_{\ell/\ell+1} = \underset{\ell}{e(d\rangle}\overset{\ell+1}{\langle d)e} \qquad (27)$$

and the row-vector of dimension d has a 1 in the ℓ^{th} position

$$\underset{\ell}{\langle} d)e = (0,\ 0 \ldots \underset{\ell^{th}\ \text{position}}{1},0,0 \ldots 0) \tag{28}$$

The matrix V can also be partitioned into its column space or row space as

$$V = \left[\ 0\rangle,\ \ e\rangle_1,\ \ e\rangle_2 2 \ldots e\rangle_d (d-1) \right] \tag{29}$$

$$V = \begin{bmatrix} {}^{2}\langle e \\ 2\ {}^{3}\langle e \\ 3\ {}^{4}\langle e \\ \vdots \\ (d-1)\ {}^{d}\langle e \\ \langle 0 \end{bmatrix} \tag{30}$$

or as factors

$$V = N\ S_{ro} \tag{31}$$

where the shift right-and-out operator is

$$S_{ro} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & 1 \\ 0 & . & . & . & 0 \end{bmatrix} = \left[\ 0\rangle,\ \ e\rangle_1,\ \ e\rangle_2,\ \ldots\ \ e\rangle_{d-1} \right] \tag{32}$$

The shift is on the identity matrix

$$\underset{dxd}{I} = \left[e\rangle_1 , \; e\rangle_2 \cdots \; e\rangle_{(d-1)} \; e\rangle_d \right] \tag{33}$$

N is a diagonal matrix of natural numbers

$$N = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \\ & & 3 & \\ & 0 & & d-1 \end{bmatrix}_d \tag{34}$$

Powers of V can be obtained from Equation (22) utilizing

$$E_{ij} \; E_{km} = \begin{cases} 0 \;\; j \neq k \\ E_{im} \;\; j=k \end{cases} \tag{35}$$

since

$$e\rangle_i \; {}^{j}\langle e \;\; e\rangle_k \; {}^{m}\langle e = e\rangle_i \; {}^{m}\langle e \;\; \delta_{jk} \tag{36}$$

where the inner-product is

$$\delta_{jk} = {}^{j}\langle e \;\; e\rangle_k = \begin{cases} 1 \;\; j=k \\ 0 \;\; j \neq k \end{cases} \tag{37}$$

δ_{jk} is the familiar Kronecker delta function. Consider the square of the expression of Equation (22)

$$V^2 = 1.2E_{13} + 2\cdot 3E_{24} + 3\cdot 4E_{35} + 4\cdot 5E_{46} + \cdots + (d-2)(d-1)\; E_{(d-2)} \tag{38}$$

or in open form

$$V^2 = \underbrace{\begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 2.3 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 3.4 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & & & \\ \vdots & \vdots & & & & & & \vdots \\ 0 & 0 & 0 & . & . & . & & (d-2)(d-1) \\ 0 & 0 & 0 & . & . & . & & 0 \\ 0 & 0 & 0 & . & . & . & & 0 \end{bmatrix}}_{dxd} \qquad (39)$$

The third power is

$$V^3 = 2,3\ E_{14} + 2.3.4\ E_{25} + 3 \cdot 4.5\ E_{36}$$
$$+ 4 \cdot 5 \cdot 6\ E_{47} + \ .\ .\ .\ (d-3)(d-2)(d-1)\ e_{(d-3)_d} \qquad (40)$$

or in open form

$$V^3 = \begin{array}{r} \\ \\ \\ \\ (d-3)^{th}\ \text{row} \\ \\ \\ \\ \end{array} \left[\begin{array}{cccccccc} 0 & 0 & 0 & 3! & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & (3+1)! & 0 & \cdots & . \\ 0 & 0 & 0 & 0 & 0 & \frac{(3+2)!}{2!} & \cdots & . \\ \vdots & & & & & & & \\ 0 & & & . & . & . & & \frac{(d-1)!}{(d-1-3)!} \\ 0 & & & . & . & . & & 0 \\ 0 & & & . & . & . & & 0 \\ 0 & & & . & . & . & & 0 \end{array}\right] \qquad (41)$$

The m^{th} power is

$$V^m = 1 \cdot 2 \cdot 3 \cdot\ mE_{1(m+1)} + 2 \cdot 3 \cdot 4\ .\ .\ .\ m(m+1)\ E_{2(m+2)} \qquad (42)$$

$$+\ .\ .\ .\ + m(m+1)(2m+1)\ E_{m(2m)} +\ .\ .\ .$$

$$+ (d-m)(d-m+1)\ .\ .\ .\ (d-2)(d-1)\ E_{(d-m)d}$$

for

$$m < d \qquad (43)$$

or in open form

$$\begin{array}{r|cccccccc|} V^m = & 0 & \cdots & m! & 0 & \cdots & 0 & \cdots & 0 \\ & 0 & \cdots & 0 & (m+1)! & & 0 & \cdots & 0 \\ & & & & & \frac{(m+2)!}{2!} & 0 & \cdots & 0 \\ & \vdots & & & & & \vdots & & \\ (d-m)\ \text{row} & & & \vdots & & & & & \frac{(d-1)!}{(d-1-m)!} \\ & \langle d)0 & & & & & & & \vdots \\ \text{m rows of zeros} & & & & & & & & 0 \\ & \langle d)0 & & & & & & & \vdots \end{array} \qquad (44)$$

the (d-1) power is

$$V^{d-1} = (d-1)!\ E_{1d} \qquad (45)$$

the d^{th} power is

$$V^d = 0 \qquad (46)$$

as an example consider the fifth degree polynomial, or

$$\langle t = (1,\ t,\ t^2,\ t^3,\ t^4,\ t^5) \qquad (47)$$

for which

$$\underset{6x6}{V} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (48)$$

$$V^2 = \begin{bmatrix} 0 & 0 & 1.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{49}$$

$$V^3 = \begin{bmatrix} 0 & 0 & 0 & 2.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.3.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.4.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{50}$$

$$V^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 2.3.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.3.4.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{51}$$

$$V^5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2.3.4.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{52}$$

$$V^6 = [0] \tag{53}$$

The dyadic expressions are

$$\underset{6x6}{V} = E_{12} + 2E_{23} + 3E_{34} + 4E_{45} + 5E_{56} \tag{54}$$

$$V^2 = 1 \cdot 2E_{13} + 2 \cdot 3E_{24} + 3 \cdot 4E_{35} + 4 \cdot 5E_{46} \tag{55}$$

$$V^3 = 1 \cdot 2 \cdot 3E_{14} + 2 \cdot 3 \cdot 4E_{25} + 3 \cdot 4 \cdot 5E_{36} \tag{56}$$

$$V^4 = 1 \cdot 2 \cdot 3 \cdot 4E_{15} + 2 \cdot 3 \cdot 4 \cdot 5E_{26} \tag{57}$$

$$V^5 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5E_{16} \tag{58}$$

$$V^6 = 0 \tag{59}$$

The coefficients are factorials and factorial relations.

The matrix V can also be written as the product Equation (32) and Equation (34) as

$$\underset{dxd}{V} = NS_{uo} \tag{60}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & & & 0 \\ 0 & & 3 & & \\ & & & \ddots & \\ . & . & . & & d \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \vdots \\ 0 & 0 & 0 & & \\ 0 & 0 & & & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{61}$$

The psuedo-inverse of the singular matrix V is used later in multiple integrals. Two expressions for the psuedo inverse are

$$V^* = V^T(VV^T)^* \tag{62}$$

$$= (V^T V)^* V^T \tag{63}$$

The transpose of Equation (61) is

$$V^T = S_{uo}^T N \tag{64}$$

The symmetric Grammian matrix of Equation (63)

$$VV^T = NS_{uo} S_{uo}^T N \tag{65}$$

The product by Equation (32) is

$$S_{uo} S_{uo}^T = \begin{bmatrix} {}^2\langle e \\ {}^3\langle e \\ \vdots \\ {}^d\langle e \\ \langle 0 \end{bmatrix} \begin{bmatrix} e\rangle_2 & \cdots & e\rangle_d, & 0\rangle \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & 0 \\ & \cdot & & & \cdot \\ & & \cdot & & \cdot \\ 0 & & & 1 & \cdot \\ \cdot & \cdot & \cdot & & 0 \end{bmatrix} \tag{66}$$

and

$$(VV^T)^* = \begin{bmatrix} 1 & \cdot & & \cdot & & \cdot \\ \cdot & 2^2 & & & & \\ \cdot & & 3^2 & & & \cdot \\ \cdot & & & \ddots & & \\ \cdot & \cdot & \cdot & & (d-1)^2 & \\ \cdot & & \cdot & \cdot & & 0 \end{bmatrix}^* \tag{67}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{(d-1)^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using Equation (67) in Equation (62)

$$V^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \vdots & & \cdot & & \cdot & & \cdot \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{d-1} & 0 \end{bmatrix} \qquad (68)$$

The m^{th} power of Equation (68) is

$$\begin{array}{r c} & (V^*)^m = \\ & \\ & \\ (m+1)^{th} \rightarrow & \\ & \\ & \\ & \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & \vdots & 0 & 0 & 0 \\ \frac{1}{m!} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{(m+1)!} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(d-m)!}{d!} & 0 & \ldots & 0 \end{bmatrix} \qquad (69)$$

The associated projectors are

$$\underset{d x d}{VV^*} = \begin{bmatrix} 1 & 0 & 0 & & & & 0 \\ 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & 0 \\ 0 & & & & & 0 & 0 \end{bmatrix} = \begin{bmatrix} \underset{(d-1)(d-1)}{I} & 0 \;(d-1\rangle \\ & \\ \langle d-1)\; 0 & 0 \end{bmatrix} \qquad (70)$$

and

$$I - VV^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (71)$$

From the preceeding we have

$$\begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \\ \cdot \\ \cdot \\ \cdot \\ x^{d-1} \end{bmatrix} = \begin{bmatrix} \langle t \\ \langle tV \\ \langle tV^2 \\ \cdot \\ \cdot \\ \cdot \\ \langle tV^{d-1} \end{bmatrix} a(d\rangle \tag{72}$$

or the d states in matrix form

$$x(d\rangle = T_v(t)\ a(d\rangle \tag{73}$$

The derivative of the state vector of (69) multiplies the package by V, that is

$$\dot{x}(d\rangle = T_v(t)\ V\ a(d\rangle$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \cdot \\ \cdot \\ \cdot \\ x^d \end{bmatrix} = \begin{bmatrix} \langle tV \\ \langle tV^2 \\ \cdot \\ \cdot \\ \cdot \\ \langle tV^{d-1} \\ \langle 0 \end{bmatrix} a\rangle \tag{74}$$

Consider the matrix $T_v(t)$ of Equation (72)

$$T_v(t) = \tag{75}$$

$$\begin{bmatrix}
1 & t & t^2 & t^3 & t^4 & t^5 & \cdot\ \cdot\ \cdot & t^{d-1} \\
0 & 1 & 2t & 3t^2 & 4t^3 & 5t^4 & \cdot\ \cdot\ \cdot & (d-1)t^{d-2} \\
0 & 0 & 2 & 2.3t & 3.4t^2 & 4.5t^3 & \cdot\ \cdot\ \cdot & (d-2)(d-1)t^{d-3} \\
0 & 0 & 0 & 2.3 & 2.3.4t & 3.4.5t^2 & \cdot\ \cdot\ \cdot & (d-3)(d-2)(d-1)t^{d-4} \\
0 & 0 & 0 & 0 & 2.3.4 & \cdot & \cdot & \cdot \\
\cdot & & & & & & & \cdot \\
\cdot & & & & & & & \cdot \\
\cdot & & & & & & & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & 1\cdot 2\cdot 3\cdot 4 & \ldots(d-3)(d-2)(d-1)
\end{bmatrix}$$

The matrix evaluated at t=0 is

$$T_v(o) = \begin{bmatrix}
1 & 0 & 0 & \cdot & & \cdot & \cdot & \cdot & 0 \\
0 & 1 & 0 & \cdot & & \cdot & & & \cdot \\
0 & 0 & 2 & & & \cdot & & & \vdots \\
0 & 0 & 0 & 3! & & & & & 0 \\
0 & 0 & 0 & 0 & 4! & & & & \cdot \\
\cdot & & & & & \cdot & & & \cdot \\
\cdot & & & & & & \cdot & & \cdot \\
\cdot & & & & & & & \cdot & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (d-1)!
\end{bmatrix} \tag{76}$$

The inverse of $T_v^{-1}(o)$ is

$$T_v^{-1}(o) = \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & \frac{1}{2} & & 0 & \\ & & & \frac{1}{3!} & & \\ 0 & & & & \ddots & \\ & & & & & \frac{1}{(d-1)!} \end{bmatrix} \tag{77}$$

The inverse of the triangular matrix of Equation (75) for a second degree polynomial is the 3x3 matrix

$$\underset{3x3}{T_v^{-1}(t)} = \begin{bmatrix} 1 & -t & t^2/2 \\ 0 & 1 & -t \\ 0 & 0 & 1/2 \end{bmatrix} \tag{78}$$

and for a third degree polynomial with a 4x4 matrix

$$\underset{4x4}{T_v^{-1}(t)} = \begin{bmatrix} 1 & -t & t^2/2 & -t^3/2\cdot 3 \\ 0 & 1 & -t & t^2/2 \\ 0 & 0 & 1/2 & -t/2 \\ 0 & 0 & 0 & 1/2\cdot 3 \end{bmatrix} \tag{79}$$

and for a fourth degree polynomial the 5x5 matrix is

$$\underset{5\text{x}5}{T_v^{-1}(t)} = \begin{bmatrix} 1 & -t & t^2/2 & -t^3/2\cdot 3 & t^4/2\cdot 3\cdot 4 \\ 0 & 1 & -t & t^2/2 & -t^3/2\cdot 3 \\ 0 & 0 & 1/2 & -t/2 & t^2/4 \\ 0 & 0 & 0 & 1/2\cdot 3 & -t/2\cdot 3 \\ 0 & 0 & 0 & 0 & 1/2\cdot 3\cdot 4 \end{bmatrix} \qquad (80)$$

The nested-nature of the size of the matrices can be seen from the above relation. A size expanding recursive inverting algorithm could be worked out (the analog of the recursive Householder inversion lemma for a matrix plus a dyad).

Time Derivatives of Two Time-Varying Bases. Polynomial bases of the form $\langle t$ or $\langle \tau$ where

$$\langle t = (1,\ t,\ t^2 \cdots t^{d-1}) \qquad (81)$$

and

$$\langle \tau = (1,\ \tau,\ \tau^2 \cdots \tau^{d-1})$$

can be referred to as monomial bases.

An arbitrary base of Equation (6) is

$$\langle d)x_b = \langle t\, B \qquad (82)$$

and the time derivative is

$$\langle d)\dot{x}_b = \left(\frac{d}{dt}\langle t\right)B + \langle t\, \dot{B} \qquad (83)$$

or by Equation (24) and Equation (83)

$$\dot{\sphericalangle}x_b = \sphericalangle x_b \left[B^{-1} V_t B + B^{-1} \dot{B} \right] \tag{84}$$

or the velocity of the $\sphericalangle x_b$ base elements in the $\sphericalangle x_b$ base is

$$\dot{\sphericalangle}x_b = \sphericalangle x_b \, V_x = \sphericalangle x_b \left[B^{-1} V_t \, B + B^{-1} \dot{B} \right] \tag{85}$$

or

$$\dot{B} = BV_x - V_t B \tag{86}$$

If $\dot{B}$=C a constant matrix one has

$$C = BV_x - V_t B \tag{87}$$

Equation (83) is exact analog of time varying-bases in classical dynamics of Gibbsian vectors.

If $V_x = -V_t$ Equation (86) becomes

$$\dot{B} = BV + VB \tag{88}$$

If the transformation between the bases is a constant then

$$BV_x = V_t B \tag{89}$$

or

$$V_x = B^{-1} \, V_t B \tag{90}$$

which is the familiar similarity transformation.

Multiple Integration of Bases and Base Dyadic Product Integration. Consider the indefinite integral of the base

$$\int \langle d) t \, dt = \left(\int dt \, , \int t \, dt, \cdots, \int t^{d-1} \, dt \right)$$

$$= \left(t, \frac{t^2}{2}, \frac{t^3}{3}, \ldots, \frac{t^d}{d} \right) \tag{91}$$

Equation (91) can be written as

$$\int \langle t \, dt = \langle t \, V^* + \frac{t^d}{d} \overset{d}{\langle} d)e \tag{92}$$

where the psuedo inverse matrix V* is given by Equation (65). Notice that the derivative of Equation (92) yields a matrix Calculus or linear product relation

$$\frac{d}{dt} \int \langle t \, dt = \langle t \, VV^* + t^{d-1} \overset{d}{\langle} d)e$$

$$= \langle t \begin{bmatrix} I & & 0 \; \overset{d}{(d-1\rangle} \\ (d-1)\mathrm{x}(d-1) & & \\ \langle d-1) & 0 & 0 \end{bmatrix} + t^{(d-1)} \overset{d}{\langle} d)e$$

$$= \langle t = (1, t, t^2, \ldots, t^{d-1}) \tag{93}$$

$$\int\int \langle t \, dt = \left[\langle t V^* + \frac{t^d}{d} \overset{d}{\langle} d)e \right] V^* + \frac{t^{(d+1)}}{(d+1)} \overset{d}{\langle} d)e$$

$$\int\int \langle t \, dt = \langle t (V^*)^2 + \frac{t^d}{d} \overset{d}{\langle} d)e \, V^* + \frac{t^{d+1}}{(d+1)} \overset{d}{\langle} d)e \tag{94}$$

Relations for higher orders can be written out.

Integration of Dyadic Product of Bases. Among the many relations occuring in polynomial analysis the integral of the dyad

$$\int t\rangle\langle t \, dt$$

is used.

It is useful to express Equation (91) as

$$\int \langle t \, dt = t\left(1,\ 1/2,\ 1/3,\ 1/4\ \ldots\ 1/d\right) D_t \tag{95}$$

where the diagonal matrix D_t is

$$\begin{bmatrix} 1 & 0 & & & 0 \\ 0 & t & & 0 & \\ 0 & 0 & t^2 & & \\ & & & \ddots & \\ 0 & & & 0 & t^{d-1} \end{bmatrix} \tag{96}$$

The integral of $t \langle t$ is

$$\int_{t_1}^{t_2} t \langle t \, dt = t^2 \left(1/2,\ 1/3,\ 1/4,\ \ldots\ 1/d,\ \frac{1}{d+1}\right) D_t \tag{97}$$

and the integral of $t^m \langle t$ is

$$\int_{t_1}^{t_2} t^m \langle t \, dt = t^{m+1} \left(1/m,\ 1/m+1,\ \ldots\ 1/m+d \right) D_t \tag{98}$$

The integral of $t^{d-1} \langle t$ is

$$\int_{t_1}^{t_2} t^{d-1} \langle t \, dt = t^{d} \left(\frac{1}{d-1} , \frac{1}{d} , \frac{1}{d+1} , \dots \frac{1}{2d-2} \right) D_t \tag{99}$$

From the above relations

$$\int_{t_1}^{t_2} t\rangle \langle t = \int_{t_1}^{t_2} \begin{bmatrix} \langle t \\ t \langle t \\ \vdots \\ t^{m} \langle t \\ \vdots \\ t^{d-1} \langle t \end{bmatrix} dt \tag{100}$$

$$= \begin{bmatrix} t \left(1 , 1/2 , 1/3 , \dots \quad 1/d \right) \\ t^{2} \left(1/2 , 1/3 , \ 1/4 , \dots \frac{1}{d+1} \right) \\ \vdots \qquad\qquad\qquad \vdots \\ t^{m+1} \left(\frac{1}{m} , \frac{1}{m+1} , \ . \quad . \quad . \quad \frac{1}{m+d} \right) \\ \vdots \qquad\qquad\qquad \vdots \\ t^{d-1} \left(\frac{1}{d-1} \right) \frac{1}{d} , \frac{1}{d+1} \qquad \frac{1}{2d-1} \Big) \end{bmatrix} D_t \tag{101}$$

or

$$\int_{t_1}^{t_2} t\rangle \langle t \, dt = tD_t \, H_{i\ell} \ D_t \Bigg|_{t_1}^{t_2} \tag{102}$$

where the well known Hilbert matrix $H_{i\ell}$ is

$$H_{i\ell} = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots & 1/d \\ 1/2 & 1/3 & 1/4 & \ddots & & 1/d+1 \\ 1/3 & 1/4 & & 1/d & & \\ 1/4 & & & & 1/d+1 & \\ \vdots & & & & & \\ 1/d & 1/d+1 & & \cdots & & 1/2d-1 \end{bmatrix}_{dxd} \tag{103}$$

Note that the Hilbert matrix is a special case of the Hankel matrix. The integral of Equation (102) is given by Equation (19) Section II.

B. INNER-PRODUCTS, BASE METRICS, HANKEL AND HILBERT MATRICES.

The classical vectors and dyads of Gibbs carried along the bases with the vectors and the operators (or transformations). Modern abstract mathematics attempted to surpress the bases and represent the vector as a row or column n-tuple and the matrix as a nxm matrix of elements (usually field elements). This was adequate for simple systems but where derivatives and many base changes are needed for large scale math-models, the matricized version of the Gibbsian representations simplifies many operations on multi-linear dynamical algebras. These extended Gibbsian techniques have been well developed for flight dynamics of multiple inner-acting bodies characterized by symmetric inertia matrices, or variance-matrices of measurements in many onboard moving bases (reference 69). The extension of these techniques to polynomial vectors, etc. provides some interesting relations and relates matrices to some very tedious age old classical analysis techniques.

Consider two polynomials

$$x_1(t) = {}_1\langle a \;\; t\rangle \tag{104}$$

$$x_1(t) = a_0 t^0 + a_1 t^1 + \ldots a_{d-1} t = {}_1\langle a \;\; t\rangle$$

(where $\langle a \;\; t\rangle$ is conventional inner product on real n-tuple vectors), and

$$x_2(t) = {}_2\langle a \;\; t\rangle = \langle t \;\; a\rangle_2 \tag{105}$$

then the "dyadic product" of the two vectors x_1 and x_2 is

$$x_1 x_2 = {}_1\langle a \;\; t\rangle\langle t \;\; a\rangle_2 \tag{106}$$

where the square dxd rank - one matrix or dyad is

$$t\rangle\langle t = \begin{pmatrix} t^1 \\ t^2 \\ \\ t^{d-1} \end{pmatrix} \left(t, \; t^1, \; t^2 \quad t^{d-1} \right) \tag{107}$$

$$t\rangle\langle t = \begin{bmatrix} 1 & t^1 & t^2 & t^3 & \cdots & t^{d-1} \\ t^1 & t^2 & t^3 & & & \\ t^2 & t^3 & & & & \\ t^3 & & & & & \\ \vdots & & & & & \\ t^{d-1} & & & & & t^{d-1} t^{d-1} \end{bmatrix} \tag{108}$$

or as a column of row vectors

$$t\rangle\langle t = \begin{bmatrix} \langle t \\ t\,\langle t \\ t^2\langle t \\ t^{d-1}\langle t \end{bmatrix} \tag{109}$$

and as a row of column vectors

$$t\rangle\langle t = \left[t\rangle, \; t\rangle\, t, \; t\rangle\, t^2 \quad t\rangle\, t^{d-1} \right] \tag{110}$$

Clearly, the symmetric dyad is a rank-one Hankel matrix

$$t\rangle\langle t = \begin{bmatrix} 1 & t & t^2 & t^3 & t^4 & \cdots & t^{d-1} \\ t & t^2 & t^3 & t^4 & & & \\ t^2 & t^3 & t^4 & & & & \\ t^3 & t^4 & & & & & \\ t^4 & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & t^{2d-1} \\ \cdot & & & & & & \\ t^{d+1} & t^{d+2} & & & & & t^{2d-2} \end{bmatrix} \tag{111}$$

The psuedo-inverse of (111) is

$$\left(t\rangle\langle t \right)^* = \frac{t\rangle\langle t}{\langle t \;\; t\rangle^2} \tag{112}$$

The inner produce of the two polynomials on the interval 0 to 1 is

$$\int_0^1 x_1 x_2 \, dt = {}_1\langle a \left(\int_0^1 t\rangle\langle t \, dt \right) a\rangle_2 \tag{113}$$

The integral over the unit interval is the Hilbert matrix

$$\int_0^1 t\rangle\langle t \; dt = \begin{bmatrix} 1 & 1/2 & 1/3 & & 1/d \\ 1/2 & 1/3 & & \cdot & \\ 1/3 & & \cdot & & 1/d+1 \\ \vdots & 1/d & & 1/d+1 & \\ 1/d & & 1/d+1 & & 1/2d-1 \end{bmatrix} = H_{i\ell} \qquad (114)$$

The matrix of inner-products between the base vectors is classically called a metric-matrix, thus the metric matrix with respect to the inner-product so defined is a Hilbert matrix.

The inner-product defined over an arbitrary normalized interval can be written by Equation (148) sec (1) as

$$\frac{1}{t_2-t_1} \int_{t_1}^{t_2} t\rangle\langle t \; dt = M_{t_1 t_2} \qquad (115)$$

$$= t \, D_t \, H_{i\ell} \, D_t \Big|_{t_1}^{t_2} \frac{1}{(t_2-t_1)} \qquad (116)$$

or

$$M_{t_1 t_2} = \left[t_2 \, D_{t_2} \, H_{i\ell} \, D_{t_2} - t_1 \, D_{t_1} \, H_{i\ell} \, D_{t_1} \right] \frac{1}{t_2 - t_1} \qquad (117)$$

For example if $t_1=0$, then

$$M_{0t_2} = D_{t_2} \, H_{i\ell} \, D_{t_2} \qquad (118)$$

(119)

$$= \begin{bmatrix} 1 & & & & & \\ & t & & & & \\ & & t_2^2 & & & \\ & & & t_2^3 & & \\ & & & & \ddots & \\ & & & & & t_2^{d-1} \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & \cdot \\ 1/3 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & & \cdot \\ \cdot & t_2 & 0 & & \cdot \\ \cdot & \cdot & t_2^2 & & \cdot \\ \cdot & 0 & \cdot & \ddots & \cdot \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & t_2^{d-1} \end{bmatrix}$$

For example for the fifth degree polynomial of Equation (3) Section I, the (0 ,1) interval of Equation (113) is

$$\underset{6x6}{H_{i\ell}} = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 & 1/10 \\ 1/6 & 1/7 & 1/8 & 1/9 & 1/10 & 1/11 \end{bmatrix} \qquad (120)$$

For the third degree polynomial the interval $(0, t_f)$, $t_2=t_f$ of Equation (118) is

$$\underset{4x4}{M_{0t_2}} = \begin{bmatrix} 1 & t_{f/2} & t^2_{f/3} & t^3_{f/4} \\ t_{f/2} & t^2_{f/3} & t^3_{f/4} & t^4_{f/5} \\ t^2_{f/3} & t^3_{f/4} & t^4_{f/5} & f^5_{f/6} \\ t^3_{f/4} & t^4_{f/5} & t^5_{f/6} & t^6_{f/7} \end{bmatrix} \qquad (121)$$

Note that the metric matrix of Equation (121) is a Hankel matrix.

By Equations (117) and (121) it is seen that the upper and lower intervals are Hankel matrices and the difference between the two Hankel matrices of Equation (117) is a Hankel matrix

$$\underset{d\times d}{M_{t_1t_2}} = \begin{bmatrix} 1 & W/2 & W^2/3 & W^3/4 & \cdots & \frac{W^{d-1}}{d} \\ W/2 & W^2/3 & & & & \\ W^2/3 & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \frac{W^{d-1}}{d} & & & & & \frac{W^{2d-2}}{2d-1} \end{bmatrix} \qquad (122)$$

where the interval width is

$$W = t_2 - t_1 \qquad (123)$$

C. CONTINUOUS TIME SHIFT BASE CHANGES, BASE TRANSITION MATRICES AND HANKEL METRIC-MATRICES FOR FIXED TIME SPAN. In Section I it is shown that a linear scaling and translation on the time axis generates a triangular-matrix base change. In Section I it is shown that the metric-matrix for a monomial base change on a normalized time interval t_1 to t_2 generates a Hankel matrix. This section will show how translation on the time axis (to the front, back, and center of a time interval) change the form of the Hankel matrix. Clearly the most simple metric would be an identity matrix, the result of a Gram-schmidt orthogonalization process (orthogonal polynomials). Smoothing over a span and estimating states at forward, backward and span-center occurs quite often.

The time changes and variables are shown in Figure (1). The variable t_o is some fixed initial time, the variable t_b is the time point at the back of the span, the time t_f is the front point, and t_c is the center or mid-point. The width of the interval $t_f - t_b$ is designated as W.

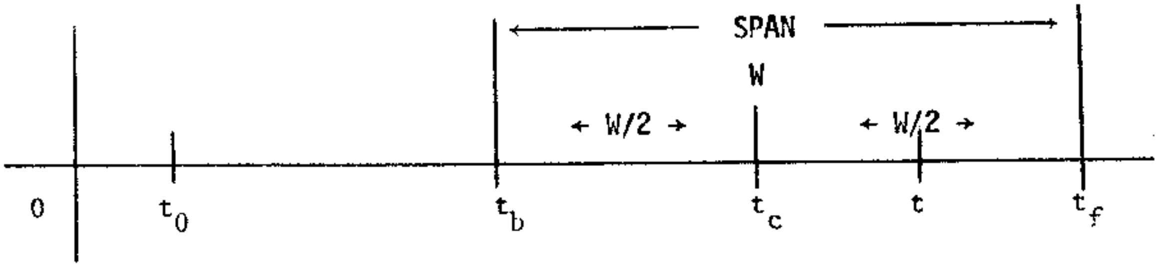


FIGURE (1)
TIME-AXIS POINTS

In discreet smoothing as well as in continuous smoothing one can use a moving span; however, for purposes of this section, one can consider the variables of integration as t, t´, t´´, t´´´ all four to be described below and considered to be varying over the span W.

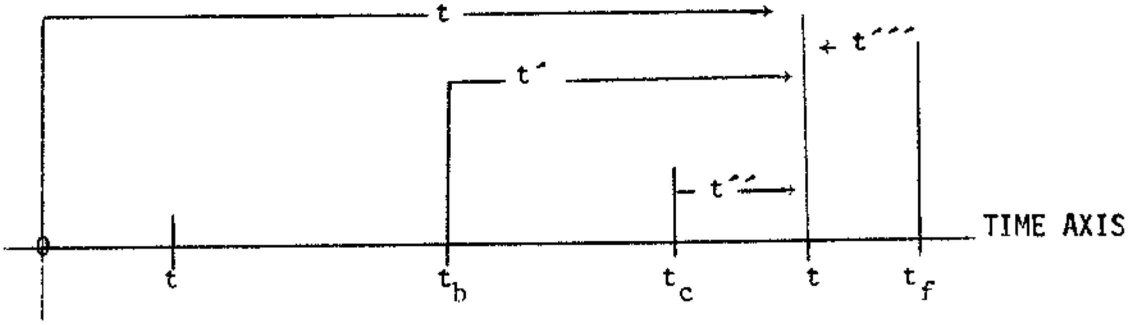


FIGURE (2)
FOUR VARIABLE TIMES

The equivalence relations by Figure (2) are

$$t = t_b + t' = t_c + t'' = t_f - t''' \tag{124}$$

If we solve each of the new variable equations

$$t' = t - t_b \tag{125}$$

$$t'' = t - t_c \tag{126}$$

$$t''' = -t + t_f \tag{127}$$

The lower and upper limits of integration are t_b and t_f for the variable t ; for the new variable by Equation (125) [parentheses meaning functions of]

$$t'(t) = t - t_b \tag{128}$$

and at the lower limits

$$t'(t_b) = t_b - t_b = 0 \tag{129}$$

and at the upper limit

$$t'(t_f) = t_f - t_b = W \tag{130}$$

In a similar manner the lower and upper limits for t'' are

$$t''(t_b) = t_b - t_c = - W/2 \tag{131}$$

and

$$t''(t_f) = t_f - t_c = W/2 \tag{132}$$

The integration limits for t''' are

$$t'''(t_b) = t_f - t_b = W \tag{133}$$

and

$$t'''(t_f) = 0 \tag{134}$$

If we define powers of the variable t′, t″, and t‴ by Equation (11) Section I we obtain for a fifth degree polynomial

$$\langle t' = [(1, t', (t')^2, (t')^3, \cdots (t')^5]$$

$$\langle t' = \langle t \begin{bmatrix} 1 & -t_b & t_b^2 & -t_b^3 & t_b^4 & -t_b^5 \\ 0 & 1 & -2t_b & 3t_b^2 & -4t_b^3 & 5t_b^4 \\ 0 & 0 & 1 & -3t_b & 6t_b^2 & -10t_b^3 \\ 0 & 0 & 0 & 1 & -4t_b & 10t_b^2 \\ 0 & 0 & 0 & 0 & 1 & -5t_b \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{135}$$

Equation (135) can be extended to higher degree polynomials, where the binomial coefficents continue along the columns.

The translations to the origin at the center of the span by Equation (126) yields

$$\langle t'' = \langle t \begin{bmatrix} 1 & -t_c & t_c^2 & -t_c^3 & t_c^4 & -t_c^5 \\ 0 & 1 & -2t_c & 3t_c^2 & -4t_c^3 & 5t_c^4 \\ 0 & 0 & 1 & -3t_c & 6t_c^2 & -10t_c^3 \\ 0 & 0 & 0 & 1 & -4t_c & 10t_c^2 \\ 0 & 0 & 0 & 0 & 1 & -5t_c \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{136}$$

and the translation to the front of the span by Equation (127) yields

$$\langle t''' = \langle t \begin{bmatrix} 1 & t_f & t_f^2 & t_f^3 & t_f^4 \\ 0 & -1 & 2t_f & -3t_f^2 & -4t_f^3 \\ 0 & 0 & 1 & 3t_f & 6t_f^2 \\ 0 & 0 & 0 & -1 & -4t_f \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{137}$$

The inverses of the three transformations of Equations (135),(136) and (137) are for 4x4 cases

$$\langle t = \langle t' \begin{bmatrix} 1 & t_b & t_b^2 & t_b^3 \\ 0 & 1 & 2t_b & 3t_b^2 \\ 0 & 0 & 1 & 3t_b \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \langle t'' \begin{bmatrix} 1 & t_c & t_c^2 & t_c^3 \\ 0 & 1 & 2t_c & 3t_c^2 \\ 0 & 0 & 1 & 3t_c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \langle t''' \begin{bmatrix} 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \\ 0 & 0 & 1 & 3t_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{138}$$

The primed base of Equation (135) and its transpose are

$$\langle t' = \langle t T(t_b)$$

$$t' \rangle = T^T(t_b)\, t\rangle \tag{139}$$

and the dyadic product is

$$t'\rangle \langle t' = T^T(t_b)\, t\rangle \langle t T^T(t_b) \tag{140}$$

The metric of the new base by Equation (139) and the integration limits of Equations (129) and (130) are for fixed t_b

$$\frac{1}{W} \int_0^W t'\rangle \langle t' \; dt' = \frac{T^T(t_b)}{t_f - t_b} \left[\int_{t_b}^{t_f} t\rangle \langle t \; dt \right] T(t_b) \tag{141}$$

The center-referenced metric matrix by Equations (131) and (132) is

$$\frac{1}{W}\int_{-W/2}^{W/2} t''\rangle\langle t'' \, dt'' = \frac{T^T(t_c)}{W}\left[\int_{t_b}^{t_f} t\rangle\langle t \, dt\right] T(t_c) \qquad (142)$$

Note the symmetric integration limits of Equation (142), it will be shown later how they simplify the Hankel matrix. The metric-matrix for the transformation of Equation (137) is

$$\int_{W}^{0} t'''\rangle\langle t''' \, dt''' = \frac{T^T(t_f)}{W}\left\{\int_{t_b}^{t_f} t\rangle\langle t \, dt\right\} T(t_f) \qquad (143)$$

The **metric-matrix** of Equation (141) by Equation (122) is

$$\frac{1}{W}\int_{0}^{W} t'\rangle\langle t' \, dt' = \begin{bmatrix} 1 & W/2 & W^2/3 & W^3/4 & \cdots \\ W/2 & W^2/3 & W^3/4 & W^4/5 & \cdots \\ W^2/3 & W^3/4 & W^4/5 & W^4/6 & \cdots \\ W^3/4 & W^4/5 & W^5/6 & W^6/7 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \qquad (144)$$

The metric-matrix for the transformation to the center of the span by Equation (142) and Equation (117) is

$$\frac{1}{W}\int_{-W/2}^{W_2} t''\rangle\langle t'' \, dt''$$

$$= \begin{bmatrix} 1 & 0 & \frac{W^2}{3\cdot 2^2} & 0 & \frac{W^4}{5\cdot 2^4} & 0 & \frac{W^6}{7\cdot 2^6} & 0 \\ 0 & \frac{W^2}{3\cdot 2^2} & 0 & \frac{W^4}{5\cdot 2^4} & 0 & \frac{W^6}{7,2^6} & 0 & \frac{W^8}{9\cdot 2^8} \\ \frac{W^2}{3\cdot 2^2} & 0 & \frac{W^4}{5\cdot 2^4} & 0 & \frac{W^6}{7\cdot 2^6} & 0 & \frac{W^8}{9\cdot 2^8} & 0 \\ 0 & \frac{W^4}{5\cdot 2^4} & 0 & \frac{W^6}{7\cdot 2^6} & 0 & \frac{W^8}{9\cdot 2^8} & 0 & \frac{W^{10}}{11,2^{10}} \\ \frac{W^4}{5\cdot 2^4} & 0 & \frac{W^6}{7\cdot 2^6} & 0 & \frac{W^8}{9\cdot 2^8} & 0 & \frac{W^{10}}{11\cdot 2^{10}} & 0 \\ 0 & \frac{W^6}{7\cdot 2^6} & 0 & \frac{W^8}{9\cdot 2^8} & 0 & \frac{W^{10}}{11\cdot 2^{10}} & 0 & \frac{W^{12}}{13\cdot 2^{12}} \\ \frac{W^6}{7\cdot 2^6} & 0 & \frac{W^8}{9\cdot 2^8} & 0 & \frac{W^{10}}{11\cdot 2^{10}} & 0 & \frac{W^{12}}{12\cdot 2^{12}} & 0 \end{bmatrix} \tag{145}$$

The third transformation Equation (143) is the negative of the first one, Equation (144) or

$$\frac{1}{W}\int_{W}^{0} t''\rangle\langle t''' \, dt''' = -\frac{1}{W}\int_{0}^{W} t\rangle\langle t' \, dt' \tag{146}$$

One can also obtain the transformations from the t´ to the t´´ variables etc., as done in Equation (138) and the related dyads of Equation (140).

D. TOEPLITZ STATE TRANSITION MATRICES IN CONTINUOUS POLYNOMIALS.

Polynomials are perhaps the most widely used approximating functions in numerical analysis approximation theory and in estimation theory. They are essential to structure studies of matrices from the spectrum point of view in that one considers characteristic and minimal polynomials over the complex field as well as matrix polynomials. The approximation of exponential functions yeilds polynomials

$$e^{A} = I + A + \frac{A^2}{2!} + \ldots \frac{A^n}{n!} + \ldots \tag{147}$$

Consider the polynomial

$$x(t) = a_0 + a_1 t + a_2 t^2 + \ldots a_{d-1}\, t^{d-1} = \langle d) t \quad a(d \rangle \tag{148}$$

and its derivatives by Equation (72) Section I

$$\dot{x} = \langle t \quad V \quad a \rangle$$

$$\ddot{x} = \langle t \quad V^2 \quad a \rangle$$

$$\vdots$$

$$x^{(d-1)} = \langle t \quad V^{d-1} \quad a \rangle$$

Packaging Equation (149) as a column vector of dimension d

$$\begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \\ \vdots \\ x^{(d-1)} \end{bmatrix} = \begin{bmatrix} \langle t \quad I \quad a \rangle \\ \langle t \quad V \quad a \rangle \\ \langle t \quad V^2 \quad a \rangle \\ \vdots \\ \langle t \quad V^{(d-1)} \quad a \rangle \end{bmatrix} = \begin{bmatrix} \langle t \\ \langle t \; V \\ \langle t \; V^2 \\ \vdots \\ \langle t \; V^{(d-1)} \end{bmatrix} a \rangle \tag{149}$$

or

$$x(t)\ (d\rangle = \underset{d\times d}{T_V(t)}\quad a\ (d\rangle \tag{150}$$

where

$$T_V(t) = \begin{bmatrix} \langle t \\ \langle t\ V \\ \langle t\ V^2 \\ \vdots \\ \langle t\ V^{d-1} \end{bmatrix} \tag{151}$$

and by Equation (21)

$$\frac{d}{dt}\ T_V(t) = \dot{T}_V(t) = T_V(t)\ V \tag{152}$$

Solving Equation (150) for the constant $a\rangle$

$$a\rangle = T_V^{-1}(t)\ x(t)\,(d\rangle \tag{153}$$

Note that the vector $a\rangle$ is a constant (usually taken constant over a time interval) and can be evaluated at points t=0 or arbitrary point t_b hence

$$a\rangle = T_V^{-1}(0)x(0)\rangle = T_V^{-1}(t_b)x(t_b)\rangle \tag{154}$$

Using Equation (154) in Equation (150)

$$x(t)\,(d\rangle = T_V(t)\ T_V^{-1}(t_b)\ x(t_b)\,(d\rangle \tag{155}$$

also

$$x(t)(d\rangle = T_v(t)\ T_v^{-1}(0)\ x(0)(d\rangle = \Phi\ (t,0)\ x\ (0)\ (d\rangle \qquad (156)$$

The product of the upper triangular matrix $T_v(t)$ and the diagonal matrix $T_v^{-1}(0)$ by Equation (75) and Equation (77) is

$$T_v(t)\ T_v^{-1}(0) \qquad (157)$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 & \cdots & t^{d-1} \\ 0 & 1 & 2t & 3t^2 & \cdots & (d-1)t^{d-2} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 0 & & & & & (d-1)! \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1/2 & 0 & & \\ & & & 1/3 & & \\ & & & & \ddots & \\ & & & & & \frac{1}{(d-1)!} \end{bmatrix}$$

or

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \\ \\ x^{(d-1)}(t) \end{bmatrix} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & & \cdots & & \frac{t^{d-1}}{(d-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & & & \frac{t^{d-2}}{(d-2)!} \\ . & & & & & & & \vdots \\ . & & & & & 0 & 1 & t \\ . & & . & & . & & 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ \dot{x}(0) \\ \ddot{x}(0) \\ \vdots \\ x^{(d-1)}(0) \end{bmatrix} \quad (158)$$

Note that Equation (158) develops the finite McLauren series expansion terms for exact polynomials. This same technique is used to develop a Taylor series approximation of functions in Section V

The state-transition matrix of Equation (155) is

$$x(t)\rangle = \Phi(t, t_b)\ x(t_b)\rangle \qquad (159)$$

where

$$\Phi(t,t_b) = T_v(t)\ T_v^{-1}(t_b)$$

By Equation (75) and Equation (80)

$$\Phi(t,t_b) =$$

$$\begin{bmatrix} 1 & t & t^2 & t^3 & t^4 \\ 0 & 1 & 2t & 3t^2 & 4t^3 \\ 0 & 0 & 2 & 6t & 12t^2 \\ 0 & 0 & 0 & 6 & 24t \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix} \begin{bmatrix} 1 & -t_b & t_b^2/2 & -t_b^3/6 & t_b^4/24 \\ 0 & 1 & -t_b & t_b^2/2 & -t_b^3/6 \\ 0 & 0 & 1/2 & -t_b/2 & t_b^2/4 \\ 0 & 0 & 0 & 1/6 & -t_b/6 \\ 0 & 0 & 0 & 0 & 1/24 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (t-t_b) & \frac{(t-t_b)^2}{2!} & \frac{(t-t_b)^3}{3!} & \frac{(t-t_b)^4}{4!} \\ 0 & 1 & (t-t_b) & \frac{(t-t_b)^2}{2!} & \frac{(t-t_b)^3}{3!} \\ 0 & 0 & 1 & (t-t_b) & \frac{(t-t_b)^2}{2!} \\ 0 & 0 & 0 & 1 & (t-t_b) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (160)$$

By Equation (160) for a 5x5 matrix

$$\Phi(t,t_b) = \begin{bmatrix} 1 & t' & \frac{(t')^2}{2} & \frac{(t')^3}{3!} & \frac{(t')^4}{4!} \\ 0 & 1 & t' & \frac{(t')^2}{2!} & \frac{(t')^3}{3!} \\ 0 & 0 & 1 & t' & \frac{(t')^2}{2!} \\ 0 & 0 & 0 & 1 & t' \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (161)$$

where

$$t' = t - t_b \qquad (162)$$

the inverse matrix of Equation (160) is

$$\Phi(t_b,t) = \phi^{-1}(t,t_b) = T_v(t_b)\ T_v^{-1}(t) \qquad (163)$$

and by Equations (80) and (75) is

$$\phi^{-1}(t,t_b) \begin{bmatrix} 1 & -t' & \frac{(t')^2}{2} & -\frac{(t')^3}{3!} & \frac{(t')^4}{4!} \\ 0 & 1 & -t' & \frac{(t')^2}{2} & -\frac{(t')^3}{3} \\ 0 & 0 & 1 & -t' & -\frac{(t')^2}{2} \\ 0 & 0 & 0 & 1 & -t' \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (164)$$

By Equations (161) and (166)

$$x(t_b)\rangle = \Phi^{-1}(t,t_b)\ x(t)\rangle \tag{165}$$

Note that the inverse transition matrix is obtained merely by a negative time sign, that is

$$-t' = -(t-t_b) = t_b - t \tag{166}$$

substituted in Equation (163).

Powers of the Transition Matrix $\Phi(t,t_b)$.

$$\Phi^2(t,t_b) = \begin{bmatrix} 1 & 2t' & \frac{(2t')^2}{2!} & \frac{(2t')^3}{3!} & \frac{(2t')^4}{4!} \\ 0 & 1 & 2t' & \frac{(2t')^2}{2!} & \frac{(2t')^3}{3!} \\ 0 & 0 & 1 & 2t' & \frac{(2t')^2}{2!} \\ 0 & 0 & 0 & 0 & 2t' \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{167}$$

$$\Phi^d = \begin{bmatrix} 1 & dt' & \frac{(dt')^2}{2!} & \frac{(dt')^3}{3!} & & \frac{(dt')^4}{4!} \\ 0 & 1 & dt' & \cdot & \cdot & \cdot \\ \cdot & \cdot & & & & \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ 0 & 0 & 0 & 0 & & 1 \end{bmatrix} \tag{168}$$

and similar for powers of the inverse

$$\Phi^{-d} = \begin{bmatrix} 1 & -dt & \frac{(dt')^2}{2!} & -\frac{(dt')^3}{3!} & & \cdots \\ 0 & 1 & dt' & . & . & . \\ . & & & & & \\ . & & . & & & . \\ . & & & & . & \\ 0 & 0 & 0 & 0 & & 1 \end{bmatrix} \qquad (169)$$

<u>Transitions over Integer Multiples of Time Intervals.</u> The transition matrix by Equation (161) from some time t_b to time t is

$$x(t)\rangle = \Phi\ (t,t_b)\ x(t_b)\rangle$$

if

$$t = nt_b \qquad (170)$$

$$t-t_b = (n-1)\ t_b = t' \qquad (171)$$

$$x(nt')\rangle = \Phi(nt_b,t_b)\ x(t_b)\rangle \qquad (172)$$

where it is easily established

$$\Phi(nt_b,t_b) = \Phi^{n-1}(t_b,0) \qquad (173)$$

also

$$\Phi(nt_b,t_b) = \Phi^{n}(t_b,0) \qquad (174)$$

E. TOEPLITZ STATE TRANSITION MATRIX IN TAYLOR SERIES APPROXIMATION.

Consider the scalar valued function of time x(t) where

$$t = t_b + \Delta t \tag{175}$$

then

$$x(t) = x(t_b) + \dot{x}(t_b)\,\Delta t + \ddot{x}(t_b)\,\frac{\Delta t^2}{2!} + \dddot{x}(t)\,\frac{\Delta t^3}{3!} + \ldots\ \varepsilon_x$$

$$\dot{x}(t) = 0\ x(t_b) + 1\ \dot{x}(t_b) + \ddot{x}(t_b)\,\Delta t + \frac{\dddot{x}(t_b)}{2!}\,\Delta t^2 + \ldots\ \varepsilon_{\dot{x}} \tag{176}$$

$$\vdots$$

$$x^{(d-1)}(t) = x^{(d-1)}(t_b) + \varepsilon_x d-1$$

In state vector form

$$\begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \\ \cdot \\ \cdot \\ \cdot \\ x^{(d-1)} \end{pmatrix}_t = \begin{bmatrix} 1 & \Delta t & \Delta t^2/2 & \Delta t^3/3! & \Delta t^4/4! & \cdots \\ 0 & 1 & \Delta t & \Delta t^2/2! & \Delta\ t^3/3! & \cdots \\ 0 & 0 & 1 & \Delta t & \Delta t^2/2! & \cdots \\ & & 0 & 1 & & \\ & & & 0 & & \Delta t^2/2! \\ & & & & & \Delta t \\ 0 & 0 & 0 & 0 & & 1 \end{bmatrix} \begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \\ \cdot \\ \cdot \\ \cdot \\ x^{(d-1)} \end{pmatrix}_{t_b}$$

$$+ \varepsilon (d \rangle \tag{177}$$

or

$$x(t)(d\rangle = \mathrm{Tay}(t-t_b)\ x(t_b(d\rangle + \varepsilon(d\rangle \tag{178}$$

The Taylor-Toeplitz-transition matrix transforms the states at time t_b to time t, and can be designated as

$$\Phi(t,t_b) = \begin{bmatrix} 1 & t-t_b & \frac{(t-t_b)^2}{2!} & \\ 0 & 1 & (t-t_b) & \\ \vdots & 0 & 1 & \\ 0 & . & . & \\ . & . & . & 1 \end{bmatrix} \tag{179}$$

If x(t) of Equation (176) is a p dimensional column vector and time differentiable then as a column of column vectors

$$\begin{pmatrix} x(t)\rangle \\ \dot{x}(t)\rangle \\ \ddot{x}(t)\rangle \\ \vdots \\ x^{(d-1)}(t)\rangle \end{pmatrix} = \begin{bmatrix} I_{p\times p} & I\ \Delta t & \frac{I\ \Delta t^2}{2!} & \cdots & & \\ 0 & I & \Delta t\ I & & & \\ 0 & 0 & I & & & \\ \vdots & \vdots & & I & & \\ & & & & I & \Delta t\ I \\ & & & & & I \end{bmatrix} \begin{pmatrix} x\rangle \\ \dot{x}\rangle \\ . \\ \vdots \\ x^{(d-1}\rangle \end{pmatrix}_{t_b}$$

$$+ \begin{pmatrix} e\rangle_{x} \\ e\rangle_{\dot{x}} \\ \vdots \end{pmatrix} \tag{180}$$

If the column vectors $x(t)(p\rangle$, $\dot{x}(t)(p\rangle$ ... etc. are packaged as a row of column vectors one obtains

$$\left(x(t)(p\rangle , \dot{x}(t)(p\rangle \ \ldots \ x^{(d-1)}(t) \ (p\rangle \right) = \underset{pxd}{X(t)}$$

and

$$\underset{pxd}{X(t)} = \left[x\rangle , \dot{x}\rangle \ \ldots \ x^{(d-1)}\rangle \right]_{t_b} \begin{bmatrix} I & 0 & 0 & 0 \\ I\Delta t & I & & \\ I\Delta t^{2}/2! & & & \\ \vdots & & & \\ & & & 0 \\ & & I\Delta t & I \end{bmatrix}$$

$$+ \underset{pxd}{E} \tag{181}$$

$$\underset{pxd}{X(t)} = \underset{pxd}{X(t_b)}\ T^T_{ay}(t-t_b) + E \tag{182}$$

Note that the state of Equation (180)are column vectors of dimension pxd; whereas the vectors of Equation (182) are rectangular matrices of size pxd.

F. STATE DYNAMICS FOR CONTINUOUS POLYNOMIALS AND POLYNOMIAL BASE CHANGE. This section shows the conventional state-vector differential equation and solution associated with polynomials, plus novel results obtained from the time derivatives of a pair of oblique bases connected by a constant matrix.

If the d^{th} derivative of a continuous polynomial (monomial form) is a constant, that is

$$\overset{(d)}{x}(t) = \overset{(d)}{x}\ (0) \tag{183}$$

and the state vector is

$$x(t)(d\rangle = \begin{pmatrix} x(t) \\ \dot{x}(t) \\ \vdots \\ \overset{(d-1)}{x}(t) \end{pmatrix} \tag{184}$$

and

$$\dot{x}\rangle = \begin{pmatrix} \dot{x} \\ \ddot{x}(t) \\ \vdots \\ x^{d}(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \\ x^{(d-1)} \end{bmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x^{(d)}(0) \end{pmatrix} \tag{185}$$

or

$$\dot{x}\rangle = F\, x\rangle + e\rangle_{d}\, x^{(d)}(0) \tag{186}$$

Note that the matrix F is nilpotent index d, that is

$$F^{d} = 0. \tag{187}$$

The companion-matrix form for a d^{th} order dynamical system with state feedback term is given as

$$x^{(d)}(t) = \left(f_{o}, f_{1}, \ldots, f_{d-1}\right) \begin{pmatrix} x \\ \dot{x} \\ \vdots \\ x^{d-1} \end{pmatrix} + g(t) \tag{188}$$

and the companion matrix is

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & & & & \vdots \\ f_{o} & f_{1} & & & f_{d-1} \end{bmatrix} = \begin{bmatrix} {}^{2}\langle e \\ {}^{3}\langle e \\ \vdots \\ \langle f \end{bmatrix} \tag{189}$$

Clearly Equation (185) holds for

$$\langle f = \langle 0 \quad . \tag{190}$$

For the general case of Equation (188) one has

$$\dot{x}(d\rangle = F\, x(d\rangle + e(\rangle_d\, g(t) \tag{191}$$

and the solution in terms of the fundamental matrix $\Phi(t)$ is

$$x(t)\rangle = \underset{dxd}{\Phi}(t)\left[x(t_0)\rangle + \int_{t_0}^{t} \Phi^{-1}(\tau)\quad e\rangle_d\, g(\tau)d\tau \right] \tag{192}$$

where

$$\Phi(t) = e^{Ft} \tag{193}$$

and

$$\Phi^{-1}(\tau) = e^{-F\tau} \tag{194}$$

or

$$x(t)\rangle = e^{Ft}x(t_0)\rangle + \int_{t_0}^{t} e^{F(t-\tau)}\; e\rangle_d\, g(\tau)d\tau \tag{195}$$

The companion matrix of Equation (189) can be expressed as the sum of the nilpotent matrix of Equation(185) plus a dyad

$$F = S_{uo} + e\rangle_d \langle f \tag{196}$$

where shift-up and out matrix S_{uo} is given by Equations (31)

$$S_{uo} = \begin{bmatrix} {}^{2}\langle e \\ {}^{3}\langle e \\ \vdots \\ {}^{d}\langle e \\ \langle 0 \end{bmatrix} \tag{197}$$

The shift operator of Equation (197) has the **powers** property given as

$$S_{uo}^{2} = \begin{bmatrix} {}^{3}\langle e \\ {}^{4}\langle e \\ \vdots \\ {}^{d}\langle e \\ \langle 0 \\ \langle 0 \end{bmatrix} \tag{198}$$

$$S_{uo}^{d-1} = \begin{bmatrix} {}^{d}\langle e \\ \langle 0 \\ \vdots \\ \langle 0 \end{bmatrix} \tag{199}$$

$$S_{uo}^{d} = [0] \quad . \tag{200}$$

By the matrix exponential expansion of Equation (1) Section A

$$e^{S_{uo}\,t} = I + S_{uo}t + \frac{S_{uo}^{2}t^{2}}{2!} + \ldots \frac{S_{uo}^{d-1}t^{d-1}}{(d-1)!} \tag{201}$$

$$\Phi(t) = (I,\ S_{uo},\ S_{uo}^{2},\ \ldots\ S_{uo}^{d-1}) \begin{pmatrix} 1 \\ t \\ \frac{t}{2!} \\ \vdots \\ \frac{t^{d-1}}{(d-1)!} \end{pmatrix} \tag{202}$$

For the 3x3 matrix case

$$\Phi(t) = I + S_{uo}t + \frac{S_{uo}^{2}t^{2}}{2!} = \begin{bmatrix} 1 & t & t^{2}/2! \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \tag{203}$$

We also have the fundamental matrix differential equation

$$\dot{\Phi} = F\ \Phi\ (t) \tag{204}$$

and for the F of Equation (185)

$$\dot{\Phi} = S_{uo}\ \Phi\ (t) \tag{205}$$

and by Equation (202)

$$\dot{\Phi} = S_{uo}\left[I, S_{uo}, S_{uo}^2, \dots\right] \begin{pmatrix} 1 \\ t \\ \frac{t}{2!} \\ \vdots \\ \frac{t^{d-1}}{(d-1)!} \end{pmatrix} \qquad (206)$$

The extended state vector is

$$\begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \\ \vdots \\ x^{(d-1)} \\ x^{(d)}(t) \end{pmatrix} = \begin{bmatrix} x(d\rangle \\ \\ \langle f\, x(d\rangle + g(t) \end{bmatrix} \qquad (207)$$

or a d+1 vector. If the $\langle f$ vector and g(t) are known then there are only d independent variables and by Equation (195) the solution vector is completely known if the initial state vector is known. If $\langle f$ is known (for example: $\langle 0$) and g(t) is assumed an unknown constant over a time interval, then one has a (d+1) dimensional state vector to estimate. As an example a constant acceleration scalar system over a time span is described by

$$\ddot{x}(t) = \ddot{x}(t_0) \qquad t \geq t_0 \qquad (208)$$

and

$$\dot{x}(2\rangle = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ \ddot{x}(t_0) \end{pmatrix} \tag{209}$$

Equation (208) implies a second degree time polynomial or

$$x(t) = (1,t,t^2)\ a(3\rangle \tag{210}$$

$$\dot{x}(t) = \langle t\ V\ a\rangle = \langle a\ V^T\ t\rangle \tag{211}$$

$$\ddot{x}(t) = \langle t\ V^2\ a\rangle = \langle a\ V^{T2}\ t\rangle \tag{212}$$

The two-dimensional state vector is

$$\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = x(2\rangle = \underset{2x3}{\begin{bmatrix} \langle a \\ \langle a\ V^T \end{bmatrix}}\ t(3\rangle = \underset{2x3}{T_V(a)}\ t(3\rangle \tag{213}$$

and the three-dimensional state vector is

$$\begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \end{pmatrix} = x(3\rangle = \begin{bmatrix} \langle a \\ \langle a\ V^T \\ \langle a\ (V^T)^2 \end{bmatrix}\ t(3\rangle = \underset{3x3}{T_V(a)}\ t(3\rangle \tag{214}$$

The derivatives of the two state vectors above are

$$\dot{x}(2\rangle = \underset{2x3}{T_V(a)}\ V^T t\rangle \tag{215}$$

and

$$\dot{x}(3\rangle = \underset{3x3}{T_v(a)}\ V^T t(3\rangle \tag{216}$$

By Equation (214)

$$t(3\rangle = \underset{3x3}{T_v^{-1}(a)}\ x(3\rangle \tag{217}$$

or Equation (217) in Equation (216)

$$\dot{x}(3\rangle = \underset{3x3}{T_v(a)}\ V\ T_v^{-1}(a)\ x(3\rangle \tag{218}$$

or

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \\ \dddot{x} \end{pmatrix} = \dot{x}(3\rangle = \underset{3x3}{F_e}\ x(3\rangle = \underset{3x3}{S_{uo}}\ x(3\rangle \tag{219}$$

Where the extended state dynamics matrix is

$$\underset{3x3}{F_e} = T_v(a)\ V\ T_v^{-1}(a) = \underset{3x3}{S_{uo}} \tag{220}$$

and clearly has rank 2, the rank of V.

The two dimensional state vector at time zero is by Equation (213)

$$x(0)(2\rangle = \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & 2a_2 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \tag{221}$$

and the solution to the two dimensional case by Equation (192) is

$$x(t)(2\rangle = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \langle a\, V^2 \tau \rangle \end{pmatrix} d\tau \qquad (222)$$

$$= \begin{bmatrix} a_0 + a_1 t + a_2 t^2 \\ a_1 + 2a_2 t \end{bmatrix} \qquad (223)$$

Note that Equation (219) is homogeneous, while Equation (209) is non-homogeneous.

The extended system (3x3) matrix of Equation (219) has the solution

$$x(t)(3\rangle = e^{S_{ou} t} x(0)(3\rangle \qquad (224)$$

$$= \left(I + S_{ou} t + S_{ou}^2 \frac{t^2}{2!} \right) x(0)\rangle \qquad (225)$$

$$= \underset{3x3}{\Phi(t)}\ x(0)(3\rangle \qquad (226)$$

By Equation (214)

$$x(0)(3\rangle = \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & 2a_2 & 0 \\ 2a_2 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ 2a_2 \end{pmatrix} \qquad (227)$$

hence Equation (227) in Equation (226)

$$x(t)(3)\rangle = \begin{bmatrix} 1 & t & t^2/2! \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ 2a_2 \end{pmatrix} . \tag{228}$$

From the above it can be seen that for a second order dynamical system with unknown force, one needs a three dimensional state vector instead of two dimensions. The generalization to higher derivatives is obvious.

STATE DYNAMIC EQUATION CHANGES DUE TO TIME-AXIS BASE CHANGE. This section derives some relations due to time-axis base changes. By Equation (157)

$$x(t)\rangle = \Phi(t,t_b)\ x(t_b)\rangle \tag{229}$$

and

$$x(t_b)\rangle = \Phi^{-1}(t,t_b)\ x(t)\rangle \tag{230}$$

Taking the derivative of Equation (229)

$$\dot{x}(t)\rangle = \dot{\Phi}(t,t_b)\ x(t_b)\rangle \tag{231}$$

and using Equation (230) in Equation (227)

$$\dot{x}(t)\rangle = \dot{\Phi}(t,t_b)\ \Phi^{-1}(t,t_b)\ x(t)\rangle \tag{232}$$

or

$$\underset{dxd}{F_e} = S_{uo} = \dot{\Phi}(t,t_b)\ \Phi^{-1}(t,t_b) . \tag{233}$$

By Equation (159)

$$\Phi\ (t,t_b) = T_v(t)\ T_v^{-1}\ (t_b) \tag{234}$$

By Equation (233)

$$\dot{\Phi}\ (t,t_b) = S_{uo}\ \Phi\ (t,t_b) \tag{235}$$

$$= S_{uo}\ T_v(t)\ T_v^{-1}(t_b)$$

$$= \dot{T}_v(t)\ T_v^{-1}(t_b)$$

By Equation (152)

$$\dot{T}_v(t) = T_v(t)\ V_t \tag{236}$$

Using Equation (236) in Equation 235)

$$\dot{\Phi}(t,t_b) = T_v(t)\ V_t\ T_v^{-1}\ (t_b)\quad . \tag{237}$$

The inverse of Equation (234) is

$$\Phi^{-1}(t,t_b) = T_v(t_b)\ T_v^{-1}(t) \tag{238}$$

Using Equations (238) and (237) in Equation (233)

$$F_e = S_{uo} = T_v(t)\ V_t\ T_v^{-1}(t) \tag{239}$$

which says that the nilpotent shift operator or the state dynamic matrix is similar to the base velocity matrix V_t.

A time axis base change by Equation (135) is

$$\langle t' = \langle t \; T_u(t_b) \tag{240}$$

where by Equation (128)

$$t = t_b + t' \tag{241}$$

For a fixed t_b and a linear time axis scale factor of 1

$$\frac{dt}{dt'} = 1 \; . \tag{242}$$

the derivative of Equation (240) with respect to t' is

$$\begin{aligned} \frac{d}{dt'} \langle t' &= \frac{d}{dt} \langle t \left(\frac{dt}{dt'}\right) T_u(t_b) \\ &= \langle t \; V_t \; T_u(b) \\ &= \langle t' \; T_u^{-1}(t_b) \; V_t \; T_u(b) \end{aligned} \tag{243}$$

and in its own base

$$\frac{d}{dt'} \langle t' = \langle t' \; V_{t'} \tag{244}$$

or

$$V_{t'} = T_u^{-1} (t_b) \; V_t \; T_u(t_b) \tag{245}$$

It is easily established by differentiation

$$\langle t' = \left[1,\ t',\ (t')^2,\ (t')^3,\ \ldots\ (t')^{d-1} \right] \tag{246}$$

$$\frac{d}{dt'} \langle t' = \langle t' \begin{bmatrix} 0 & 1 & 0 & 0 & \cdot\ \cdot\ \cdot & & \\ 0 & 0 & 2 & 0 & \cdot\ \cdot\ \cdot & & \\ & & 0 & & & & \\ & 0 & & & 0 & & d-1 \\ \cdot & \cdot & \cdot & \cdot & & & 0 \end{bmatrix} \tag{247}$$

and by Equation (22) Section I, one can see that

$$V_{t'} = V_t \tag{248}$$

Continuous Time Base Change on the State Vector. The variable x(t) can be written as by Equation (240)

$$x(t) = x(t') = \langle t a \rangle = \langle t'\ T_u^{-1}\ (t_b)\ a \rangle \tag{249}$$

or

$$x(t') = \langle t' \quad a' \rangle \tag{250}$$

where

$$a' \rangle = T_u^{-1}\ (t_b)\ a \rangle\ . \tag{251}$$

The derivative of Equation (250) with respect to the new time variable is

$$\frac{dx}{dt'} = \langle t' \quad V a' \rangle \tag{252}$$

and the state vector becomes

$$x(t')\rangle = \begin{bmatrix} x(t') \\ \frac{dx}{dt'} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \langle t' \\ \langle t' V \\ \cdot \\ \cdot \\ \cdot \\ \langle t' V^{(d-1)} \end{bmatrix} a'\rangle \qquad (253)$$

or

$$x(t')(d\rangle = T_V(t')\ a'\rangle \qquad (254)$$

at

$$t = t_b$$

$$t_b = t_b + t'$$

or

$$t' = 0.$$

hence

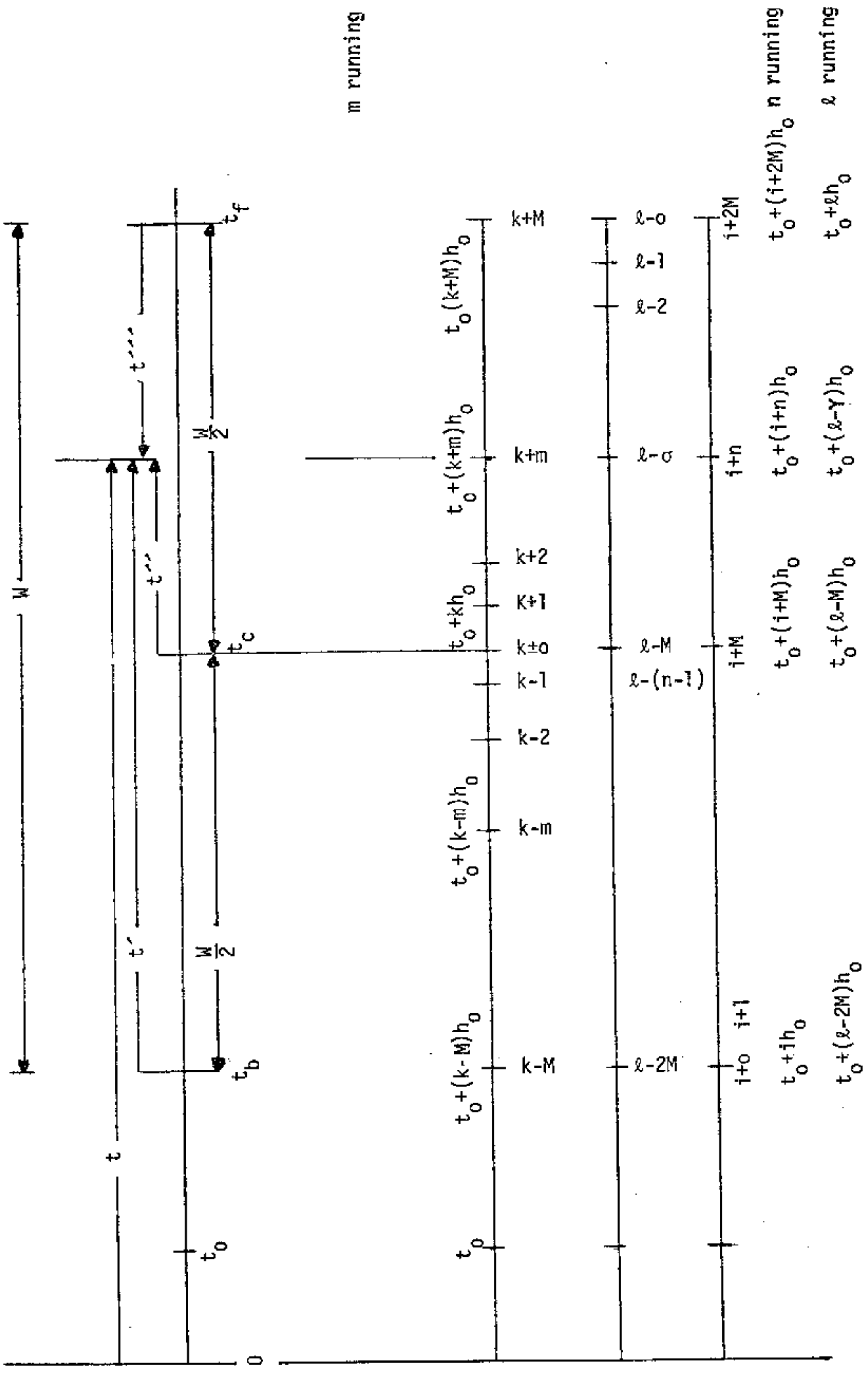
$$x(t' = 0)\rangle = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 2 & & 0 & \\ & & & 3! & & \\ & & 0 & & \ddots & \\ & & & & & (d-1)! \end{bmatrix} a'\rangle \qquad (255)$$

or

$$a'\Big\rangle = T_V^{-1}\,(t'=0)\;\; x(t'=0)\Big\rangle \tag{256}$$

$$x(t')(d)\Big\rangle = T_V(t')\;\; T_V^{-1}\,(t'=0)\;\; x(t'=0)\Big\rangle \tag{257}$$

$$x(t')(d)\Big\rangle = \Phi\;(t',\; t'=0)\;\; x(t'=0)\Big\rangle \tag{258}$$


m running
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t_b
t_o
0
$t_o(k+M)h_o$
$t_o+(k+m)h_o$
t_o+kh_o
$t_o+(k-m)h_o$
$t_o+(k-M)h_o$
t_o
k+M
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K+1
k±o
k-1
k-2
k-m
k-M
ℓ-o
ℓ-1
ℓ-2
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ℓ-M
ℓ-(n-1)
ℓ-2M
i+2M
i+n
i+M
i+o
i+1
$t_o+(i+2M)h_o$
$t_o+(i+n)h_o$
$t_o+(i+M)h_o$
t_o+ih_o
$t_o+\ell h_o$
$t_o+(\ell-\gamma)h_o$
$t_o+(\ell-M)h_o$
$t_o+(\ell-2M)h_o$

FIGURE (1)

Section 5

DISCREET "METRIC MATRIX" FOR FRONT, BACK AND CENTER OF SPAN. The previous sections derived a **normalized metric matrix for continuous time bases via integration of the dyadic product, namely**

$$M_{t_2 t_1} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} t\rangle\langle t \, dt \tag{1}$$

In dealing with discreet time points on continuous functions one normally replaces integration by finite or countably infinite summations. Consider the dyad at time t_n that is

$$t\rangle_n {}^n\langle t = \begin{bmatrix} 1 & t_n & \cdots & t_n^{d-1} \\ t_n & t_n^2 & & t_n^{d} \\ \vdots & & & \vdots \\ t_n^{d-1} & & \cdots & t_n^{d-1} \end{bmatrix} \tag{2}$$

and the sum (arbitrarily chosen non-normalized)

$$\sum_{n=0}^{M} t(d\rangle_n {}^n\langle d)t = t\rangle_0 {}^0\langle t + t\rangle_1 {}^1\langle t + \cdots + t\rangle_N {}^M\langle t \tag{3}$$

The summation of N+1 rank one dxd dyads can be expressed instead of a sum as a product

$$\sum t(d\rangle_{n}\,{}^{n}\langle d)t = \left[t\rangle_{0},\ t\rangle_{1}\ \cdots\ t\rangle_{N} \right] \begin{bmatrix} {}^{0}\langle t \\ {}^{1}\langle t \\ \vdots \\ {}^{N}\langle t \end{bmatrix} \qquad (4)$$

$$\underset{dxd}{M} = \underset{dx(N+1)xd}{T^{T}\ T} \qquad (5)$$

Where the (N+1)xd matrix T is a column of row vectors

$$\underset{(N+1)d}{T} = \begin{bmatrix} {}^{0}\langle t \\ {}^{1}\langle t \\ \vdots \\ {}^{N}\langle t \end{bmatrix}$$

and the transpose T^T is a row of column vectors,

$$\underset{dx(N+1)}{T^{T}} = \left[t\rangle_{0},\ t\rangle_{1},\ \cdots\ t\rangle_{N} \right]$$

For a time span of discreet data points, the notation of Figure (1) will be used to generate the matrices commonly occurring in discreet least squares estimation.

The arbitrary time point t of Figure (1) is expressed as

$$t = t_b + t' \qquad (6)$$

$$= t_c + t'' $$

$$= t_f - t''' $$

where the instantanious continuous time point t is expressed as a function of the primed variables, t', t'', and t'''. The parameters or variables t_b, t_c, and t_f reference the back, center and front of the span. It is seen by the figure that the point at the beginning of the span

$$t_b = t_o + ih_o \qquad (7)$$

and similarly for center and front points

$$t_c = t_o + kh_o \qquad (8)$$

$$t_f = t_o + \ell h_o \qquad (9)$$

For a moving span, one develops recursive relations for i, k, and ℓ varying also. In this section the structure of the matrices for the variables within the span(n, m, and γ)are developed.

The discreet analog of Equation (6) by using Equations (7),(8) and (9) in Equation (6)

$$t(i,n) = t_o + (i+n)\ h_o \qquad (10)$$

$$i = 0, 1, 2, \ldots$$

$$n = 0, 1, 2, \ldots, N=2M$$

$$t(k,m) = t_o + (k \pm m)\, h_o \tag{11}$$

$k = 0, 1, 2, . . .$

$m = 0, 1, 2, 3, ..., M$

$$t(\ell,\gamma) = t_o + (\ell-\gamma)\, h_o \tag{12}$$

$\ell = 0, 1, 2, ...$

$\gamma = 0, 1, 2, . . ., 2M$

Equations (10), (11), and (12) can also be written as

$$t(i,n) = h_o[\beta+\tau(i,n)] \tag{13}$$

$$t(k,m) = h_o[\beta+\tau(k,m)] \tag{14}$$

$$t(\ell,\gamma) = h_o[\beta+\tau(\ell,\gamma)] \tag{15}$$

where

$$\beta = t_o/h_o \tag{16}$$

$$\tau(i,n)= i + n \tag{17}$$

$$\tau(k,m)= k \pm m \tag{18}$$

$$\tau(\ell,\gamma)= \ell - \gamma \tag{19}$$

and τ is non-dimensionalized time.

In order to obtain the inverse matrices one needs

$$\tau(i,n) = (i+n) = \frac{1}{h_o}\,(t(i,n)-t_o) \tag{20}$$

$$\tau(k,m) = (k \pm m) = \frac{1}{h_o}\,(t(k,m)-t_o) \tag{21}$$

$$\tau(\ell,\gamma) = (\ell-\gamma) = \frac{1}{h_o}\,(t(\ell,\gamma)-t_o) \tag{22}$$

Note that at the back of the span that is $t=t_b$ by Equation (13),(14) and (15)

$$t(i,o) = h_o [\beta+i] \quad (23)$$

$$= t(k,-M) = h_o[\beta+k-M] \quad (24)$$

$$= t(\ell,2M) = h_o[\beta+\ell-2M] \quad (25)$$

and the above equivalence relations imply

$$i = k-M = \ell-2M \quad (26)$$

at the center of the span

$$t(i,M) = h_o[\beta+(i,M)] \quad (27)$$

$$= t(k,o) = h_o[\beta+k] \quad (28)$$

$$= t(\ell,M) = h_o[\beta+\ell-M] \quad (29)$$

at the front of the span one has

$$t(i,2M) = h_o[\beta+i+2M] \quad (30)$$

$$= t(k,M) = h_o[\beta+k+M] \quad (31)$$

$$= t(\ell,o) = h_o[\beta+\ell] \quad (32)$$

The relationship between the width of the span W occurring in the matrices of Section (4) for the continuous case and the number of points in the span of Figure (1) is

$$\int_{t_b}^{t_f} 1dt = t_f-t_b=W=N\Delta t \quad (33)$$

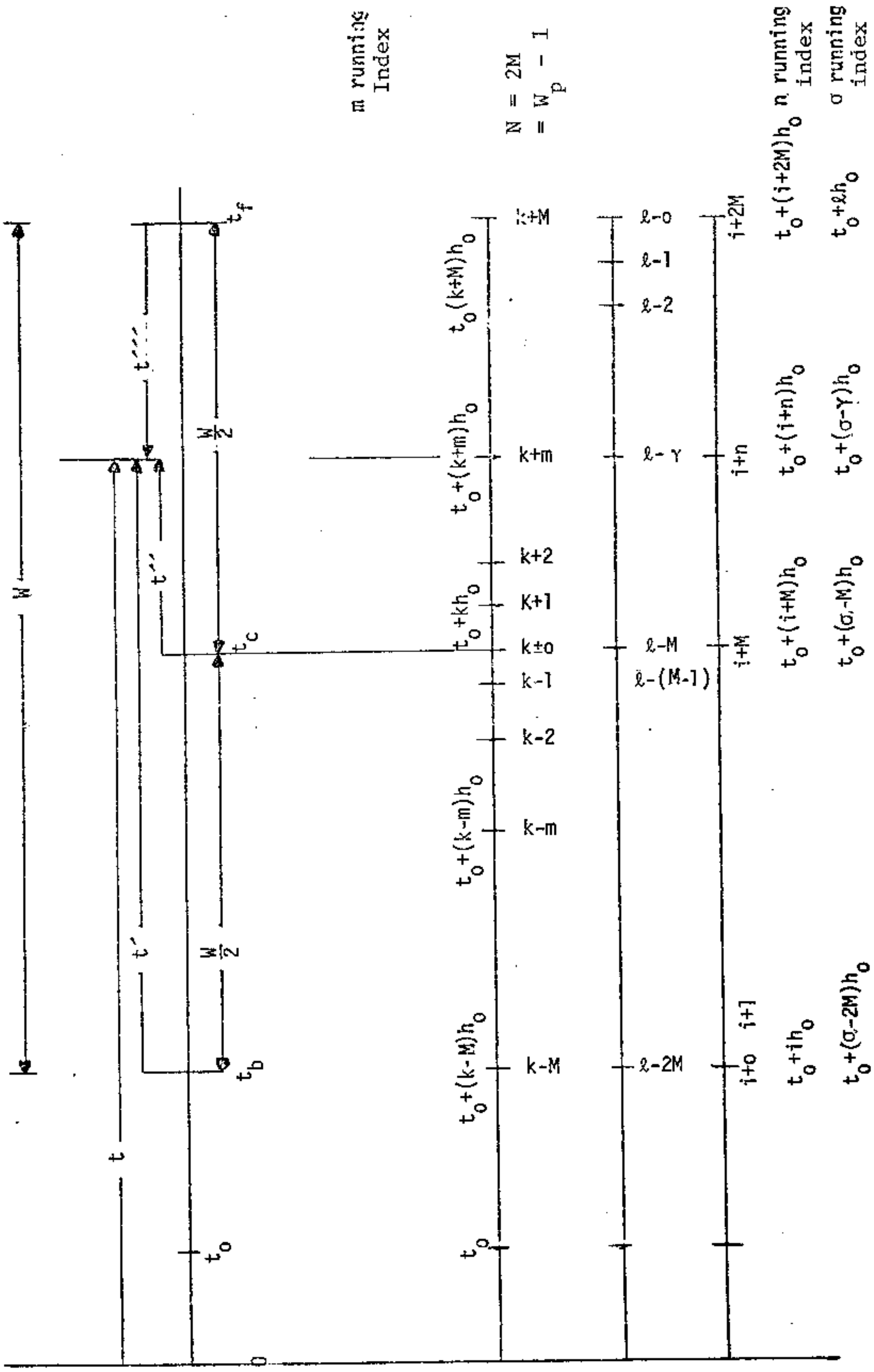

FIGURE (1)
TIME AXIS - INDICES

that is, there are N segments of length Δt but N+1 points.

By Equation (10) and Equation (17) we obtain

$$\langle t(i,n) = \langle \tau(i,n) \begin{bmatrix} 1 & \beta & \beta^2 & \beta^3 & \beta^4 & \beta^5 \\ 0 & 1 & 2\beta & 3\beta^2 & 4\beta^3 & 5\beta^4 \\ 0 & 0 & 1 & 3\beta & 6\beta^2 & 10\beta^3 \\ 0 & 0 & 0 & 1 & 4\beta & 10\beta^2 \\ 0 & 0 & 0 & 0 & 1 & 5\beta \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & \vdots & & \vdots \end{bmatrix} X$$

$$\begin{bmatrix} 1 & & & & & \\ & h_o & & 0 & & \\ & & h_o^2 & & & \\ & 0 & & h_o^4 & & \\ & & & & & h_o^5 \end{bmatrix} \quad (34)$$

$$= \langle \tau\, T_u(\beta)\, D(h_o) \quad (35)$$

and similar for Equations (11) and (12).

The inverse transformations and the ones of prime interest here are obtained as the discrete analog of Equations (135) Section (4). By Equation (20) using the notation.

$$\Big\langle(i+n) = (1,(i+n),(i+n)^2 \cdots (i+n)^{d-1}) \equiv \Big\langle\tau(i,n) \tag{36}$$

we obtain

$$\Big\langle\tau(i,n) = \Big\langle t(i,n)\ D^{-1}(h_o) \begin{bmatrix} 1 & -t_o & t_o^2 & -t_o^3 & . \\ 0 & 1 & -2t_o & 3t_o^2 & . \\ 0 & 0 & 1 & -3t_o & . \\ 0 & 0 & 0 & 1 & . \\ 0 & \cdots & & 0 & . \\ \vdots & \vdots & & \vdots & \vdots \end{bmatrix} \tag{37}$$

also by Equation (36) and Equation (17) the left hand side of Equation (37) is

$$\Big\langle\tau(i,n) = \Big\langle n \begin{bmatrix} 1 & i & i^2 & i^3 & i^4 & i^5 \\ 0 & 1 & 2i & 3i^2 & 4i^3 & 5i^4 \\ 0 & 0 & 1 & 3i & 6i^2 & 10i^3 \\ 0 & 0 & 0 & 1 & 4i & 10i^2 \\ 0 & 0 & 0 & 0 & 1 & 5i \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{38}$$

$$= \langle n \, T_u(i) \qquad (39)$$

$$\langle n = (1, n, n^2, n^3, \cdots, n^{d-1}) \qquad (40)$$

Equations (21) and (22) have the same right hand terms as Equation (37); however, the left hand terms differ. The left hand term by Equation (21) and Equation (18) is

$$\langle \tau(k,m) = \langle m \begin{bmatrix} 1 & k & k^2 & k^3 & . \\ 0 & 1 & 2k & 3k^2 & . \\ 0 & 0 & 1 & 3k & . \\ 0 & 0 & 0 & 1 & \\ 0 & \multicolumn{2}{c}{\cdots} & 0 & \\ \vdots & . & & \vdots & \vdots \end{bmatrix} \qquad (41)$$

where

$$\langle m = (1, \pm m, m^2, \pm m^3, m^4, \pm m^5, \cdots) \qquad (42)$$

and the transformation to front of the span by Equation (22) is

$$\langle \tau(\ell,\gamma) = \langle \gamma \begin{bmatrix} 1 & \ell & \ell^2 & \ell^3 & . \\ 0 & 1 & 2\ell & 3\ell^2 & . \\ 0 & 0 & 1 & 3\ell & . \\ 0 & 0 & 0 & 1 & . \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & 0 & . \end{bmatrix} \qquad (43)$$

and

$$\langle\gamma = (1,\ -\gamma,\ \gamma^2,\ -\gamma^3,\ \gamma^4,\ -\gamma^5,\ \cdots) \tag{44}$$

The metric-matrix of Equation (144) Section (4)

$$\frac{1}{w}\int_0^w t'\rangle\langle t'\, dt' = \begin{bmatrix} 1 & w/2 & w/3 & \cdots \\ w/2 & 2/3 & & \\ w/3 & & & . \\ \vdots & . & . & . \end{bmatrix} \tag{45}$$

has the discreet analog

$$\frac{1}{w}\sum_{n=0}^{2M=N} \tau(i,n)\rangle\langle\tau(i,n) \tag{46}$$

$$= \frac{1}{2M+1}\Big[\tau(i,o)\rangle\langle\tau(i,o) + \tau(i,1)\rangle\langle\tau(i,1) + \cdots \tag{47}$$

$$+ \tau(i\ ,2M)\rangle\langle\tau(i,2M)\Big]$$

$$= \frac{1}{2M+1}\ T_u^T(i)\left[\sum_{n=0}^{N} n\rangle\langle n\right] T_u(i) \tag{48}$$

The grammian matrix

$$\underset{dx(n+1)(n+1)xd}{N^T N} = \sum_{n=0}^{N} n\rangle\langle n = \left[n\rangle_0,\ n\rangle_1,\ n\rangle_2,\ \cdots\ n\rangle_N \right] \begin{bmatrix} {}^{0}\langle n \\ {}^{1}\langle n \\ \vdots \\ {}^{N}\langle n \end{bmatrix} \tag{49}$$

The matrix N^T is

$$\underset{d\text{x}N+1}{N^T} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & & N \\ 0 & 1 & 2^2 & 3^2 & & N^2 \\ 0 & 1 & 2^3 & 3^3 & & N^3 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2^{d-1} & 3^{d-1} & & N^{d-1} \end{bmatrix} \tag{50}$$

If we partition N as shown below

$$\underset{d\text{x}N+1}{N^T} = \left[\underset{1}{e(d\rangle}, \underset{d\text{x}N}{C^T} \right] \tag{51}$$

where the partitioning of the matrix C into its row space yields row vectors whose coordinates are powers of the first N natural numbers, that is

$$\underset{d\text{x}n}{C^T} = \begin{bmatrix} (1, & 1, & 1, & \cdots & 1) \\ (1, & 2, & 3, & \cdots & N) \\ (1, & 2^2, & 3^2, & \cdots & N^2) \\ \vdots & & & & \\ (1, & 2^{d-1}, & 3^{d-1}) & \cdots & N^{d-1}) \end{bmatrix} \tag{52}$$

or using the notation

$$C^T = \begin{bmatrix} \langle 1 \\ \langle C \\ \langle C^2 \\ \vdots \\ \langle C^{(d-1)} \end{bmatrix} \qquad (53)$$

$$\langle M)\, c = (1, 2, 3, \cdots N) \qquad (54)$$

Transposing Equation (51)

$$\underset{(N+1)xd}{N} = \begin{bmatrix} {}^{1}\langle d)e \\ C \end{bmatrix} \qquad (55)$$

and the Grammian of Equation (49) becomes

$$N^T N = e\rangle_1 \, {}^1\langle e + \underset{dxd}{C^T C} \qquad (56)$$

and one can obtain the inverse of the Grammian matrix of Equation (60) via the Householder inversion lemma of the inverse of a matrix plus a dyad.

The transpose of Equation (53) is

$$C = \left[1\rangle, \; c\rangle, \; c^2\rangle \cdots \; c^{(d-1)}\rangle \right] \qquad (57)$$

and

$$\underset{d\times d}{C^{T}C} = \begin{bmatrix} N & \langle 1c \rangle & \langle 1c^{2} \rangle & \cdots & \langle 1c^{d-1} \rangle \\ \langle c1 \rangle & \langle 1c^{2} \rangle & \langle c^{3}1 \rangle & & \langle 1c^{d} \rangle \\ \langle c^{2}1 \rangle & \langle c^{3}1 \rangle & & & \\ \vdots & & \cdot & & \vdots \\ \langle 1c^{d-1} \rangle & \langle c^{d}1 \rangle & \cdot & \cdot \quad \cdot & \langle c^{2d-2}1 \rangle \end{bmatrix} \qquad (58)$$

note the properties

$$\langle cc \rangle = \langle c^{2}1 \rangle = \langle 1c^{2} \rangle \qquad (59)$$

or the sum of the squares of the first N natural numbers, also

$$\langle c^{d}1 \rangle = 1 + 2^{d} + 3^{d} + \cdots + N^{d} \qquad (60)$$

is the sum of the d^{th} powers of the first N natural numbers.

Adding Equation (58) to Equation (56)

$$N^{T}N = \begin{bmatrix} N+I & \langle 1c \rangle & \langle 1c^{2} \rangle & \cdots \\ \langle c1 \rangle & \langle c^{2}1 \rangle & & \\ \langle c^{2}1 \rangle & & & \cdot \\ \cdot & & \cdot & \cdot \end{bmatrix} \qquad (61)$$

The matrix

$$e \rangle \overset{1}{\underset{1}{}} \langle e = \begin{bmatrix} 1 & 0 & 0 & . \\ 0 & & & . \\ . & & & \\ . & & & . \\ . & & & \\ 0 & \ldots\ldots & & 0 \end{bmatrix} \qquad (62)$$

contributed to the first row first column term in Equation (61), and Equation (61) is clearly a Hankel-matrix. The sums of the powers of the natural numbers are given in Appendix (C).

The discreet analog of the metric-matrix of Equation (145) sec (4) is Equation (41)

$$\frac{1}{w} \sum_{m=-M}^{M} \tau(k,m) \rangle \langle \tau(k,m) \qquad (63)$$

$$= \frac{1}{w} T_u^T(k) \left(\sum_{m=-M}^{M} m \rangle \langle m \right) T_u(k) \qquad (64)$$

The sum of dyads of Equation (64) can be written as

$$\sum_{m=-M}^{M} m \rangle \langle m = \underset{dxd}{M^T M} \qquad (65)$$

The matrix M^T is

$$M^T = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ -M & -(M-1) & & -2 & -1 & 0 & 1 & 2 & & M \\ M^2 & (M-1)^2 & & 2^2 & 1 & 0 & 1 & 2^2 & & M^2 \\ -M^3 & -M-1)^3 & & -2^3 & -1 & & 1 & 2^3 & & M^3 \\ M^4 & \cdot & & & & \cdot & & & & \cdot \\ \vdots & \cdot & & & & \cdot & & & & \cdot \\ & \cdot & & & & \cdot & & & & \cdot \\ (-M)^{(d-1)} & [-(M-1)]^{d-1} & & (-2)^{(d-1)} & (-1)^{d-1} & 0 & 1 & 2^{d-1} & & M^{d-1} \end{bmatrix} \qquad (66)$$

Partion M^T into the form

$$\underset{dx2M+1}{M^T} = \begin{bmatrix} \underset{dxM}{C^T(-1)\, L_C^T,\, e\rangle_1}, & \underset{dxM}{C^T} \end{bmatrix} \tag{67}$$

where as before the C^T matrix is

$$\underset{dxM}{C^T} = \begin{bmatrix} 1 & 1 & . & . & . & 1 \\ 1 & 2 & 3 & & & M \\ 1 & 2^2 & 3^2 & & \cdots & M^2 \\ \vdots & & \vdots & & & \vdots \\ 1 & 2^{d-1} & & & \cdots & M^{d-1} \end{bmatrix} = \begin{bmatrix} \langle 1 \\ \langle c \\ \langle c^2 \\ \vdots \\ \langle c^{d-1} \end{bmatrix} \tag{68 }$$

$$\langle M) c \equiv (1, 2, 3, \cdots M)$$

and the alternating sign matrix

$$\underset{dxM}{C^T(-1)} = \begin{bmatrix} \langle 1 \\ -\langle c \\ \langle c^2 \\ -\langle c^3 \\ \vdots \\ \langle c^{(d-1)} \end{bmatrix} = \underset{dxd}{I(-1)}\, C^T \tag{69}$$

and the linear convolution matrix is

$$L_c^T = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & 1 & 0 \\ & & \cdot & \\ 0 & \cdot & & \\ 1 & \cdot\ 0 & 0 & 0 \end{bmatrix} \tag{70}$$

with the property

$$L_c^T L_c = I \tag{71}$$

Transposing Equation (67)

$$\underset{(2M+1)xd}{M} = \begin{bmatrix} L_c \underset{Mxd}{C(-1)} \\ \overset{1}{\langle} d) c \\ \underset{Mxd}{C} \end{bmatrix} \tag{72}$$

and the Grammian is

$$M^T M = C^T(-1)\ C(-1) + e\rangle_1 \overset{1}{\langle} e + C^T C \tag{73}$$

The Hankel matrix $C^T(-1)\ C(-1)$ has alternating signs

$$C^T(-1)\ C(-1) = \begin{bmatrix} M & -\langle 1c(m \rangle & \langle c^2 1 \rangle & -\langle c^3 1 \rangle & \cdots \\ -\langle 1c \rangle & \langle c^2 1 \rangle & -\langle c^3 1 \rangle & & \\ \langle c^2 1 \rangle & & & & \\ -\langle c^3 1 \rangle & & & & \vdots \\ \vdots & & & & \\ \langle c^{d-1} 1 \rangle & & \cdots & & \langle c^{2d-2} 1 \rangle \end{bmatrix} \tag{74}$$

The positive sign matrix C^TC is

$$\underset{d\times d}{C^TC} = \begin{bmatrix} M & \langle 1c \rangle & \langle c^2 1 \rangle & \cdots & \langle c^{d-1} 1 \rangle \\ \langle 1c \rangle & \langle c^2 1 \rangle & & & \\ \langle c^2 1 \rangle & & & & \vdots \\ \vdots & & & & \\ \langle c^{d-1} 1 \rangle & & \cdots & & \langle c^{2d-2} 1 \rangle \end{bmatrix} \tag{75}$$

Adding the matrices of Equation (73)

$$\underset{d\times d}{M^{T}M} = \begin{bmatrix} 2M+1 & 0 & 2\left\langle M)c^{2}_{1}\right\rangle & 0 & 2\left\langle c^{4}_{1}\right\rangle \\ 0 & 2\left\langle c^{2}_{1}\right\rangle & 0 & 2\left\langle c^{4}_{1}\right\rangle & 0 \\ 2\left\langle c^{2}_{1}\right\rangle & 0 & 2\left\langle c^{4}_{1}\right\rangle & 0 & \\ 0 & 2\left\langle c^{4}_{1}\right\rangle & 0 & & \\ 2\left\langle c^{4}_{1}\right\rangle & 0 & & & \\ \vdots & & \cdots & & \vdots \end{bmatrix} \tag{76}$$

with alternating left-slant diagonals being zero. Once again MM^{T} is a Hankel matrix. Only even-powers of the natural numbers occur. Compare this matrix with continuous one of Equation (145) Section (4).

The transformation to the front of the span with the running index γ running backwards by Equation (146) Section (4) has the discrete analog by Equation (43)

$$\frac{1}{W} \sum_{\gamma=2M}^{0} \tau(\ell,\gamma)\rangle\langle\tau(\ell,\gamma) = \frac{1}{W}\; T^{T}_{u}(\ell) \sum_{\gamma=2M}^{0} \gamma\rangle\langle\gamma \; T_{u}(\ell) \tag{77}$$

The summation term of Equation (77) is

$$\frac{1}{2M+1} \sum_{\gamma=2M}^{0} \gamma\rangle\langle\gamma = \frac{1}{(2M+1)} \underset{dx(2M+1)xd}{\Gamma^{T}\Gamma} \tag{78}$$

The matrix Γ is by Equation (44)

(79)

$$\Gamma = \begin{bmatrix} 1 & -2M & (2M)^2 & (-2M)^3 & (2M)^4 & & \cdots \\ 1 & -(2M-1) & (2M-1)^2 & -(2M-1)^3 & (2M-1)^4 & & \vdots \\ \vdots & & & & & & \\ 1 & -2 & 2^2 & -2^3 & 2^4 & \cdots & (-2)^{d-1} \\ 1 & -1 & 1 & -1 & \cdots & & (-1)^{d-1} \\ 1 & 0 & 0 & & \cdots & & 0 \end{bmatrix}$$

Partitioning Γ

$$\underset{(N+1)xd}{\Gamma} = \begin{bmatrix} \underset{nxd}{C_L(-1)} \\ \\ I \\ \langle d)e \end{bmatrix} \tag{80}$$

where convolved vectors are: (81)

$$\underset{nxd}{C_L(-1)} = \left[1(N\rangle, -L_C C\rangle , L_C C^2\rangle, -L_C C^3\rangle \cdots (-1)^{d-1} L_C C^{(d-1)}\rangle \right]$$

$$C_L^T(-1) = \begin{bmatrix} \langle 1 \\ -\langle C L_C \\ \langle C^2 L_C \\ -\langle C^3 L_C \\ \vdots \\ (-1)^{d-1} \langle C^{(d-1)} L_C \end{bmatrix} \tag{82}$$

Transposing Equation (80)

$$\underset{d\times(N+1)}{\Gamma^T} = \begin{bmatrix} \underset{d\times N}{C_L^T(-1)}, & e(d\rangle_1 \end{bmatrix} \tag{83}$$

and the Grammian is

$$\Gamma^T\Gamma = \underset{d\times d}{C_L^T(-1)C_L(-1)} + e(d\rangle_1{}^1\langle d)e \tag{84}$$

The Grammian

$$C_L^T(-1)C_L(-1) = \begin{bmatrix} N & -\langle 1c\rangle & \langle 1c^2\rangle & \cdots & (-1)^d\langle 1c^{d-1}\rangle \\ -\langle 1c\rangle & \langle 1c^2\rangle & -\langle 1c^3\rangle & & \\ \langle 1c^2\rangle & -\langle 1c^3\rangle & & & \\ -\langle 1c^3\rangle & & & & \vdots \\ \cdots & & & & \end{bmatrix} \tag{85}$$

and

$$\Gamma^T\Gamma = \begin{bmatrix} N+1 & -\langle 1c\rangle & \langle 1c^2\rangle & -\langle 1c^3\rangle & \cdots \\ -\langle 1c\rangle & \langle 1c^2\rangle & -\langle 1c^3\rangle & & \\ \langle 1c^2\rangle & -\langle 1c^3\rangle & & & \\ -\langle 1c^3\rangle & & & & \\ \cdots & & & & \vdots \end{bmatrix} \tag{86}$$

Note the alternating signs on the left diagonals of the Hankel matrix of Equation (86).

Summarizing the three Grammian matrices for spans with indices over the natural numbers running from back, center and front of the spans are by Equations (61), (76) and (86)

$$N^TN = \begin{bmatrix} N+1 & \langle 1N \rangle & \langle 1N^2 \rangle & \langle 1N^3 \rangle & \langle 1N^4 \rangle & \cdots \\ \langle 1N \rangle & \langle 1N^2 \rangle & \langle 1N^3 \rangle & \langle 1N^4 \rangle & & \\ \langle 1N^2 \rangle & \langle 1N^3 \rangle & \langle 1N^4 \rangle & & & \\ \langle 1N^3 \rangle & \langle 1N^4 \rangle & \cdots & & & \\ \langle 1N^4 \rangle & & & & & \\ \vdots & & & & & \end{bmatrix} \qquad (87)$$

$$M^TM = \begin{bmatrix} 2M+1 & 0 & 2\langle M^2 1 \rangle & 0 & 2\langle M^4 1 \rangle & 0 & \cdots \\ 0 & 2\langle M^2 1 \rangle & 0 & 2\langle M^4 1 \rangle & 0 & 2\langle M^6 1 \rangle & \cdots \\ 2\langle M^2 1 \rangle & 0 & 2\langle M^4 1 \rangle & 0 & 2\langle M^6 1 \rangle & & \\ 0 & 2\langle M^4 1 \rangle & 0 & 2\langle M^6 1 \rangle & & & \\ 2\langle M^4 1 \rangle & 0 & 2\langle M^6 1 \rangle & & & & \\ 0 & 2\langle M^6 1 \rangle & & & & & \\ 2\langle M^6 1 \rangle & & & & & & \\ 0 & & & & & & \\ \vdots & & & & & & \end{bmatrix} \qquad (88)$$

$$S_2(N) = \frac{W_p}{6}(W_p-1)(2W_p-1) \tag{92}$$

$$S_3(N) = \frac{W_p^{\,2}}{4}(W_p-1)^2 \tag{93}$$

$$S_4(N) = \frac{W_p}{30}(W_p-1)(2W_p-1)(3W_p^{\,2}-3W_p-1) \tag{94}$$

and

$$W_p = N+1 = 2M+1 \ .$$

The center span matrix is

$$M^T M = \begin{bmatrix} W_p & 0 & 2S_2(M) \\ 0 & 2S_2(M) & 0 \\ 2S_2(M) & 0 & 2S_4(M) \end{bmatrix} \tag{95}$$

where by (C - 21 and 23)

$$S_2(M) = \frac{1}{24} W_p (W_p^2-1) \tag{96}$$

$$S_4(M) = \frac{W_p}{(30)(16)} (W_p+1)(W_p-1)(3W_p^2-7) \tag{97}$$

$$\Gamma^T\Gamma = \begin{bmatrix} N+1 & -\langle 1N \rangle & \langle 1N^2 \rangle & -\langle 1N^3 \rangle & \langle 1N^4 \rangle & \cdots \\ -\langle 1N \rangle & \langle 1N^2 \rangle & -\langle 1N^3 \rangle & \langle 1N^4 \rangle & \cdots & \\ \langle 1N^2 \rangle & -\langle 1N^3 \rangle & \langle 1N^4 \rangle & \cdots & & \\ -\langle 1N^3 \rangle & \langle 1N^4 \rangle & \cdots & & & \\ \langle 1N^4 \rangle & -\langle 1N^5 \rangle \cdots & & & & \\ -\langle 1N^5 \rangle & & & & & \\ \vdots & & & & & \end{bmatrix} \tag{89}$$

For 3x3 matrices (or second degree polynomials) using the notation of section (C) for sums of powers we have

$$N^TN = \begin{bmatrix} W_p & S_1(N) & S_2(N) \\ S_1(N) & S_2(N) & S_3(N) \\ S_2(N) & S_3(N) & S_4(N) \end{bmatrix} \tag{90}$$

where by (C - 5 thru 8)

$$S_1(N) = \frac{W_p}{2}(W_p-1) \tag{91}$$

and for front span case

$$\Gamma^T\Gamma = \begin{bmatrix} W_p & -S_1(N) & S_2(N) \\ -S_1(N) & S_2(N) & -S_3(N) \\ S_2(N) & -S_3(N) & S_4(N) \end{bmatrix} \tag{98}$$

For a second degree polynomial and W_p=7 points - we have

$$S_1(N) = \frac{7}{2}(6) = 21 \tag{99}$$

$$S_2(N) = \frac{7}{6}(6)(13) = \frac{7}{6}(13) = \frac{91}{6} \tag{100}$$

$$S_3(N) = \frac{49}{4}(36) = 49(9) = 441 \tag{101}$$

$$S_4(N) = \frac{7}{30}(6)(13)(125) = 2275 \tag{102}$$

$$S_2(M) = \frac{7}{24}(48) = 7(2) = 14 \tag{103}$$

$$S_4(M) = \frac{7(70)}{5} = \frac{490}{5} = 98 \tag{104}$$

and

$$N^TN = \begin{bmatrix} 7 & 21 & 91/6 \\ 21 & 91/6 & 441 \\ 91/6 & 441 & 2275 \end{bmatrix} \tag{105}$$

$$M^TM = \begin{bmatrix} 7 & 0 & 28 \\ 0 & 28 & 0 \\ 28 & 0 & 196 \end{bmatrix} \tag{106}$$

and

$$\Gamma^T\Gamma = \begin{bmatrix} 7 & -21 & 91/_6 \\ -21 & 91/_6 & -144 \\ 91/_6 & -144 & 2275 \end{bmatrix} \tag{107}$$

Consider the factors of the alternating sign case

$$\begin{aligned} \langle\gamma &= (1,\ -\gamma,\ \gamma^2,\ -\gamma^3,\ \gamma^4,\ -\gamma^5\ \cdots) \\ &= (1,\ \gamma,\ \gamma^2, \gamma^3, \gamma^4, \gamma^5) \begin{bmatrix} 1 & 0 & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & -1 \end{bmatrix} \end{aligned} \tag{108}$$

or

$$= \langle\gamma_+\ \ I(\pm 1) \tag{109}$$

transposing

$$\gamma\rangle = I(\pm 1)\gamma_+\rangle$$

Taking the dyadic product

$$\gamma\rangle\langle\gamma = I(\pm1)\ \underset{+}{\gamma\rangle}\langle\gamma_{+}\ I(\pm1) \tag{110}$$

Note that

$$\underset{+}{\overset{+}{\gamma\rangle\langle\gamma}} = n\rangle\langle n \tag{111}$$

and

$$I^{-1}(\pm1) = I(\pm1) \tag{112}$$

Equation (110) by (111) and (112)

$$\gamma\rangle\langle\gamma = I(\pm1)\ n\rangle\langle n\ \ I(\pm1) \tag{113}$$

$$I(\pm1)\ \ \gamma\rangle\langle\gamma\ I(\pm1) = n\rangle\langle n \tag{114}$$

Summing all dyads

$$\sum n\rangle\langle n = I(\pm1)\ \left(\sum \gamma\rangle\langle\gamma\right)\ I(\pm1) \tag{115}$$

and inverting the matrix

$$\left(\sum n\rangle\langle n\right) = I(\pm1)\ \left(\sum \gamma\rangle\langle\gamma\right)^{-1} I(\pm1) \tag{116}$$

For arbitrary span of N+1 points Equation (90) is

$$\underset{3x3}{N^TN} = \begin{bmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{bmatrix}$$

(117)

$$N^TN = \begin{bmatrix} N+1 & \frac{N}{2}(N+1) & \frac{N}{6}(N+1)(2N+1) \\ \frac{N}{2}(N+1) & \frac{N}{6}(N+1)(2N+1) & \frac{N^2}{4}(N+1)^2 \\ \frac{N}{6}(N+1)(2N+1) & \frac{N^2}{4}(N+1)^2 & \frac{N}{30}(N+1)(2N+1)(3N^2+3N-1) \end{bmatrix}$$

and the inverse is

(118)

$$(N^TN)^{-1} = \frac{3}{(N+1)(N+2)(N+3)} \begin{bmatrix} 3N^2+3N+2 & -(12N+6) & 10 \\ -(12N+6) & \frac{4(2N+1)(8N-3)}{N(N-1)} & \frac{-60}{(N-1)} \\ 10 & \frac{-60}{(N-1)} & \frac{60}{N(N-1)} \end{bmatrix}$$

The matrix for N+1 points in the span and using the midpoint metric of Equation (95) with the summation formulas for the powers of the natural numbers of Equation (95) is

(119)

$$M^TM = \begin{bmatrix} N+1 & 0 & \frac{N(N+1)(N+2)}{12} \\ 0 & \frac{N(N+1)(N+2)}{12} & 0 \\ \frac{N(N+1)(N+2)}{12} & 0 & \frac{N(N+1)(N+2)(3N^2+6N-4)}{240} \end{bmatrix}$$

and the inverse is

(120)

$$(M^TM)^{-1} = \begin{bmatrix} \frac{3(3N^2+6N-4)}{4(N^2-1)(N+3)} & 0 & \frac{-15}{(N^2-1)(N+3)} \\ 0 & \frac{12}{N(N+1)(N+2)} & 0 \\ \frac{-15}{(N^2-1)(N+3)} & 0 & \frac{180}{N(N+2)(N+3)(N^2-1)} \end{bmatrix}$$

The matrix for front span of Equation (92) for N+1 points in the span is:

(121)

$$\Gamma^T\Gamma = \begin{bmatrix} N+1 & -\frac{N}{2}(N+1) & \frac{N}{6}(N+1)(2N+1) \\ -\frac{N}{2}(N+1) & \frac{N}{6}(N+1)(2N+1) & -\frac{N^2}{4}(N+1)^2 \\ \frac{N}{6}(N+1)(2N+1) & -\frac{N^2}{4}(N+1)^2 & \frac{N}{30}(N+1)(2N+1)(3N^2+3N-1) \end{bmatrix}$$

and the inverse of 121 using Equation (116) and Equation (118) is

$$(\Gamma^T\Gamma)^{-1} = g_o \begin{bmatrix} 3N^2+3N+2 & 12N+6 & 10 \\ 12N+6 & \frac{4(2N+1)(8N-3)}{N(N-1)} & \frac{60}{N-1} \\ 10 & \frac{60}{N-1} & \frac{60}{N(N-1)} \end{bmatrix} \qquad (122)$$

where

$$g_o = \frac{3}{(N+1)(N+\ \ )(N+3)}$$

Note the sign reversals in Equation (118) and (122) in the inverse matrices.

The Grammian inverse matrices and their psuedo inverses will be used in later sections.

By Equations (35) and (38) for the d=3 case and back to front of span indices we have

$$\langle t(i,n) = \langle n\ T_u(i)T_u(\beta)D(h_o) = \langle nT \qquad (123)$$

and for n=0,1,2,···,N

$$\begin{bmatrix} \langle t(i,0) \\ \langle t(i,1) \\ \langle t(i,2) \\ \vdots \\ \langle t(i,N) \end{bmatrix} = \underset{(N+1)\times 3}{T} = \underset{(N+1)\times 3}{N}\ \underset{3\times 3}{T(i,h_o)} \qquad (124)$$

and the transpose is

$$T^T = T^T(i,h_o)N^T \tag{125}$$

and the discrete metric is

$$\underset{3\times 3}{T^T T} = T^T(i,h_o)N^T NT(i,h_o) \tag{125}$$

where the purely "number theoretic" metric $N^T N$ is given by Equation (61) or Equation (119) for the 3x3 case.

The inverse of the discrete metric of Equation (126) is

$$(T^T T)^{-1} = T^{-1}(i,h_o)(N^T N)^{-1}\ T^{-T}(i,h_o) \tag{127}$$

where the inverse $(N^T N)^{-1}$ is given by Equation (118).

For the center of span case we have by Equation (35), Equation (41) and Equation (42)

$$\langle t(k,m) = \langle m T_u(k)T_u(\beta)D(h_o) \tag{128}$$

and packagewise for the points over the span

$$\begin{bmatrix} \langle t(k,-M) \\ \vdots \\ \langle t(k,-1) \\ \langle t(k,0) \\ \langle t(k,1) \\ \vdots \\ \langle t(k,M) \end{bmatrix} = \underset{(N+1)\times 3}{T} = \underset{(N+1)\times 3}{M} T_u(k)T_u(\beta)D(h_o) = MT(k,h_o) \tag{129}$$

and the metric is

$$\underset{3\times 3}{T^{T}T} = T^{T}(k,h_{o})M^{T}MT(k,h_{o}) \tag{130}$$

where the "number-theoretic" metric $M^{T}M$ is given by Equation (76), and the inverse is

$$(T^{T}T)^{-1} = T^{-1}(k,h_{o})(M^{T}M)^{-1}T^{-T}(k,h_{o}) \tag{131}$$

The front of span to back case by Equation (35) and Equation (43) is

$$\langle t(\ell,\gamma) = \langle\gamma t_{u}(\ell)t_{u}(\beta)D(h_{o}) \tag{132}$$

$$\langle t(\ell,\gamma) = \langle\gamma T(\ell,h_{o}) \tag{133}$$

$$\langle\gamma = (1,-\gamma,\gamma^{2},-\gamma^{3},\cdots) \tag{134}$$

and for

$$\gamma = 0,1,2,\cdots,N$$

we have

$$\begin{bmatrix} \langle t(\ell,0) \\ \langle t(\ell,1) \\ \langle t(\ell,2) \\ \vdots \\ \langle t(\ell,\gamma) \\ \vdots \\ \langle t(\ell,N) \end{bmatrix} = \underset{(N+1)\times 3}{T} = \underset{(N+1)\times 3}{\Gamma} T_{u}(\ell)T_{u}(\beta)D(h_{o}) \tag{135}$$

or

$$\underset{(N+1)\times 3}{T} = \underset{(N+1)\times 3}{\Gamma} T(\ell,h_{o}) \tag{136}$$

where Γ is given by Equation (79).

The metric is

$$(T^TT) = T^T(\ell,h_o)\Gamma^T\Gamma T(\ell,h_o) \tag{137}$$

and the inverse is

$$(T^TT)^{-2} = T^{-1}(\Gamma^T\Gamma)^{-1}T^{-T} \tag{138}$$

Where the metric $\Gamma^T\Gamma$ and its inverse are given by Equation (86) or Equation (121) and Equation 123).

DISCRETE "METRIC-MATRIX" FOR FORWARD AND BACKWARD EXPONENTIALLY WEIGHTED MONOMIAL BASE. This section develops the discrete metric-matrix for the base vectors which when "Gram-Schmitted" yield the Laguerre base over the interval (0,∞) on discrete point sets. The following section will derive the discrete Laguaerre polynomials utilizing these matrices.

Forward Exponential Weight Case. The back-to-front case will be considered first with the index N going to ∞. By Equation (30), Section (2)

$$\langle f_e = \langle t_e = (1,t,t^2,\cdots,t^{d-1})e^{-at/2} \tag{139}$$

and by Equation (13)

$$t(i,n) = h_o[\beta+\tau(i,n)] \tag{140}$$

or by Equation (17) in Equation (140)

$$t(i,n) = h_o[\beta+i+n] \tag{141}$$

and Equation (141) and used in the exponential

$$e^{-\frac{at}{2}} = e^{-\frac{ah_o[\beta+i+n]}{2}} = e^{-\frac{ah_on}{2}}e^{-\frac{ah_o(\beta+i)}{2}} \tag{142}$$

or

$$e^{-\frac{at(i,n)}{2}} = e^{-\frac{\lambda n}{2}} e^{-b_o} \tag{143}$$

where

$$\lambda = ah_o \tag{144}$$

and

$$b_o = \frac{ah_o(\beta+i)}{2} \tag{145}$$

Let

$$\theta^{1/2} = e^{-\frac{\lambda}{2}} = e^{-\frac{ah_o}{2}} \tag{146}$$

then

$$\theta = e^{-\lambda} \tag{147}$$

and

$$\theta^n = e^{-\lambda n} \tag{148}$$

Packaging Equation (139) by use of Equation (141)

$$\begin{bmatrix} \langle t(i,o)_e \\ \langle t(i,1)_e \\ \langle t(i,2)_e \\ \vdots \\ \langle t(i,n)_e \\ \vdots \\ \langle t(i,N)_e \end{bmatrix} = \begin{bmatrix} e^{-\frac{at(i,o)}{2}} & & & & \\ & e^{-\frac{at(i,1)}{2}} & & & \\ & & \ddots & & \\ & & & e^{-\frac{at(i,n)}{2}} & \\ & & & & \ddots \\ & & & & & e^{-\frac{at(i,N)}{2}} \end{bmatrix} \begin{bmatrix} \langle t(i,o) \\ \langle t(i,1) \\ \vdots \\ \langle t(i,n) \\ \vdots \\ \langle t(i,N) \end{bmatrix} \qquad (149)$$

or by Equation (124) in Equation (149)

$$\underset{(N+1)xd}{T_e} = e^{-b_o} \begin{bmatrix} e^{-\frac{\lambda o}{2}} & & & & \\ & e^{-\frac{\lambda 1}{2}} & & & \\ & & e^{-\frac{\lambda 2}{2}} & & \\ & & & \ddots & \\ & & & & e^{-\frac{\lambda N}{2}} \end{bmatrix} \underset{(N+1)xd}{NT} \underset{d^{x}d}{(i,h_o)} \qquad (150)$$

Define the matrix

$$W^{1/2} = \begin{bmatrix} 1 & & & & \\ & \theta^{1/2} & & & \\ & & \theta^{2/2} & & \\ & & & \theta^{3/2} & \\ & & & & \ddots & \\ & & & & & \theta^{N/2} \end{bmatrix} \qquad (151)$$

and

$$\underset{(N+1)}{W^{\frac{1}{2}}} \quad \underset{(N+1)\times d}{N} = \underset{(N+1)\times d}{N_W} \tag{152}$$

Form the metric

$$\underset{d\times d}{N_w^T N_w} = \underset{d\times (N+1)}{N^T} \quad \underset{(N+1)\times d}{WN} \tag{153}$$

The elements of Equation (152) are by Equation (50)

$$W^{\frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & & 1 \\ 1 & 2 & 2^2 & & 2^{d-1} \\ 1 & 3 & 3^3 & & 3^{d-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & N & N^2 & & N^{d-1} \end{bmatrix} = W^{\frac{1}{2}} \left[n\rangle_0 \;,\; n\rangle_1 \;, \cdots, n\rangle_{d-1} \right] \tag{154}$$

and the metric of Equation (153) is

$$N^T WN = \begin{bmatrix} \langle N+1)n_0 \\ \langle N+1)n_1 \\ \vdots \\ \langle N+1)n_{d-1} \end{bmatrix} W \left[n(N+1\rangle_0 ,\; n(N+1\rangle_1 , \cdots n(N+1\rangle_{d-1} \right] \tag{155}$$

The i, jth elements of Euqation (155) is

$$m_{ij} = \underset{i}{\langle} N+1)n \;W\; n(N+1 \underset{j}{\rangle} = \underset{i}{\langle} n \;\text{dig}\left(n \underset{j}{\rangle} \right) w(N+1\rangle \tag{156}$$

or

$$m_{ij} = \underset{i+j}{\langle} n\cdots w \rangle \tag{157}$$

where

$$\langle N+1)n = (0, \langle N)c) \tag{158}$$

and the counting number vector of dimension N is

$$\langle N)c = (1,2,3,\cdots N) \tag{159}$$

By Equation (158) in Equation (156)

$$m_{ij} = \underset{i+j}{\langle N)c} \; w(N\rangle \tag{160}$$

and in the limit

$$\underset{N\to\infty}{\ell m} \; \underset{i+j}{\langle N)c} \; w(N\rangle = \sum_{c=0}^{\infty} C^{i+j} W^{c} \tag{161}$$

where i,j=0,1,2$\cdots$d-1 and

$$0<\theta = \omega<1 \tag{162}$$

we have for i,j=0,1,2

$$\sum_{c=0}^{\infty} \theta^{c} = \frac{1}{1-\theta} \tag{163}$$

$$\sum_{c=0}^{\infty} C\theta^{c} = \frac{1}{(1-\theta)^{2}} \tag{164}$$

$$\sum_{c=0}^{\infty} C^{2}\theta^{c} = \frac{\theta(1+\theta)}{(1-\theta)^{3}} \tag{165}$$

$$\sum_{c=0}^{\infty} C^{3}\theta^{c} = \frac{\theta(1+4\theta+\theta^{2})}{(1-\theta)^{4}} \tag{166}$$

$$\sum_{c=0}^{\infty} C^{4}\theta^{c} = \frac{(1+11\theta+11\theta^{2}+\theta^{3})}{(1-\theta)^{5}} \tag{167}$$

and the 3x3 metric matrix is (the non-dimensionalized portion)

$$M_{\tau\tau\theta} = \begin{bmatrix} \frac{1}{1-\theta} & \frac{\theta}{(1-\theta)^2} & \frac{\theta(1+\theta)}{(1-\theta)^3} \\ \frac{\theta}{(1-\theta)^2} & \frac{\theta(1+\theta)}{(1-\theta)^3} & \frac{\theta(1+4\theta+\theta^2)}{(1-\theta)^4} \\ \frac{\theta(1+\theta)}{(1-\theta)^3} & \frac{\theta(1+\theta4+\theta^2)}{(1-\theta)^4} & \frac{\theta(1+11\theta+11\theta^2+\theta^3)}{(1-\theta)^5} \end{bmatrix} \tag{168}$$

Backward Exponential-Weight Case. The front of time-span to back in the limit as $N\to\infty$ case of Equation (132) to Equation (137) will be developed. By Equation (15) and Equation (19)

$$t(\ell,\gamma) = h_o\ [\beta+\ell-\gamma] \tag{169}$$

$$= h_o\ (\beta+\ell) - h_o\gamma \tag{170}$$

By Equation (23) in Appendix D we see that for the negative infinity limit we need a weight factor e^{at} where $a>0$, hence Equation (139) becomes

$$\langle fe = \langle te = (1,t,t^2,t^3,\ldots)\ e^{at/2} \tag{171}$$

Using Equation (170) in the exponential factor of Equation (171)

$$e^{\frac{at(\ell,\gamma)}{2}} = e^{\frac{ah_o(\beta+\ell)}{2}}\ e^{\frac{-ah_o\gamma}{2}} \tag{172}$$

$$e^{\frac{-at(\ell,\gamma)}{2}} = e^{\frac{\alpha o}{2}}\ \psi^{\frac{\gamma}{2}} \tag{173}$$

where

$$e^{\frac{\alpha o}{2}} = e^{\frac{ah_o(\beta+\ell)}{2}} \tag{174}$$

and

$$\psi^{\frac{1}{2}} = e^{\frac{-ah_o}{2}} \tag{175}$$

By Equation (173) in Equation (171) and Equation (132)

$$\Big\langle fe = e^{\frac{\alpha o}{2}} \; \psi^{\frac{1}{2}} \Big\langle \gamma T_u(\ell) T_u(\beta) D(h_o) \tag{176}$$

where by Equation (134)

$$\Big\langle \gamma = (1, -\gamma, \gamma^2, -\gamma^3, \gamma^4, \cdots (-1)^{d-1} \gamma^{d-1}) \tag{177}$$

or packagewise by Equation (135)

$$\underset{(N+1)\times d}{T_e} = e^{\frac{\alpha o}{2}} \; W^{\frac{1}{2}} \, \Gamma T(\ell, h_o) \tag{178}$$

and the weighted metric of Equation (137) becomes

$$T_e^T T_e = T^T(\ell, ho) \; \Gamma^T W \Gamma T(\ell, ho) e^{\alpha o} \tag{179}$$

The Γ matrix is given by Equation (79) as

$$\underset{(N+1)\times d}{\Gamma} = \begin{bmatrix} 1 & -N & N^2 & -N^3 & (-1)^{d-1}(N)^{d-1} \\ 1 & -(N-1) & (N-1)^2 & -(N-1)^3 & (-1)^{d-1}(N-1)^{d-1} \\ 1 & . & . & . & . \\ \vdots & . & . & . & . \\ & . & . & . & . \\ 1 & -2 & 2^2 & -2^2 & -2^{d-1} \\ 1 & -1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{180}$$

Equation (180) can be written as Equation (154)

$$\underset{(N+1)\times d}{\Gamma} = \left[n(n+1\Big\rangle_0, \;\; \overset{-L}{(N+L)(N+1)} \, n(N+1\Big\rangle_1, \;\; L \; n\Big\rangle_2, \;\; -L \; n\Big\rangle_3, \cdots (-1)^{d-1} Ln\Big\rangle_{d-1} \right] \tag{181}$$

or the metric is

$$\underset{d\times d}{\Gamma_W^T\Gamma_W} = \begin{bmatrix} \underset{0}{\langle (N+1)n} \\ \underset{1}{\langle n\ L} \\ -\underset{2}{\langle n\ L} \\ \vdots \\ (-1)^{d-1}\underset{d-1}{\langle n\ L} \end{bmatrix} W \left[\underset{0}{n\rangle},\ -L\underset{1}{n\rangle},\ L\underset{2}{n\rangle}\cdots\ (-1)^{d-1}L\underset{d-1}{n\rangle} \right] \tag{182}$$

The 3×3 matrix case becomes as $N\to\infty$

$$M_{\tau\tau}(1) = \begin{bmatrix} \frac{1}{1-\psi} & \frac{-\psi}{(1-\psi)^2} & \frac{\psi(1+\psi)}{(1-\psi)^3} \\ \frac{-\psi}{(1-\psi)^2} & \frac{\psi(1+\psi)}{(1-\psi)^3} & \frac{-\psi(1+4\psi+\psi^2)}{(1-\psi)^4} \\ \frac{(1+\psi)}{(1-\psi)^3} & \frac{-\psi(1+4\psi+\psi^2)}{(1-\psi)^4} & \frac{\psi(1+11\psi+11\psi^2+\psi^3)}{(1-\psi)^5} \end{bmatrix} \tag{183}$$

which is the same as Equation (168) except for alternating signs, and

$$0<\psi = e^{-ah_o} <1 \tag{184}$$

By Equation (147) we see that

$$\psi = \theta \tag{185}$$

ORTHOGONAL POLYNOMIALS OVER DISCRETE POINT SETS FRONT, CENTRAL, AND BACK; GRAM, LEGENDRE AND LAGUARRE. Morrison says that one approach to obtaining the form of the polynomials so defined would be to use a Schmidt orthogonalization procedure...as many of the polynomials as we have energy for, each to within an unspecified constant. He says the drawback to the above method is that the general form of the polynomials is not then obtained. Morrison derives the discrete polynomials by applying summation by parts. He says he has extended Hildebrand's approach for obtaining the discrete Gram polynomials to obtain the discrete Lengendre and Laguerre polynomials. Morrison says that Milne has an alternate and extremely elegant derivation of the discrete Lengendre polynomials. In Erdelyea (page 222) one finds also the derivation via summation by parts with a reference to an alternate route via the methods of generating functions. The applications of Laguarre polynomials to trajectory estimation problems in Reference (16) by Brown use matrices which though not called as such is the metric matrix of Equation (183). Since I have not found references deriving the orthogonal polynomials via the different variations of Gram-Schmidt procedures they are developed here for the 3×3 case, the case most commonly applied in trajectory estimation corresponding to constant accelerations over data spans. Also many of the submatrices inverted here-in will be applied later to the exponentially weighted filter equations. And finally with respect to the simple minded vector space concepts thus far developed at this stage of the report; the Gram-Schmidt procedure is the most natural way to develop said polynomials.

Three cases of the discrete monomial base will be developed:

1. Back to front of span,

2. Mid-point span or central, and

3. Front-to-back of span.

One additional case will be developed for the exponential weighting from front-to-back as the back point of span goes to negative infinity. This later case is the case of smoothing over all data from real-time (front) to back.

The Gram-Schmidt procedure will be applied first to the matrices over the integers, N, M and Γ and the Γ_W Laguarre case; and then to derive the classical polynomials. The necessary transformations on these orthogonal polynomials to take care of the matrices $T(ih_o)$ $T(k,h_o)$ and $T(\ell,h_o)$ will be done in a separate section.

Orthogonal Polynomials With Respect to Summation Over Discrete Point Sets Indexed from Back-to-Front of Span for Matrix N. Consider the second degree polynomial case generating the N matrix of Equation (50)

$$\underset{(N+1)\times 3}{N} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ \vdots & \vdots & \vdots \\ 1 & N & N^2 \end{bmatrix} = \begin{bmatrix} n(N+1)\rangle_0, & n(N+1)\rangle_1, & n(N+1)\rangle_2 \end{bmatrix} \tag{186}$$

$$N^T N = \begin{bmatrix} N+1 & \frac{N}{2}(N+1) & \frac{N}{6}(N+1)2N+1) \\ \frac{N}{2}(N+1) & \frac{N}{6}(N+1)(2N+1) & \frac{N^2}{4}(N+1)^2 \\ \frac{N}{6}(N+1)(2N+1) & \frac{N^2}{4}(N+1)^2 & \frac{N}{30}(N+1)(2N+1)(3N^2+3N-1) \end{bmatrix} \tag{187}$$

Let the first Gram-Schmidt vector be

$$g(N+1)\rangle_0 = n(N+1)\rangle_0 = \begin{bmatrix} n\rangle_0, & n\rangle_1, & n\rangle_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{188}$$

The second Gram-Schmidt vector with unit first coordinate

$$g(N+1)\rangle_1 = \begin{bmatrix} n\rangle_0, & n\rangle_1 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}_1 \tag{189}$$

and the orthogonal constraint

$$\langle n\; g\rangle_{0\,1} = 0 = \begin{bmatrix} \langle n\; n\rangle_{0\,0}, & \langle n\; n\rangle_{0\,1} \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}_1 \tag{190}$$

or

$$\lambda_1 = \frac{-\left\langle \underset{0}{n}\ \underset{0}{n} \right\rangle}{\left\langle \underset{0}{n}\ \underset{1}{n} \right\rangle} = \frac{-(N+1)}{\frac{N}{2}(N+1)} = \frac{-2}{N} \tag{191}$$

hence

$$g(N+1\Big\rangle_{1} = \begin{bmatrix} \underset{0}{n}\Big\rangle, & \underset{1}{n}\Big\rangle, & \underset{2}{n}\Big\rangle \end{bmatrix} \begin{bmatrix} 1 \\ \frac{-2}{N} \\ 0 \end{bmatrix} \tag{192}$$

The third Gram-Schmidt vector with first coordinate constrained to unity is

$$g(N+1\Big\rangle_{2} = \begin{bmatrix} \underset{0}{n}\Big\rangle, & \underset{1}{n}\Big\rangle, & \underset{2}{n}\Big\rangle \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}_2 \tag{193}$$

The constraint that it be perpendicular to the first two vectors is

$$\begin{pmatrix} \Big\langle \underset{0}{n} \\ \Big\langle \underset{1}{n} \end{pmatrix} g(N+1\Big\rangle_{2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{matrix} n_{00} & n_{01} & n_{02} \\ n_{10} & n_{11} & n_{12} \end{matrix} \begin{pmatrix} 1 \\ \lambda(2\Big\rangle \end{pmatrix}_2 \tag{194}$$

where

$$n_{ij} = \left\langle \underset{i}{n}\ \underset{j}{n} \right\rangle \tag{195}$$

or

$$\lambda(\underset{2}{2}\Big\rangle = -\begin{bmatrix} n_{01} & n_{02} \\ n_{11} & n_{12} \end{bmatrix}^{-1} \begin{bmatrix} n_{00} \\ n_{10} \end{bmatrix} \tag{196}$$

The $2^{x}2$ submatrix by Equation (90) is

$$\begin{bmatrix} S_1(N) & S_2(N) \\ S_2(N) & S_3(N) \end{bmatrix}^{-1} = \frac{1}{\text{DET()}} \begin{bmatrix} S_3(N) & -S_2(N) \\ -S_2(N) & S_3(N) \end{bmatrix} \tag{197}$$

where

$$\text{det()} = S_1(N)\ S_3(N)\ -S_2^2(N) \tag{198}$$

Using Equation (197) and Equation (198) in Equation (196)

$$\mu(2\Big\rangle_{2} = (-1) \begin{bmatrix} S_3(N)\ S_0(N) & -S_2(N)S_1(N) \\ -S_2(N)\ S_0(N) & +S_1^2(N) \end{bmatrix} \frac{1}{\text{det()}} \tag{199}$$

or

$$\mu_{12} = (-1) \left[\frac{(N+1)\ \frac{N^2}{4}\ (N+1)^2 - \frac{N}{2}\ (N+1)\ \frac{N}{6}\ (N+1)\ (2N+1)}{\frac{N}{2}\ (N+1)\ \frac{N^2}{4}\ (N+1)^2\ -\ (\frac{N}{6})^2\ (N+1)^2\ (2N+1)^2} \right] \tag{200}$$

Cancelling terms and collecting one obtains

$$\mu_{12} = -\frac{6}{N-1} \tag{201}$$

and in a similar manner

$$\mu_{22} = \frac{6}{N(N-1)} \tag{202}$$

or

$$g)N+1\Big\rangle_{2} = \begin{bmatrix} n\rangle_0, & n\rangle_1, & n\rangle_2 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{6}{N-1} \\ \frac{6}{N(N-1)} \end{bmatrix} \tag{203}$$

The first three Gram-Schmidt vectors by Equation (188), Equation (192) and Equation (203) in package form are

$$\left[\underset{0}{g(N+1)\rangle}, \ \underset{1}{g(N+1)\rangle}, \ \underset{2}{g(N+1)\rangle} \right] = \left[\underset{0}{n\rangle}, \ \underset{1}{n\rangle}, \ \underset{2}{n\rangle} \right] \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{-2}{N} & \frac{-6}{N-1} \\ 0 & 0 & \frac{6}{N(N-1)} \end{bmatrix} \tag{204}$$

or

$$\underset{(N+1)\times 3}{G(N)} = \underset{(N+1)\times 3}{N} \ \underset{3\times 3}{B_g(N)} \tag{205}$$

where

$$\underset{3\times 3}{B_g(N)} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{-2}{N} & \frac{-6}{N-1} \\ 0 & 0 & \frac{6}{N(N-1)} \end{bmatrix} \tag{206}$$

If one now partitions Equation (205) into its row space

$$\begin{bmatrix} \overset{0}{\langle 3)}g \\ \overset{1}{\langle 3)}g \\ \vdots \\ \overset{n}{\langle 3)}g \\ \vdots \\ \overset{N}{\langle 3)}g \end{bmatrix} = \begin{bmatrix} (1\ 0\ 0) \\ (1\ 1\ 1) \\ (1,2,2^2) \\ \vdots \\ (1,n,n^2) \\ \vdots \\ (1,N,N^2) \end{bmatrix} B_g(N) \tag{207}$$

and equating the (m+1)th element

$$\overset{n}{\Big<} 3) g = (1, n, n^2) \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{-2}{N} & \frac{-6}{N-1} \\ 0 & 0 & \frac{6}{N(N-1)} \end{bmatrix} \tag{208}$$

We obtain the first three of the general form for what Milne calls the discrete Lengendre polynomials. The first six are given on page 267 of reference [ ] as (for n=0,1,2,···N)

$$\begin{aligned}
g_{o,N}(n) &= 1 \\
g_{1,N}(n) &= 1 - \frac{2n}{N} \\
g_{2,N}(n) &= 1 - \frac{6n}{N} + \frac{6n(n-1)}{N(N-1)} \\
g_{3,N}(n) &= 1 - \frac{12n}{N} + \frac{30n(n-1)}{N(N-1)} - \frac{20n(n-1)(n-2)}{N(N-1)(N-2)} \\
g_{4,N}(n) &= 1 - \frac{20n}{N} + \frac{90n(n-1)}{N(N-1)} - \frac{140n(n-1)(n-2)}{N(N-1)(N-2)} \\
&\quad + \frac{70n(n-1)(n-2)(n-3)}{N(N-1)(N-2)(N-3)} \\
g_{5,N}(n) &= 1 - \frac{30n}{N} + \frac{210n(n-1)}{N(N-1)} - \frac{560n(n-1)(n-2)}{N(N-2)(N-1)} \\
&\quad + \frac{630n(n-1)(n-2)(n-3)}{N(N-1)(N-2)(N-3)} \\
&\quad - \frac{252n(n-1)(n-2)(n-3)(n-4)}{N(N-1)(N-2)(N-3)(N-4)}
\end{aligned} \tag{209}$$

The general term is

$$g_{\delta,N}(n) = \sum_{\varepsilon=0}^{\delta} (-1)^{\varepsilon} \binom{\delta}{\varepsilon}\binom{\delta+\varepsilon}{\varepsilon} \frac{n^{(\varepsilon)}}{N^{(\varepsilon)}} \tag{210}$$

where

$$\delta = 0,1,2,\cdots,d-1$$

the degree of the polynomial and

$$n^{(\varepsilon)} = n(n-1)(n-2)\cdots(n-\varepsilon+1)$$

is the backward factoral function given in Appendix (B). Milne utilizes the Newton polynomials in his derivations.

The general term for the elements of the diagonal metric matrix G^TG is given on page (268) of Milne as

$$\sum_{n=0}^{N} g_{\delta,N}^{2}(n) = \frac{(N+\delta+1)(\delta+N)^{(\delta)}}{(2\delta+1)\ N^{(\delta)}} \tag{211}$$

and for the first three

$$\sum_{n=0}^{N} g_{0,N}^{2}(n) = N + 1$$

$$\sum_{n=0}^{N} g_{1,N}^{2}(n) = \frac{(N+2)(N+1)^{(1)}}{(2+1)\ N^{(1)}} = \frac{(N+2)(N+1)}{3N} = \frac{(N+2)(N+1)}{3} \tag{212}$$

and

$$\sum_{n=0}^{N} g_{2,N}^{2}(n) = \frac{(N+2+1)(2+N)^{(2)}}{(4+1)\ N^{(2)}}$$

or

$$\sum_{n=0}^{N} g_{2,N}^{2}(n) = \frac{(N+3)(N+2)(N+1)}{5N(N-1)} \tag{213}$$

These diagonal metric elements can also be derived via Equation (188), Equation (192) and Equation (203). By Equation (188)

$$\underset{0}{\Big\langle (N+1)g}\quad \underset{0}{g(N+1)\Big\rangle} = \underset{0}{\big\langle n}\quad \underset{0}{n\big\rangle} = N + 1$$

By Equation (192)

$$\underset{1}{\langle g}\quad \underset{1}{g\rangle}\ (1, -\frac{2}{N}) \begin{bmatrix} N+1 & \frac{N(N+1)}{2} \\ \frac{N(N+1)}{2} & \frac{N(N+1)(2N+1)}{6} \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{2}{N} \end{bmatrix} \tag{214}$$

$$\underset{1}{\langle g}\quad \underset{1}{g\rangle} = \frac{(N+2)(N+1)}{3N} \tag{215}$$

and the square of the norm of the $g\rangle_2$ vector is likewise obtained yielding the discrete metric-matrix

$$M_{gg} = \underset{3\text{x}(N+1)}{G^T(N)}\ \underset{(N+1)\text{x}3}{G(N)} = \begin{bmatrix} N+1 & 0 & 0 \\ 0 & \frac{(N+2)(N+1)}{3N} & 0 \\ 0 & 0 & \frac{(N+3)(N+2)(N+1)}{N5(N-1)} \end{bmatrix} \tag{216}$$

The first three ortho-normal discrete Legendre vectors are

$$\underset{(N+1)\text{x}3}{G(N)}\ M_{gg}^{\frac{1}{2}} = \underset{(N+1)\text{x}3}{S(N)} = \underset{(N+1)\text{x}3}{N}\ B_g\ M_{gg}^{-1} = N\ B_s(N) \tag{217}$$

where

$$B_s(N) = B_g(N)\ M_{gg}^{-\frac{1}{2}} \tag{218}$$

By Equation (205)

$$\underset{(N+1)\text{x}3}{G(N)} = \underset{(N+1)\text{x}3}{N}\ \underset{3\text{x}3}{B_g(N)} \tag{219}$$

or

$$G(N)\ B_g^{-1} = N \qquad (220)$$

and the metrics are related

$$\underset{3\times3}{N^TN} = B_g^{-T}\ G^TG\ B_g^{-1} \qquad (221)$$

$$= (B_g\ M_{gg}^{-1}\ B_g^T)^{-1} \qquad (222)$$

and inverting

$$(N^TN)^{-1} = B_g\ M_{gg}^{-1}\ B_g^T \qquad (223)$$

and for the 3x3 case by Equation (206) and Equation (216) in Equation (223)

$$(N^TN)^{-1} = \frac{3}{(N+1)(N+2)(N+3)} \begin{bmatrix} 3N^2+3N+2 & -(12N+6) & 10 \\ -(12N+6) & \frac{4(2N+1)(8N-3)}{N(N-1)} & \frac{-60}{N-1} \\ 10 & \frac{-60}{N-1)} & \frac{60}{(N-1)} \end{bmatrix} \qquad (224)$$

The inverse of the matrix $B_g(N)$ can be obtained via Equation (395) of Section (1) as

$$B_g^{-1} = D_g^{-1}\ B_g^T\ N^TN \qquad (225)$$

Orthogonal Polynomials With Respect to Summation Over Discrete Point Sets Indexed From Midpoint of Span. The matrix of integers of Equation (129) M is given by Equation (66) for the second-degree polynomial case as

$$\underset{(2M+1)\times 3}{M} = \begin{bmatrix} 1 & -M & M^2 \\ 1 & -(M-1) & (M-1)^2 \\ \vdots & \vdots & \vdots \\ 1 & -2 & 2^2 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ \vdots & \vdots & \vdots \\ 1 & (M-1) & (M-1)^2 \\ 1 & M & M^2 \end{bmatrix} \tag{226}$$

The vector of the row space is given by Equation (42) as

$$\langle 3)m = (1, \pm m, m^2) \tag{227}$$

for m=0,1,2,$\cdots$M.

The metric is given by Equation (119) as a function of N=2M as

$$\underset{3\times 3}{M^T M} = \begin{bmatrix} N+1 & 0 & \frac{N(N+1)(N+2)}{12} \\ 0 & \frac{N(N+1)(N+2)}{12} & 0 \\ \frac{N(N+1)(N+2)}{12} & 0 & \frac{N(N+1)(N+2)(3N^2+6N-4)}{240} \end{bmatrix} \tag{228}$$

or as a function of M as

$$\underset{3\times 3}{M^T M} = \begin{bmatrix} 2M+1 & 0 & \frac{M(M+1)(2M+1)}{3} \\ 0 & \frac{M(M+1)(2M+1)}{3} & 0 \\ \frac{M(M+1)(2M+1)}{3} & 0 & \frac{M(M+1)(2M+1)(3M^2+3M-1)}{15} \end{bmatrix} \tag{229}$$

The Gram-Schmidt procedure will be run on the metric of Equation (228). Consider the three vectors in the column space of Equation (226)

$$\underset{(N+1)\times 3}{M} = \left[m(N+1)\rangle_{0},\ m(N+1)\rangle_{1},\ m(N+1)\rangle_{2} \right] \tag{230}$$

Take the first Gram-Schmidt vector to be

$$g\rangle_{0} = m(N+1)\rangle_{0} \tag{231}$$

If we attempt to constrain the first coordinate of the second vector to be unity we obtain

$$g(N+1)\rangle_{1} = \left[m(N+1)\rangle_{0},\ m(N+1)\rangle_{1} \right] \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}_{1} \tag{232}$$

The orthogonal constraint yields

$$\langle \underset{0}{m}\ \underset{1}{g} \rangle = 0 = \left[\langle \underset{0}{m}\ \underset{0}{m} \rangle, \langle \underset{0}{m}\ \underset{1}{m} \rangle \right] \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}_{1} \tag{233}$$

and by Equation (228)

$$0 = [N+1,\ 0] \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} \tag{234}$$

or

$$-(N+1) = 0(\lambda_1) \tag{235}$$

Clearly by Equation (228)

$$\begin{bmatrix} \Big\langle (N+1)m_{0} \\ \Big\langle (N+1)m_{1} \\ \Big\langle (N+1)m_{2} \end{bmatrix} \begin{bmatrix} m(N+1)\Big\rangle_{0}, & m(N+1)\Big\rangle_{1}, & m(N+1)\Big\rangle_{2} \end{bmatrix} = \begin{bmatrix} m_{00} & 0 & m_{01} \\ 0 & m_{11} & 0 \\ m_{21} & 0 & m_{22} \end{bmatrix} \tag{236}$$

The following vectors are perpendicular to each other

$$\Big\langle (N+1)m_{0} \quad m(N+1)\Big\rangle_{1} = \Big\langle (N+1)m_{1} \quad m(N+1)\Big\rangle_{2} = 0 \tag{237}$$

Since the second vector of Equation (228) is perpendicular to the first

$$g\rangle_{1} = m\rangle_{1} \tag{238}$$

or

$$g(N+1)\Big\rangle_{2} = \begin{bmatrix} m\rangle_{0}, & m\rangle_{1} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{239}$$

If we attempt to constrain the first coordinate to unity of the next vector we obtain

$$g(N+1)\Big\rangle_{2} = M \begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}_2 \tag{240}$$

with the orthogonal constraint

$$\begin{bmatrix} \langle m_0 \\ \langle m_1 \end{bmatrix} g\rangle_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{bmatrix} N+1 & 0 & \frac{N(N+1)(N+2)}{12} \\ 0 & \frac{N(N\ 1)(N+2)}{12} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}_2 \tag{241}$$

or

$$-\begin{pmatrix} N+1 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 & \frac{N(N+1)(N+2)}{12} \\ \frac{N(N+1)(N+2)}{12} & 0 \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{242}$$

and by analogy to Equation (235) we see by Equation (242) that we cannot solve for the two unknowns. By Equation (237) we see that the vector $m\rangle_2$ is perpendicular to $m\rangle_1$, hence by Equation (238) for the $g\rangle_2$ vector to be perpendicular to the previous two $g\rangle_0$, $g\rangle_1$, we must have $g\rangle_2$ lie in the space spanned by $m\rangle_0$ and $m\rangle_2$.

$$g(N+1)\rangle_2 = \begin{bmatrix} m\rangle_0, & m\rangle_1, & m\rangle_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ 0 \\ \lambda_2 \end{bmatrix} \tag{243}$$

The previous observations are not necessary if we proceed with the Gram-Schmidt process with the main diagonal unity constants, or

$$g(N+1)\rangle_2 = M \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ 1 \end{bmatrix}_2 \tag{244}$$

or

$$\begin{bmatrix} \langle m_0 \\ \langle m_1 \end{bmatrix} g\rangle_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{245}$$

or

$$\begin{bmatrix} N+1 & 0 \\ 0 & \frac{N(N+1(N+2)}{12} \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = - \begin{bmatrix} \frac{N(N+1)(N+2)}{12} \\ 0 \end{bmatrix} \qquad (246)$$

or

$$\lambda(2\rangle_{2} = - \begin{bmatrix} \frac{N(N+2)}{12} \\ 0 \end{bmatrix} \qquad (247)$$

or

$$g(N+1\rangle_{2} = M \begin{bmatrix} \frac{-N(N+2)}{12} \\ 0 \\ 1 \end{bmatrix} \qquad (248)$$

Packaging Equation (231), Equation (239) and Equation (248)

$$\underset{(N+1)\times 3}{G} = \underset{(N+1)\times 3}{M} \; B_g(M) \qquad (249)$$

where

$$B_g(M) = \begin{bmatrix} 1 & 0 & \frac{-N(N+2)}{12} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (250)$$

The orthonormal set are next obtained; via Equation (231)

$$\langle g_0 \;\; g_0 \rangle = N + 1 \qquad (251)$$

By Equation (239)

$$\langle \underset{1}{g}\ \underset{1}{g} \rangle = (0,1) \begin{bmatrix} N+1 & 0 \\ 0 & \frac{N(N+1)(N+2)}{12} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (252)$$

$$\langle \underset{1}{g}\ \underset{1}{g} \rangle = \frac{N(N+1)(N+2)}{12} \qquad (253)$$

and

$$\langle \underset{2}{g}\ \underset{2}{g} \rangle = \frac{N(N+1)(N+2)(N^2+2N-3)}{180} \qquad (254)$$

The diagonal metric of the Gram-Schmidt vector is

$$\underset{3x3}{G^TG} = \begin{bmatrix} N+1 & 0 & 0 \\ 0 & \frac{N(N+1)(N+2)}{12} & 0 \\ 0 & 0 & \frac{N(N+1)(N+2)(N^2+2N-3)}{180} \end{bmatrix} \qquad (255)$$

and the orthonormal set is

$$\underset{(N+1)x3}{S} = \underset{(N+1)x3}{G(N)} \quad (G^TG)^{-\frac{1}{2}} \qquad (256)$$

or

$$\underset{N+1x3}{S} = G \begin{bmatrix} \frac{1}{\sqrt{N+1}} & 0 & 0 \\ 0 & \frac{\sqrt{12}}{\sqrt{N(N+1)(N+2)}} & 0 \\ 0 & 0 & \frac{\sqrt{180}}{\sqrt{N(N+1)(N+2)(N^2+2N-3)}} \end{bmatrix} \qquad (257)$$

Using Equation (249) in Equation (257)

$$\underset{(N+1)\times 3}{S} = \underset{(N+1)\times 3}{M} B_s(N) \tag{258}$$

where

$$B_s(N) = B_g(N) \ (G^T G)^{-\frac{1}{2}} \tag{259}$$

or

$$B_s(N) = \begin{bmatrix} \frac{1}{\sqrt{N+1}} & 0 & \frac{-N(N+2)\sqrt{180}}{12\sqrt{N(N+1)(N+2)(N^2+2N-3}} \\ 0 & \frac{\sqrt{12}}{\sqrt{N(N+1)(N+2)}} & 0 \\ 0 & 0 & \frac{\sqrt{180}}{\sqrt{N(N+1)(N+2)(N^2+2N-3)}} \end{bmatrix} \tag{260}$$

or in terms of the number of points in the span $W_p = 2N+1$

$$B_s(W_p) = \begin{bmatrix} \frac{1}{\sqrt{W_p}} & 0 & \frac{-\sqrt{180}(W_p^2-1)}{12\sqrt{W_p(W_p^2-1)(W_p^2-4)}} \\ 0 & \frac{\sqrt{12}}{\sqrt{W_p(W_p^2-1)}} & 0 \\ 0 & 0 & \frac{\sqrt{180}}{\sqrt{W_p(W_p^2-1)(W_p^2-4)}} \end{bmatrix} \tag{261}$$

The inverse of the discrete midpoint metric can be obtained via Equation (346), Section (1) as

$$\underset{3\times 3}{(M^T M)^{-1}} = B_s(N) \ B_s^T(N) \tag{262}$$

or for the $3^{x}3$ case

$$(M^TM)^{-2} = \begin{bmatrix} 3(3N^2+6N-4) & 0 & \frac{-15}{(N^2-1)(N+3)} \\ 0 & \frac{12}{N(N+1)(N+2)} & 0 \\ \frac{-15}{(N^2-1)(N+3)} & 0 & \frac{180}{N(N+2)(N+3)(N^2-1)} \end{bmatrix} \tag{263}$$

In a similar manner the inverse of B_s by Equation (358), Section (1)

$$B_s^{-1} = B_s^T \, (M^TM) \tag{264}$$

In a similar manner one can obtain the Gram-Schmidt as the front to back span case.

Orthogonal Polynomials With Respect to Summation Over Discrete Point Sets Indexed From Front to Back of Span for Matrix Γ. The metric is given for the $3^{x}3$ case by Equation (121) and the first coordinates will be taken as unity, hence

$$g(N+1\underset{0}{\rangle} = \gamma(N+1\underset{0}{\rangle} \tag{265}$$

and

$$g(N+1\underset{1}{\rangle} = \begin{bmatrix} \gamma(N+1\underset{0}{\rangle}, & \gamma(N+1\underset{1}{\rangle} \end{bmatrix} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}_1 \tag{266}$$

and the orthogonal constraint is

$$\begin{bmatrix} \underset{0}{\langle} N+1)\gamma \\ \underset{1}{\langle} N+1)\gamma \end{bmatrix} g(N+1\underset{0}{\rangle} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\lambda_{11} = \frac{-\langle\underset{0}{\gamma}\;\;\underset{0}{\gamma}\rangle}{\langle\underset{0}{\gamma}\;\;\underset{1}{\gamma}\rangle} = \frac{-(N+1)}{-\frac{N}{2}(N+1)} = \frac{2}{N} \qquad (267)$$

or

$$g\underset{1}{(N+1\rangle} = \begin{bmatrix}\underset{0}{\gamma\rangle}, & \underset{1}{\gamma\rangle}\end{bmatrix}\begin{bmatrix}1\\ \frac{2}{N}\end{bmatrix} \qquad (268)$$

also

$$g\underset{2}{(N+1\rangle} = \underset{(N+1)\times 3}{\Gamma}\begin{bmatrix}1\\ \lambda_1\\ \lambda_2\end{bmatrix}_2$$

Solving for $\lambda(2\rangle$

$$\underset{2}{\lambda(2\rangle} = -\begin{bmatrix}-\frac{N}{2}(N+1) & \frac{N}{6}(N+1)(2N+1)\\ \frac{N}{6}(N+1)(2N+1) & -\frac{N^2}{4}(N+1)^2\end{bmatrix}^{-1}\begin{pmatrix}N+1\\ -\frac{N}{2}(N+1)\end{pmatrix} \qquad (269)$$

$$= -\begin{bmatrix}-S_1 & S_2\\ S_2 & -S_3\end{bmatrix}^{-1}\begin{pmatrix}S_0\\ -S_1\end{pmatrix} \qquad (270)$$

observe by analogy with Equation (196) and Equation (197) that

$$\begin{bmatrix}-S_1 & S_2\\ S_2 & -S_3\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}S_1 & S_2\\ S_2 & S_3\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \qquad (271)$$

and

$$\begin{bmatrix} -S_1 & S_2 \\ S_2 & -S_3 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S_1 & S_2 \\ S_2 & S_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{272}$$

and

$$\begin{bmatrix} S_0 \\ -S_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \end{bmatrix} \tag{273}$$

and using Equation (273) and Equation (272) in Equation (270)

$$\mu(2\Big\rangle_{2} = \begin{bmatrix} \frac{-6}{N-1} \\ \\ \frac{-6}{N(N-1)} \end{bmatrix} \tag{274}$$

or

$$g(N+1\Big\rangle_{2} = \Gamma \begin{bmatrix} 1 \\ \\ \frac{-6}{N-1} \\ \\ \frac{-6}{N(N-1)} \end{bmatrix} \tag{275}$$

Packaging Equation (265), Equation (268) and Equation (275)

$$\underset{(N+1)\times 3}{G(N)} = \underset{(N+1)\times 3}{\Gamma} B_g(N) \tag{276}$$

where

$$B_g(N) = \begin{bmatrix} 1 & 1 & 1 \\ \\ 0 & \frac{2}{N} & \frac{-6}{N-1} \\ \\ 0 & 0 & \frac{-6}{N(N-1)} \end{bmatrix} \tag{277}$$

The inverse of the metric matrix is by Equation (348), Section (1)

$$(\Gamma^T\Gamma)^{-1} = B_g \ (G^TG)^{-1} \ B_g^T \qquad (278)$$

and is for the 3^x3 case

$$(\Gamma^T\Gamma)^{-1} = g_0 \begin{bmatrix} 3N^2+3N+2 & 12N+6 & 10 \\ 12N+6 & \frac{4(2N+1)(8N-3)}{N(N-1)} & \frac{60}{N-1} \\ 10 & \frac{60}{N-1} & \frac{60}{N(N-1)} \end{bmatrix} \qquad (279)$$

where

$$g_0 = \frac{3}{(N+1)(N+2)(N+3)} \qquad (280)$$

Compare Equation (279) with Equation (224) and note the only difference is some negative signs; now compare the metrics of Equation (121) with the metric of Equation (187), and again note the differences of signs in the same row and column positions.

Orthogonal Polynomials for Exponentialy Weighted Polynomials Over Discrete Time Points from Front to Back of Span. The non-dimensionalized time points generate the weighted vectors of Equation (181) as

$$\underset{(N+1)^x3}{\Gamma_W} = \underset{(N+1)(N+1)}{W^{\frac{1}{2}}} \begin{bmatrix} n(N+1\rangle_0, & -Ln(N+1\rangle_1, & -Ln(N+1\rangle_3 \end{bmatrix} \qquad (281)$$

or

$$\Gamma_W = \begin{bmatrix} \gamma(N+1\rangle_{W0}, & \gamma(N+1\rangle_{W1}, & \gamma(N+1\rangle_{W2} \end{bmatrix} \qquad (282)$$

where the corresponding 3^x3 metric is given by Equation (183) on terms of θ as

$$M_{\tau\tau e}(-1) = \begin{bmatrix} \frac{1}{1-\theta} & \frac{-\theta}{(1-\theta)^2} & \frac{\theta(1+\theta)}{(1-\theta)^3} \\ \frac{-\theta}{(1-\theta)^2} & \frac{\theta(1+\theta)}{(1-\theta)^3} & \frac{-\theta(1+4\theta+\theta^2}{(1-\theta)^4} \\ \frac{\theta(1+\theta)}{(1-\theta)^3} & \frac{-\theta(1+4\theta+\theta^2)}{(1-\theta)^4} & \frac{\theta(1+11\theta+11\theta^2+\theta^3}{(1-\theta)^5} \end{bmatrix} \tag{283}$$

Set the first Gram-Schmidt vector to be

$$g(N+1\rangle_0 = \gamma(N+1\rangle_{W0} = \Gamma_{(N+1)\times 3} \; e(3\rangle_1 \tag{284}$$

and the norm square is

$$\langle g_0 \;\; g_0\rangle = \langle e_1 \; (\Gamma^T\Gamma) \; e_1\rangle = \frac{1}{1-\theta} \tag{285}$$

The second vector is with unity constraint on first coordinate

$$g(N+1\rangle_1 = \left[\gamma(N+1\rangle_{W0}, \; \gamma(N+1\rangle_{W1}\right] \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}_1 \tag{286}$$

and the orthogonal condition

$$\langle \gamma_{W0} \;\; g_1\rangle = 0 = \left[\frac{1}{1-\theta}, \; \frac{-\theta}{(1-\theta)^2}\right] \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}_1 \tag{287}$$

or

$$\lambda_{11} = \frac{1-\theta}{\theta} \tag{288}$$

or

$$g\rangle_{1} = \Gamma_{(N+1)\times 3} \begin{bmatrix} 1 \\ \frac{1-\theta}{\theta} \\ 0 \end{bmatrix} \tag{289}$$

and the square of the norm is

$$\langle g_{1} \quad g_{1}\rangle = (1, \frac{1-\theta}{\theta}, 0)\ \Gamma_W^T \Gamma_W \begin{bmatrix} 1 \\ \frac{1-\theta}{\theta} \\ 0 \end{bmatrix} = \frac{1}{\theta(1-\theta)^2} \tag{290}$$

The third Gram-Schmidt vector is

$$g\rangle_{2} = \Gamma_{(N+1)\times 3} \begin{bmatrix} 1 \\ \lambda(2\rangle \end{bmatrix}_{2} \tag{291}$$

and the orthogonal constraint

$$\begin{bmatrix} \langle \gamma_W{}_{0} \\ \langle \gamma_W{}_{1} \end{bmatrix} g\rangle_{2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{292}$$

or

$$\lambda(2\rangle_{2} - - \begin{bmatrix} \gamma_{01} & \gamma_{02} \\ \gamma_{11} & \gamma_{12} \end{bmatrix}^{-1} \begin{bmatrix} \gamma_{00} \\ \gamma_{10} \end{bmatrix} \tag{293}$$

where

$$\langle \gamma_{i} \quad \gamma_{j}\rangle \quad \gamma_{ij} \tag{294}$$

By the metric matrix of Equation (283)

$$\begin{bmatrix} \frac{-\theta}{(1-\theta)^2} & \frac{(1+\theta)}{(1-\theta)^3} \\ \frac{\theta}{(1-\theta)^3} & \frac{-(1+4\theta+\theta^2)}{(1-\theta)^4} \end{bmatrix}^{-1} = \frac{(1-\theta)^2}{2\theta^2} \begin{bmatrix} -(1+4\theta+\theta^2) & -(1+\theta)(1-\theta) \\ -(1+\theta)(1-\theta) & -(1-\theta)^2 \end{bmatrix} \quad (295)$$

and Equation (295) in Equation (293)

$$\lambda(2\rangle_{2} = \begin{bmatrix} \frac{-(\theta-1)(3\theta+1)}{2\theta^2} \\ \frac{(\theta-1)^2}{2\theta^2} \end{bmatrix} \quad (296)$$

Using Equation (296) in Equation (291)

$$g(N+1\rangle_{2} = \underset{(N+1)\times 3}{\Gamma} \begin{bmatrix} 1 \\ \frac{-\theta(\theta-1)(3\theta+1)}{2\theta^2} \\ \frac{(\theta-1)^2}{2\theta^2} \end{bmatrix} \quad (297)$$

Packaging the Gram-Schmidt vectors of Equation (284), Equation (289) and Equation (297)

$$\underset{(N+1)\times 3}{G} = \underset{(N+1)\times 3}{\Gamma_W} B_g \quad (298)$$

where

$$B_g = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{-(\theta-1)}{\theta} & \frac{-(\theta-1)(3\theta+1)}{2\theta^2} \\ 0 & 0 & \frac{(\theta-1)^2}{2\theta^2} \end{bmatrix} \tag{299}$$

<u>Orthogonal Polynomials for Exponential Weight Over Discrete Time Points from Back to Front of Span.</u> The non-dimensionalized time points generates the weighted vectors of Equation (152) and Equation (50) as

$$\underset{(N+1)x3}{N_W} = \underset{(N+1)(N+1)}{W^{\frac{1}{2}}} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ \vdots & \vdots & \vdots \\ 1 & N & N^2 \end{bmatrix} \tag{300}$$

with the 3×3 metric given by Equation (168) as

$$M_{tt\theta} = \begin{bmatrix} \frac{1}{1-\theta} & \frac{\theta}{(1-\theta)^2} & \frac{\theta(1+\theta)}{(1-\theta)^3} \\ \frac{\theta}{(1-\theta)^2} & \frac{\theta(1+\theta)}{(1-\theta)^3} & \frac{\theta(1+4\theta+\theta^2)}{(1-\theta)^4} \\ \frac{\theta(1+\theta)}{(1-\theta)^3} & \frac{\theta(1+4\theta+\theta^2)}{(1-\theta)^4} & \frac{\theta(1+11\theta+11\theta^2+\theta^3)}{(1-\theta)^5} \end{bmatrix} \tag{301}$$

The G-S procedure with respect to the metric of Equation (301) yields

$$\underset{(N+1)x3}{G} = \underset{(N+1)(N+1}{N_W} \underset{x3}{B_g} \tag{302}$$

where

$$B_{g\,3\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -\frac{(\theta-1)}{\theta} & -\frac{(\theta-1)(3\theta+1)}{2\theta^2} \\ 0 & 0 & \frac{(\theta-1)^2}{2\theta^2} \end{bmatrix} \qquad (303)$$

<u>Relations of These "Discrete Laguerre Polynomials" with Those in the Literature</u>. Morrison in reference (61) derives the discrete Laguerre polynomials (his terminology) via summation by parts and refers the interested reader to Gottlieb's paper (reference 35). Gottlieb refer to polynomials of the "Laguerre Type" and gives the formula

$$\ell_\delta(n) = e^{-\lambda\delta} \sum_{\varepsilon=0}^{\delta} (1-e^{\lambda})^{\varepsilon} \binom{\delta}{\varepsilon}\binom{n}{\varepsilon} \qquad (304)$$

where δ=0, 1, 2, ... d-1 is the degree of the polynomial and n = 0, 1, 2, 3, ...N.

If we make the substitution of Equation (147)

$$\theta = e^{-\lambda} \qquad (305)$$

the term $1-e^{\lambda}$ of Equation (304) is

$$1 - e^{\lambda} = 1 - \frac{1}{\theta} = \frac{\theta-1}{\theta} \qquad (306)$$

and using Equation (306) in Equation (304) we obtain the expression used by Morrison reference (61) page (65) as

$$\ell_\delta(n) = \theta^{\delta} \sum_{\varepsilon=0}^{\delta} \left(\frac{\theta-1}{\theta}\right)^{\varepsilon} \binom{\delta}{\varepsilon}\binom{n}{\varepsilon} \qquad (307)$$

The first three of these polynomials are

$$\ell_0(n) = \theta^0 \qquad (308)$$

$$\ell_1(n) = \theta \sum_{\varepsilon=0}^{1} \left(\frac{\theta-1}{\theta}\right)^{\varepsilon} \binom{1}{\varepsilon}\binom{n}{\varepsilon} \qquad (309)$$

$$\ell_1(n) = \theta[1+(\frac{\theta-1}{\theta})n] \tag{310}$$

$$\ell_1(n) = (1,n) \begin{bmatrix} \theta \\ (\theta-1) \end{bmatrix} \tag{311}$$

and

$$\ell_2(n) = \theta^2 \sum_{\varepsilon=0}^{2} (\frac{\theta-1}{\theta})^{\varepsilon} \binom{2}{\varepsilon}\binom{n}{\varepsilon} \tag{312}$$

$$\ell_2(n) = \theta^2 \left[1+(\frac{\theta-1}{\theta})2n+(\frac{\theta-1}{\theta})^2 \frac{n(n-1)}{2} \right]$$

$$\ell_2(n) = (1,n,n^2) \begin{bmatrix} \theta^2 \\ \\ \frac{1}{2}(\theta-1)(3\theta+1) \\ \\ \frac{(\theta-1)^2}{2} \end{bmatrix} \tag{314}$$

Morrison gives the next two also as

$$\ell_3(n) = \theta^3 \; [1-3 \; (\frac{1-\theta}{\theta}) \; n+3 \; (\frac{1-\theta}{\theta})^2 \; \frac{n(n-1)}{2!} - (\frac{1-\theta}{\theta})^3 \; \frac{n(n-1)(n-2)}{3!} \;] \tag{315}$$

$$\ell_4(n) = \theta^4 \; [1-4 \; (\frac{1-\theta}{\theta}) \; n+6 \; (\frac{1-\theta}{\theta})^2 \; \frac{n(n-1)}{2!} - 4 \; (\frac{1-\theta}{\theta})^3 \; \frac{n(n-1)(n-2)}{3!} + (\frac{1-\theta}{\theta})^4 \; \frac{n(n-1)(n-2)(n-3)}{4!}] \tag{316}$$

Morrison states that this set of polynomials is seen to have an infinite number of elements in contrast to those of the previous section, which form a finite set for any given value of N. However one must take this statement with a grain of salt! First of all we observe by Equation (308), Equation (311) and Equation (314) that

$$[\ell_0(n), \ell_1(n), \ell_2(n)] = [1, n, n^2] \begin{bmatrix} 1 & \theta & \theta^2 \\ 0 & \theta-1 & \frac{(\theta-1)(3\theta+1)}{2} \\ 0 & 0 & \frac{(\theta-1)^2}{2} \end{bmatrix} \tag{317}$$

and that the 3×3 matrix of Equation (317) can be written as

$$\begin{bmatrix} 1 & \theta & \theta^2 \\ 0 & \theta-1 & \frac{1}{2}(\theta-1)(3\theta+1) \\ 0 & 0 & \frac{1}{2}(\theta-1)^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{\theta-1}{\theta} & \frac{(\theta-1)(3\theta+1)}{2\theta^2} \\ 0 & 0 & (\frac{\theta-1}{\theta})^2 \frac{1}{2} \end{bmatrix} D(\theta) \tag{318}$$

where

$$D(\theta) = \begin{bmatrix} \theta^0 & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta^2 \end{bmatrix} \tag{319}$$

The matrix of Equation (318) with unity coordinates is the Gram-Schmidt matrix of Equation (303) corresponding to the back to front (going to infinity) span metric. Thus, the set of fitting polynomials used by Morrison on page 500 of Reference [ ] seems rather nonsensical. The front to back of span metric should be used for then when one uses these approximating functions in estimation theory, the data before say time equals zero can be assumed to be zero, and one has a smoother. Brown in Reference [16] uses the metric of Equation (283) which generated the B_g matrix for the first three polynomials of Equation (299). Observe that the matrices differ in sign on some coordinates hence in applications to fitting functions in data processing one will get different results.

Furthermore if we partition these Laguerre-type polynomials as we did for the discrete Legendre polynomials of Equation (208) (though not too mathematically meaningful) we obtain from Equation (302)

$$\underset{(N+1)\times 3}{G} = \begin{bmatrix} \overset{0}{\langle 3)}g \\ \overset{1}{\langle 3)}g \\ \vdots \\ \overset{n}{\langle 3)}g \\ \vdots \\ \overset{N}{\langle 3)}g \end{bmatrix} = \begin{bmatrix} \theta^{0/2}(1,0,0) \\ \theta^{1/2}(1,1,1) \\ \theta^{2/2}(1,2,3) \\ \vdots \\ \theta^{n/2}(1,n,n^2) \\ \vdots \\ \theta^{N/2}(1,N,N^2) \end{bmatrix} \underset{3\times 3}{B_g} \qquad (320)$$

and equating elements

$$\overset{n}{\langle 3)}g = \theta^{n/2}(1,n,n^2)\ B_g \qquad (321)$$

If we use the concepts from the continuous Laguerre polynomials versus the continuous Laguerre functions, we can define the discrete polynomials as

$$[\ell_0(n)\ell_1(n),\ell_2(n)] = \langle 3)\ell(n) = (1,n,n^2) \qquad (322)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{\theta-1}{\theta} & \frac{(\theta-1)(3\theta+1)}{2} \\ 0 & 0 & (\frac{\theta-1}{\theta})^2\frac{1}{2} \end{bmatrix}$$

for n=0,1,2,$\cdots N\to\infty$, hence we do not need to interprete Equation (322) as an infinite set of polynomials, for one can deal with a set of three polynomials of degree two.

The connections between the Gram-Schmidt polynomials is obtained by Equation (318)

$$B_g = B_{mo} \; D^{-1}(\theta) \tag{323}$$

where B_{mo} is the matrix obtained from Morrison's polynomials of Equation (318); using Equation (323) in Equation (302)

$$\underset{(N+1)\times 3}{G} = N_w \; B_g = N_w \; B_{mo} \; D^{-1}(\theta) \tag{324}$$

and

$$\underset{3\times 3}{G^T G} = D^{-1}(\theta) \; B^T_{mo} \; N^T_w \; N_w \; B_{mo} \; D^{-1}(\theta) \tag{325}$$

and the metric given by Morrison page 66.

$$\sum_{n=0}^{\infty} \ell_\delta(n) \; \ell_\beta(n) \; \theta^n = \begin{bmatrix} 0; \delta \neq \beta \\ \\ \frac{\theta^\delta}{1-\theta}; \delta=\beta \end{bmatrix} \tag{326}$$

or in terms of the g elements of Equation (321), that is

$$(g_0, g_1, g_2) = \theta^{n/2} \; (1, n, n^2) \; B_g \tag{327}$$

$$\sum_{n=0}^{\infty} g_\delta(n) \; g_\beta(n) = \begin{bmatrix} 0; \delta \neq \beta \\ \\ \frac{\theta^\delta}{1-\theta}; \delta=\beta \end{bmatrix} \tag{328}$$

Specifically one should write Equation (328) as

$$\underset{N\to\infty}{\ell m} \; \sum_{n=0}^{N} g_\delta(n) \; g_\beta(n) = \sum_{n=0}^{\infty} \; g_\delta(n) \; g_\beta(n) \tag{329}$$

Using the first three elements of Equation (328) in Equation (325)

$$\underset{3\times3}{G^TG} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\theta & 0 \\ 0 & 0 & 1/\theta^2 \end{bmatrix} \begin{bmatrix} \frac{1}{1-\theta} & 0 & 0 \\ 0 & \frac{\theta}{1-\theta} & 0 \\ 0 & 0 & \frac{\theta^2}{1-\theta} \end{bmatrix} \qquad (330)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\theta & 0 \\ 0 & 0 & 1/\theta^2 \end{bmatrix}$$

$$G^TG = \begin{bmatrix} \frac{1}{1-\theta} & 0 & 0 \\ 0 & \frac{1}{\theta(1-\theta)} & 0 \\ 0 & 0 & \frac{1}{\theta^2(1-\theta)} \end{bmatrix} \qquad (331)$$

or the general term for a d-1 degree polynomial yields

$$\underset{d\times d}{G^TG} = \begin{bmatrix} \frac{1}{1-\theta} & 0 & & 0 \\ 0 & \frac{1}{\theta(1-\theta)} & & \\ & 0 & \ddots & \frac{1}{\theta^{d-1}(1-\theta)} \end{bmatrix} \qquad (332)$$

Note that the only difference in the forward and backward case is a sign change, hence the magnitudes of the vector are the same, hence Equation (332) holds for both cases.

Orthonormal Discrete Laguerre Polynomials. The two cases back to front and front to back orthogonal vectors will be normalized to obtain the unit magnitude sets. By Equation (305) Section (1) we have for the back of span to front case of Equation (302)

$$\underset{(N+1)\times 3}{S} = N_w B_g (G^T G)^{-1/2} = N_w B_s \tag{333}$$

where

$$B_s = B_g (G^T G)^{-1/2} \tag{334}$$

or by Equations (332) and (303)

$$B_s = \sqrt{1-\theta} \begin{bmatrix} \frac{1}{\sqrt{1-\theta}} & \sqrt{\theta} & \theta \\ 0 & \frac{\theta-1}{\sqrt{\theta}} & \frac{(\theta-1)(3\ +1)}{2\theta} \\ 0 & 0 & \frac{(\theta-1)^2}{2} \end{bmatrix} \tag{335}$$

The orthonormal vectors of Equation (333) for the case of front to back of span is the same as Equation (335) except for the sign changes in the positions of the matrix given by Equation (229) or

$$B_s(-1) = \begin{bmatrix} 1 & \sqrt{\theta} & \theta \\ 0 & \frac{-(\theta-1)}{\sqrt{\theta}} & \frac{-(\theta-1)(3\theta+1)}{2\theta^2} \\ 0 & 0 & \frac{(\theta-1)^2}{2\theta^2} \end{bmatrix} \tag{336}$$

One can obtain the same orthnormal vector case of Equation (335) by using the orthogonal vectors of Equation (318) with powers of θ on the first row of the matrix with the square root of the inverse of the diagonal metric of Equation (328)

Observe also that the Gram-Schmidt process applied to the vectors with first coordinates constrained to be the powers of θ yields the matrix corresponding to those polynomials of Morrison (or of Equation (328)). For example

$$g\,(N+1\rangle_{1} = [\; n\rangle_{0w} \;,\; n\rangle_{1w} \;]\begin{bmatrix}\theta \\ \lambda_1\end{bmatrix}_1 \tag{337}$$

and

$$g\,(N+1\rangle_{2} = \underset{(N+1)\times 3}{N_w}\begin{bmatrix}\theta^2 \\ \lambda_1 \\ \lambda_2\end{bmatrix}_2 \tag{338}$$

etc.

<u>Forward and Backward Exponentially Weighted Inverse Metrics.</u> The inverse of the metric matrix of Equation (168) can be obtained by Equation (348) Section (1) as

$$(N_w{}^T N_w)^{-1} = B_g\, M_{gg}^{-1} B_g^T \tag{339}$$

and by Equation (332) and Equation (303) is

$$(N_w^T N_w)^{-1} = \begin{bmatrix} (1-\theta)(1+\theta+\theta^2) & -\frac{3}{2}(1+\theta)(1-\theta)^2 & \frac{(1-\theta)^3}{2} \\ -\frac{3}{2}(1+\theta)(1-\theta)^2 & \frac{(1-\theta)^3(1+10\theta+9\theta^3)}{4\theta^2} & \frac{-(1-3\theta)(1-\theta)^4}{4\theta^2} \\ \frac{(1-\theta)^3}{2} & \frac{-(1-3\theta)(1-\theta)^4}{4\theta^2} & \frac{+(1-\theta)^5}{4\theta^2} \end{bmatrix} \tag{340}$$

where by Equation (168)

$$N_w^T N_w = \begin{bmatrix} \frac{1}{1-\theta} & \frac{\theta}{(1-\theta)^2} & \frac{(1+\theta)}{(1-\theta)^3} \\ \frac{\theta}{(1-\theta)^2} & \frac{\theta(1+\theta)}{(1-\theta)^3} & \frac{\theta(1+4\theta+\theta^2)}{(1-\theta)^4} \\ \frac{\theta(1+\theta)}{(1-\theta)^3} & \frac{\theta(1+4\theta+\theta^2)}{(1-\theta)^4} & \frac{\theta(1+11\theta+11\theta^2+\theta^3)}{(1-\theta)^5} \end{bmatrix} \qquad (341)$$

The inverse of the front to back metric of Equation (183) that is

$$\Gamma_w^T \Gamma_w = \begin{bmatrix} \frac{1}{1-\theta} & \frac{-\theta}{(1-\theta)^2} & \frac{(1+\theta)}{(1-\theta)^3} \\ \frac{-\theta}{(1-\theta)^2} & \frac{\theta(1+\theta)}{(1-\theta)^3} & \frac{-\theta(1+4\theta+\theta^2)}{(1-\theta)^4} \\ \frac{(1+\theta)}{(1-\theta)^3} & \frac{-\theta(1+4\theta+\theta^2)}{(1-\theta)^4} & \frac{\theta(1+11\theta+11\theta^2+\theta^3)}{(1-\theta)^5} \end{bmatrix} \qquad (342)$$

has as inverse by Equation (332) and Equation (299) in Equation (339)

$$(\Gamma_w^T \Gamma_w)^{-1} = \begin{bmatrix} (1-\theta)(1+\theta+\theta^2) & \frac{3}{2}(1+\theta)(1-\theta)^2 & \frac{(1-\theta)^3}{2} \\ \frac{3}{2}(1+\theta)(1-\theta)^2 & \frac{(1-\theta)^3(1+10\theta+9\theta^2)}{4\theta^2} & \frac{(1-\theta)^4(1+3\theta)}{4\theta^2} \\ \frac{(1-\theta)^3}{2} & \frac{(1-\theta)^4(1+3\theta)}{4\theta^2} & \frac{(1-\theta)^5}{4\theta^2} \end{bmatrix} \qquad (343)$$

The inverses of the two cases by G_g matrix of Equation (299) and Equation (303) can also be obtained via Equation (348) Section (1).

The Discrete Laguerre Polynomials Orthogonal over the Back to Front Span in Matrix Factor Form. The continuous Laguerre polynomials of Section (2) Equation (54) are expressed in terms of the binomial matrix, factorial matrix, etc., hence the following derivations will cast the discrete polynomials in similar forms. Consider Equation (307)

$$\ell_\delta(n) = \theta^\delta \sum_{\varepsilon=0}^{\delta} (-1)^\varepsilon b^\varepsilon \binom{\delta}{\varepsilon}\binom{n}{\varepsilon} \tag{344}$$

where δ=0,1,2···d-1 the polynomials degree and

$$\left(\frac{\theta-1}{\theta}\right)^\varepsilon = (-1)^\varepsilon \left(\frac{1-\theta}{\theta}\right)^\varepsilon = (-1)^\varepsilon\, b^\varepsilon \tag{345}$$

The first d polynomials can be written as

$$\begin{aligned} \ell_0(n) &= \theta^0 \sum_{\varepsilon=0}^{0} (\ ) \\ \ell_1(n) &= \theta^1 \sum_{\varepsilon=0}^{1} (\ ) \\ \ell_2(n) &= \theta^2 \sum_{\varepsilon=0}^{2} (\ ) \\ &\vdots \\ \ell_{d-1}(n) &= \theta^{d-1} \sum_{\varepsilon=0}^{d-1} (\ ) \end{aligned} \tag{346}$$

or as a row vector

$$(\ell_0(n),\ell_1(n),\cdots\ell_{d-1}(n)) = \left(\sum_{\varepsilon=0}^{0} (\ ),\ \sum_{\varepsilon=0}^{0},\cdots \sum_{\varepsilon=0}^{d-1} \right) D(\theta) \tag{347}$$

where the diagonal matrix is

$$D(\theta) = \begin{bmatrix} \theta^0 & & & 0 \\ & \theta^1 & & \\ & & \ddots & \\ 0 & & & \theta^{d-1} \end{bmatrix} \tag{348}$$

The summation terms can be written as inner-products by Equation (24) Appendix (B).

$$\binom{n}{\varepsilon} = \frac{n^{(\varepsilon)}}{\varepsilon!} \text{ for } \varepsilon > 0 \tag{349}$$

as

$$\sum_{\varepsilon=0}^{\delta} (-1)^{\delta}\ b^{\delta}\binom{\delta}{\varepsilon}\ \frac{n^{(\varepsilon)}}{\varepsilon!} \tag{350}$$

or

$$\sum_{\varepsilon=0}^{0} (\ ) = \frac{n^{(0)}}{0!}\ \binom{0}{0}\ b^0(-1)^0 \tag{351}$$

$$\sum_{\varepsilon=0}^{1} (\ ) = [n^{(0)}, n^{(1)}] \begin{bmatrix} \frac{(-1)^0 b^0}{0!} & 0 \\ 0 & (-1)^1 b^1 \end{bmatrix} \begin{bmatrix} \binom{1}{0} \\ \binom{1}{1} \end{bmatrix} \tag{352}$$

$$\sum_{\varepsilon=0}^{2} (\ ) = [n^{(0)}, n^{(1)}, n^{(2)}] \begin{bmatrix} \frac{(-1)^0 b^0}{0!} & 0 & 0 \\ 0 & \frac{(-1)^1 b^1}{1!} & 0 \\ 0 & 0 & \frac{(-1)^2 b^2}{2!} \end{bmatrix} \begin{bmatrix} \binom{2}{0} \\ \binom{2}{1} \\ \binom{2}{2} \end{bmatrix} \tag{353}$$

$$\sum_{\varepsilon=0}^{d-1}(\ ) = [n^{(0)},n^{(1)},n^{(2)},\cdots n^{(d-1)}]\begin{bmatrix}\frac{(-1)^0b^0}{0!} & & \\ & \ddots & \\ & & \frac{(-1)^{d-1}b^{d-1}}{(d-1)!}\end{bmatrix}\begin{bmatrix}\binom{d-1}{0}\\ \binom{d-1}{1}\\ \vdots\\ \binom{d-1}{d-1}\end{bmatrix} \quad (354)$$

or

$$(\ell_0(n),\ell_1(n),\cdots\ell_{d-1}(n)) = (n^{(0)},n^{(1)},\cdots n^{(d-1)}\ I(-1)\ \mathbb{F}^{-1}\ D(b)\ BD(\theta) \quad (355)$$

or

$$\langle d)\ell(n) = \langle d)n^{(\ )}I(-1)\ \mathbb{F}^{-1}\ D(b)\ BD(\theta) \quad (356)$$

where

$$D(\theta) = \begin{bmatrix}(\frac{1-\theta}{\theta})^0 & & & & \\ & (\frac{1-\theta}{\theta})^1 & & 0 & \\ & 0 & (\frac{1-\theta}{\theta})^2 & & \\ & & & \ddots & \\ & & & & (\frac{1-\theta}{\theta})^{d-1}\end{bmatrix} \quad (357)$$

and B is the binomial matrix

$$B = \begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \cdots & \binom{d-1}{0} \\ & \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & & \vdots \\ & & \binom{2}{2} & \binom{3}{2} & & \\ & & & \binom{3}{3} & & \\ & & & & & \binom{d-1}{d-1} \end{bmatrix} \tag{358}$$

The Discrete Laguerre Polynomials Orthogonal Over the Front to Back Span in Matric Factor Form. By Equation (298) and Equation (302) we see that the B_g matrices for these two cases differin signs in alternating rows, hence by inspection it is clear that positive axis span is carried into the negative axis span via

$$B(-1) = \begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \end{bmatrix} B(+1) \tag{359}$$

or element-wise we have by analogy with Equation (307)

$$\ell_\delta(n) = \theta^\delta \sum_{\varepsilon=0}^{\delta} (-1)^\varepsilon \left(\frac{\theta-1}{\theta}\right)^\varepsilon \binom{\delta}{\varepsilon} \binom{n}{\varepsilon} \tag{360}$$

The row-vector expression of Equation (360) is by analogy with Equation (355)

$$(\ell_0(n), \ell_1(n), \cdots \ell_{d-1}(n)) = [1, n^{(1)}, n^{(2)}, \cdots n^{(d-1)}] \, I(-1)$$

$$\ddagger^{-1} \text{ dia } \left(\frac{\theta-1}{\theta}\right) BD(\theta) \tag{361}$$

$$\text{dia}\left(\frac{\theta-1}{\theta}\right) = \begin{bmatrix} \left(\frac{\theta-1}{\theta}\right)^0 & & & & \\ & \left(\frac{\theta-1}{\theta}\right) & & & 0 \\ & & \left(\frac{\theta-1}{\theta}\right)^2 & & \\ & 0 & & \ddots & \\ & & & & \left(\frac{\theta-1}{\theta}\right)^{d-1} \end{bmatrix} \tag{363}$$

<u>Discrete Gram-Schmidt on the Matrix T</u>. The discrete orthogonal polynomials of the previous cases are with respect to the non-dimensionalized time spans. By Equation (124).

$$\underset{(N+1)\times 3}{T} = \underset{(N+1)\times 3}{N} \; T_u \underset{3\times 3}{(i)} D(h_0) \tag{363}$$

By Equation (205)

$$\underset{(N+1)\times 3}{G_N} \; B_{gN}^{-1} = N \tag{364}$$

Using Equation (304) in Equation (363)

$$\underset{(N+1)\times 3}{T} = \underset{(N+1)\times 3}{G_N} \; B_{gN}^{-1} \, T_u(i) \; D(h_0) \tag{365}$$

$$\underset{(N+1)\times 3}{T} = \underset{(N+1)\times 3}{G_N} \; \underset{3\times 3}{B_T} \tag{366}$$

where B_T is upper triangular since

$$B_T = B_{gN}^{-1} \, T_u(i) \; D(h_0) \qquad (367$$

and inverse of an upper triangular matrix is upper triangular and also $T_n(i)$ is upper triangular and products of upper-triangular matrices are upper triangular. Inverting Equation (366)

$$G_N = \underset{(N+1)\times 3}{T} \; B_T^{-1} = T \; D^{-1}(h_0) \; T_u^{-1}(i) \; B_{gN} \tag{368}$$

By Equation (368) we see that an orthogonalization procedure on the column vectors of T using the columns of T as a base for the sub-space of dimension three, yields the three column vectors of G with coordinates given by B_T^{-1} in the T base. Hence one can obtain the discrete orthogonal polynomials in many forms, and as shown by Equation (367) the upper-triangular matrix is a function of the fixed time interval h_0, and the index i.

Section 6 SOME GENERALIZATIONS OF THE DISCRETE DIFFERENCES OF POLYNOMIALS AND THE BASE FITTING FUNCTIONS.

a. Generalities

In this section discrete first differences and higher differences of sequences of x(i) and of the fitting function f(i) will be developed.

Corresponding to the three different indexing notations of Figure (1) Section (4) we have

$$\begin{pmatrix} x(i,0) \\ x(i,1) \\ x(i,2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x(i,N) \end{pmatrix} \qquad \begin{pmatrix} x(k-n) \\ \cdot \\ \cdot \\ \cdot \\ x(k-1) \\ x(k,0)) \\ x(k,1) \\ \cdot \\ \cdot \\ \cdot \\ x(k,n) \end{pmatrix} \qquad \begin{pmatrix} x(1,N) \\ x(1,N)-1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x(1,o) \end{pmatrix} \tag{1}$$

and the corresponding relations on the fitting functions of Eq. (124), Eq. (129) and Eq. (135).

$$\begin{pmatrix} t(i,0) \\ t(i,1) \\ \cdot \\ \cdot \\ \cdot \\ t(i,N) \end{pmatrix} \qquad \begin{pmatrix} t(k-n) \\ \cdot \\ \cdot \\ \cdot \\ t(k,0) \\ \cdot \\ \cdot \\ \cdot \\ t(k,n) \end{pmatrix} \qquad \begin{pmatrix} t(1,N) \\ t(1,N-1) \\ \cdot \\ \cdot \\ \cdot \\ t(1,o) \end{pmatrix} \tag{2}$$

One can use selector matrices and evolve an "operational matrix" calculus in the same manner authors introduce operators, for example for the first vector sequence of Eq. (1)

$$\begin{aligned} \langle e_1 x(i)(N+1\rangle &= x(i,0) \\ \langle e_1 S_{uo} x(i)\rangle &= x(i,1) \\ &\vdots \\ \langle e_1 S_{uo}^{n} x(i)\rangle &= x(i,n) \\ &\vdots \\ \langle e_1 S_{uo}^{N} x(i)\rangle &= x(i,N) \end{aligned} \tag{3}$$

where the select vector

$$\langle \mathbf{e}_1 = \underset{1\text{x}(N+1)}{(1,\ 0,\ 0\ .\ .\ .\ 0)} \tag{4}$$

and the shift matrix is for example 3 x 3

$$\underset{3\text{x}3}{S_{uo}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{5}$$

For the midpoint span sequence of Eq. (1) one can map-down via the vector

$$\langle (N+1)\mathbf{e}_c = (0,\ 0\ .\ .\ .\ 1,\ 0,\ 0\ .\ .\ .) = (\langle M)0,\ 1,\ \langle M)0) \tag{6}$$

or

$$\begin{aligned} \langle \mathbf{e}_c x(k)\rangle &= x(k,0) \\ \langle \mathbf{e}_c S_{uo} x(k)\rangle &= x(k,1) \\ \langle \mathbf{e}_c S_{do} x(k)\rangle &= x(k,-1) \end{aligned} \tag{7}$$

where the shift-down matrix is (e.g.)

$$S_{do} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = S_{uo}^{T} \tag{8}$$

If we write Eq. (3) in terms of power of the shift operator we obtain the obvious identity

$$\begin{pmatrix} x(i,0) \\ x(i,1) \\ x(i,2) \\ \cdot \\ \cdot \\ \cdot \\ x(i,n) \\ \cdot \\ \cdot \\ \cdot \\ x(i,N) \end{pmatrix} \qquad \begin{pmatrix} \langle \mathbf{e}_1 \\ \langle \mathbf{e}_1 S_{uo} \\ \langle \mathbf{e}_1 S_{uo}^2 \\ \cdot \\ \cdot \\ \cdot \\ \langle \mathbf{e}_1 S_{uo}^n \\ \cdot \\ \cdot \\ \cdot \\ \langle \mathbf{e}_1 S_{uo}^N \end{pmatrix} \qquad x(i)\rangle \tag{9}$$

where N + 1

$$\underset{(N+1)(N+1)}{S_{uo}} = \underset{(N+1)(N+1)}{[0]} \tag{10}$$

Consider some vectors and matrices generated by moving fixed-span procedures, for example the sequence of variables for fixed N, with the index i varying

$$\langle x(i) = (x(i,0) \; . \; . \; . \; x(i,1) \; . \; . \; . \; x(i,n) \; . \; . \; . \; (i,N) \qquad (11)$$

and

$$\begin{aligned} \langle x(i+1) &= (x(i,1),\ x(i,2) \; . \; . \; . \; x(i,N) \; . \; . \; . \; x(iN+1) \\ &= (x(i+1,\ 0) \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; x(i+1,N) \end{aligned} \qquad (12)$$

or

$$\langle x(i+1) = \langle x(i)\ S^{T}_{uo} + x(i+1,N) \underset{N+1}{\langle e} \qquad (13)$$

where the shift matrix is

$$S^{T}_{uo} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \\ 0 & 1 & & \\ 0 & 0 & & \\ & & 1 & 0 \end{bmatrix} \qquad (14)$$

and

$$\underset{N+1}{\langle e} = \underset{1 \text{ x } (N+1)}{(0,\ 0 \; . \; . \; . \; 1)} \qquad (15)$$

Depending upon how one orders or indexes and packages the variables, various matrix types appear, for example for a span of N + 1, N = 3 data points packaged as a column

$$\begin{pmatrix} x(i,0) \\ x(i,1) \\ x(i,2) \\ x(i,3) \end{pmatrix} = x(i)(4\rangle \qquad (16)$$

we have shifting up and out the first point and shifting in the new point, we have

$$x(i+1)(4\rangle = S_{uo} x(i)(4\rangle + (4\rangle_{4} x(i,3) \qquad (17)$$

or

$$x(i+1)(4\rangle = \begin{pmatrix} x(i,1) \\ x(i,2) \\ x(i,3) \\ x(i,4) \end{pmatrix} \qquad (18)$$

etc., for a row of column vectors

$$\begin{bmatrix} x(i,0) & x(i,1) & x(i,2) & x(i,3) \\ x(i,1) & x(i,2) & x(i,3) & x(i,4) \\ x(i,2) & x(i,3) & x(i,4) & x(i,5) \\ x(i,3) & x(i,4) & x(i,5) & x(i,6) \end{bmatrix} = \underset{4x4}{X} \qquad (19)$$

which is a Hankel matrix.

If we package the "data" or variables with the "real-time" or most advanced time point at the top and "shift down" we have for the array of spans

$$\begin{bmatrix} x(i,3) & x(i,4) & x(i,5) & x(i,6) \\ x(i,2) & x(i,3) & x(i,4) & x(i,5) \\ x(i,1) & x(i,2) & x(i,3) & x(i,4) \\ x(i,0) & x(i,1) & x(i,2) & x(i,3) \end{bmatrix} \qquad (20)$$

or a Teoplitz matrix.

Minimal Set of First Differenced

Consider next the sequence under U

$$[x(i,0),\ x(i,1),\ x(i,2)\ .\ .\ .\ x(i,N)] \underset{(N+1)\ x\ N}{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & \\ 0 & 1 & -1 & \\ 0 & 0 & 1 & \\ \cdot & \cdot & 0 & -1 \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ 0 & 0 & & 1 \end{bmatrix}}$$

$$= \langle N)\ \Delta^{(1)}x(i) = (\Delta^{(1)}x(i,0),\ \Delta x^{(1)}(i,1)\ .\ .\ .\ \Delta x^{(1)}(i,N-1) \qquad (21)$$

or

$$\langle x(i)\ \underset{(N+1)N}{U} = \langle \Delta x(i) \qquad (22)$$

where

$$\underset{(N+1)N}{U} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & \\ 0 & 0 & 1 & \\ \cdot & \cdot & 0 & \\ \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \qquad (23)$$

The same relations hold when x are vectors $\tilde{x}$ and the geometry is easier to construct, for example see Figure (1).

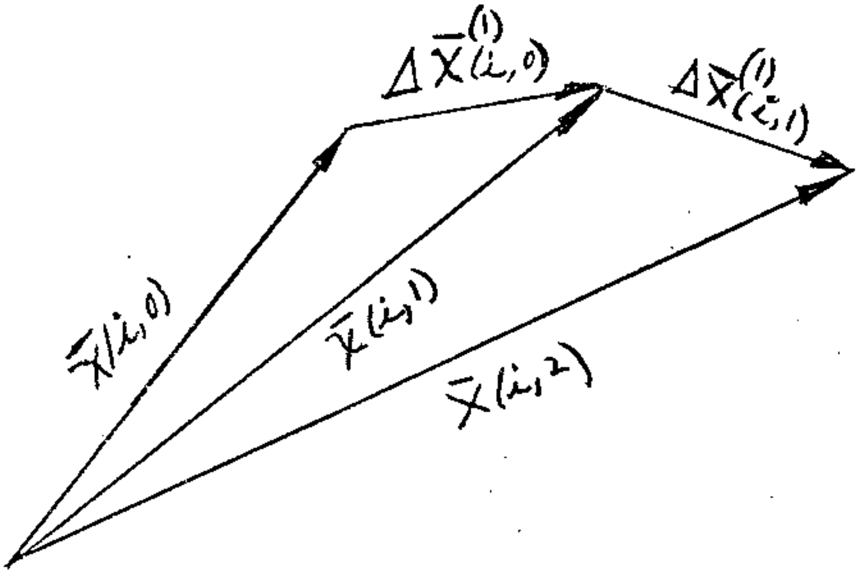

Figure 1 First Forward Difference Vectors

The first differences

$$\Delta^{(1)}x(i,n) = x(i,n+1) - x(i,n) \qquad (24)$$

are referred to as the first forward differences. The same methods developed herein can also be applied to the so called backward and central difference formulas, but will not be done here. The rather standard difference-operator techniques used in Morrison's text and others is not "in the spirit" of the "matrix of the operator" state space approach and is considered very tedious by the author of this report.

Return now to Eq. (9) substituted into the transpose of Eq. (21)

$$\underset{N \times (N+1)}{\begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & & \\ 0 & -1 & 1 & 0 & \cdots & & \\ & & & & & & \\ & & & & & & \\ & & & & & -1 & 1 \end{bmatrix}} \begin{matrix} \langle e_1 x(i)\rangle \\ \langle e_1 S_{uo} x(i)\rangle \\ \langle e_1 S_{uo}^2 x(i)\rangle \\ \vdots \\ \langle e_1 S_{uo}^N x(i)\rangle \end{matrix} = U^T x(i)\rangle \qquad (25)$$

$$\begin{bmatrix} \left\langle \mathbf{e}\,[S_{uo}-I]x(i)\right\rangle_1 \\ \left\langle \mathbf{e}\,S_{uo}[S_{uo}-I]x(i)\right\rangle_1 \\ \left\langle \mathbf{e}\,S_{uo}^{2}[S_{uo}-I]x(i)\right\rangle_1 \\ \vdots \\ \left\langle \mathbf{e}\,S_{uo}^{N-1}[S_{uo}-I]x(i)\right\rangle_1 \end{bmatrix} = \begin{bmatrix} \Delta^{(1)}x(i,0) \\ \Delta^{(1)}x(i,1) \\ \Delta^{(1)}x(i,2) \\ \vdots \\ \Delta^{(1)}x(i,N-1) \end{bmatrix} \tag{26}$$

where

$$S_{uo} - I = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & & \\ 0 & -1 & 1 & 0 & 0 & & \\ 0 & 0 & -1 & 1 & 0 & & \\ & & & & & -1 & 1 \\ & & & & & & -1 \end{pmatrix} \tag{27}$$

We see that this matrix is the same as Eq.(110) of Section (8) for b = 1 and its inverse is, for example 4 x 4,

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{28}$$

and the negative of the above matrices yields

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{29}$$

If we use the "so called" push-down sequence (column vector where the "real-time" point is at the top we have the connection via the "linear convolution" matrix.

$$\begin{pmatrix} x(i,N) \\ x(i,N-1) \\ \vdots \\ x(i,0) \end{pmatrix} \begin{pmatrix} \bigcirc & \\ & \\ & \bigcirc \end{pmatrix} \begin{pmatrix} x(i,0) \\ x(i,1) \\ x(i,1) \\ \vdots \\ x(i,n) \\ \vdots \\ x(i,N) \end{pmatrix} \tag{30}$$

and the matrix $-U^T$ where

$$-U^T = \begin{pmatrix} 1 & -1 & 0 & \cdots & & \\ 0 & 1 & -1 & & & \\ 0 & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & & & & 1 & -1 \end{pmatrix} \tag{31}$$

$$\underset{N \times (N+1)}{\begin{pmatrix} 1 & -1 & 0 & 0 & & & \\ 0 & 1 & -1 & 0 & & & \\ 0 & 0 & 1 & -1 & & & \\ & & & & & & \\ & & & & 0 & 1 & -1 \end{pmatrix}} \begin{pmatrix} x(i,N) \\ x(i,N-1) \\ \cdot \\ \cdot \\ \cdot \\ x(i,0) \end{pmatrix} = \underset{N \times 1}{\begin{bmatrix} x(i,N) - x(i,N-1) \\ \cdot \\ \cdot \\ \cdot \\ x(i,n) - x(i,n-1) \\ \cdot \\ \cdot \\ \cdot \\ x(i,1) - x(i,0) \end{bmatrix}} \tag{32}$$

$$= \begin{bmatrix} \Delta^{(1)}x(i,N-1) \\ \Delta^{(1)}x(i,N-2) \\ \cdot \\ \cdot \\ \cdot \\ \Delta^{(1)}x(i,n) \\ \cdot \\ \cdot \\ \cdot \\ \Delta^{(1)}x(i,0) \end{bmatrix} = L_c \Delta x^{(1)}\Big\rangle \tag{33}$$

If we add the x(i,0) element to the bottom of the vector of Eq. (33) we have a full rank matrix.

$$\begin{pmatrix} \Delta^{(1)}x(i,N-1) \\ \cdot \\ \cdot \\ \cdot \\ \Delta^{(1)}x(i,0) \\ x(i,0) \end{pmatrix} = \begin{bmatrix} 1 & -1 & & \bigcirc & \\ & 1 & -1 & & \\ & & & & \\ \bigcirc & & & 1 & -1 \\ & & & & 1 \end{bmatrix} \begin{pmatrix} x(i,N) \\ x(i,N-1) \\ \cdot \\ \cdot \\ \cdot \\ x(i,0) \end{pmatrix} \tag{34}$$

The inverse of the matrix of Eq. (34) is given by Eq. (29) and we have

$$\begin{pmatrix} x(i,N) \\ x(i,N-1) \\ \cdot \\ \cdot \\ \cdot \\ x(i,n) \\ \cdot \\ \cdot \\ \cdot \\ x(i,0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ & & & & & \\ & & & 1 & \cdots & \\ & \bigcirc & & & & \\ & & & & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta^{(1)}x(i,N-1) \\ \Delta^{(1)}x(i,N-2) \\ \cdot \\ \cdot \\ \cdot \\ \Delta^{(1)}x(i,0) \\ x(i,o) \end{pmatrix} \tag{35}$$

The inverse can also be computed step-wise as,

$$(I-S_{uo})^{-1} = (I-S_{uo})^T \; [(I-S_{uo})(I-S_{uo}{}^T]^{-1} \tag{36}$$

or the symmetric Gram matrix of Eq. (36) is for a 4 x 4 case

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \tag{37}$$

The inverse of the n x n matrix of Eq. (37) is given by Charters and Stapleman reference (101) as

$$\begin{pmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & 0 & & \\ 0 & -1 & 2 & -1 & & \\ & 0 & & & & \\ & & \vdots & & & \\ & & & & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & 3 & \cdots & 3 \\ 1 & 2 & 3 & 4 & \cdots & 4 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ 1 & 2 & 3 & 4 & & n \end{pmatrix} \tag{38}$$

which is the <u>Frankel matrix</u> of reference (90). Using Eq. (38) in Eq. (36) one obtains for the 4 x 4 case example (one verifies)

$$\begin{aligned} (I-S_{uo})^{-1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{39}$$

Consider the first difference vector of Eq. (21)

$$\langle N)\Delta x^{(1)}(i) = \langle N+1)\ x(i)\ \underset{(N+1)xN}{U} \tag{40}$$

Multiply on right of Eq. (40) with U^{*}

$$\langle \Delta x^{(1)} U^{*} = \langle x(i)\ UU^{*} = \langle x(i) \underset{(N+1)(N+1)}{P_{uu*}} \tag{41}$$

where the psuedo inverse of U is

$$\underset{N(N+1)}{U^{*}} = \underset{NxN}{(U^TU)^{-1}} \underset{Nx(N+1)}{U^T} \tag{42}$$

and by Eq. (21)

$$\underset{N\ N}{U^TU} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \dots & & \\ 0 & -1 & 1 & 0 & 0 & & \\ 0 & 0 & -1 & 1 \dots & & & \\ & & & & & & \\ 0 & 0 & \dots & \dots & \dots & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & & \\ 0 & 1 & -1 & & \\ 0 & 0 & 1 & & \\ \dots & \dots & 0 & & -1 \\ \dots & \dots & \dots & \dots & 1 \end{pmatrix} \tag{43}$$

or

$$U^TU = \underset{N\ x\ N}{\begin{pmatrix} 2 & -1 & 0 & 0 & & \\ -1 & 2 & -1 & 0 & & \\ 0 & -1 & 2 & -1 & & \\ 0 & 0 & -1 & 2 & & \\ & & & & & -1 \\ & & & & -1 & 2 \end{pmatrix}} \tag{44}$$

The matrix of Eq. (38) is called the Frank-matrix by Westlake in reference (90)

It is clear that for the (N+1)(N+1) case of Eq. (36) one replaces n = N+1 in Eq. (36).

If we call the matrix of Eq. (34) A, that is

$$A = I - S_{uo} \tag{45}$$

then for an n x n matrix

$$\underset{nxn}{U^TU} = AA^T + e\rangle_n\langle e \tag{46}$$

and using the <u>Householder inversion</u> lemma for inverting a matrix plus a rank one dyad, we have

$$(U^TU)^{-1} = (AA^T)^{-1}\left[I - \frac{e\rangle_n^{n}\langle e\ (AA^T)^{-1}}{1 + {}_n\langle e\,(AA^T)^{-1}\, e\rangle_n}\right] \tag{47}$$

Consider the elements in the parenthesis of Eq. (47), by Eq. (38)

$$ {}_n\langle e\,(AA^T)^{-1} = (0,\ 0,\ 0\ .\ .\ .\ 1)\begin{pmatrix} 1 & 1 & \cdots & & 1 \\ 1 & 2 & \cdots & & 2 \\ \cdot & \cdot & & & 3 \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ 1 & 2 & 3 & \cdots & n \end{pmatrix} \qquad (48) $$

$$ = (1,\ 2,\ 3\ .\ .\ .\ n) = \langle n)c $$

where n)c is the vector of counting numbers. We have also

$$ {}_n\langle e\,(AA^T)^{-1}\,e\rangle_n = \langle c\ e\rangle_n = n \qquad (49) $$

Using Eq. (48) and Eq. (49) in Eq. (47) we have the factors

$$ (U^TU)^{-1} = (AA^T)^{-1}[I - \frac{e\rangle_n\langle c}{1+n}] \qquad (50) $$

The matrix in brackets is

$$ I - \frac{e\rangle_n\langle c}{1+n} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ & & & \\ \frac{1}{1+n} & \frac{-2}{1+n} & \frac{-3}{1+n} & 1-\frac{n}{1+n} \end{pmatrix} \qquad (51) $$

Transposing Eq. (48) and using it in Eq. (50) we also have

$$ (U^TU)^{-1} = (AA^T)^{-1} - \frac{c\rangle\langle c}{1+n} \qquad (52) $$

The dyad is

$$ \frac{c\rangle\langle c}{1+n} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ \cdot \\ \cdot \\ \cdot \\ n \end{pmatrix} (1,\ 2,\ 3,\ 4\ .\ .\ .\ n)\,\frac{1}{1+n} \qquad (53) $$

or

$$ \frac{c\rangle\langle c}{1+n} = \begin{bmatrix} 1 & 1 & 3 & 4 & \cdots & n \\ 2 & 4 & 6 & 8 & \cdots & 2n \\ 3 & 4 & 9 & 12 & \cdots & 3n \\ 4 & 6 & 12 & 16 & \cdots & 4n \\ \cdot & 8 & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ n & 2n & 3n & & & n^2 \end{bmatrix} \quad \frac{1}{n+1} \qquad (54) $$

Using Eq. (52) and the Frank matrix of Eq. (36) in Eq. (50) we obtain

$$(U^TU)^{-1} = \begin{bmatrix} \frac{n}{n+1} & \frac{n-1}{n+1} & \frac{n-2}{n+1} & \frac{n-3}{n+1} \cdots & \frac{1}{n+1} \\ \frac{n-1}{n+1} & \frac{2(n-1)}{n+1} & \frac{2(n-2)}{n+1} & \frac{2(n-3)}{n+1} & \\ \frac{n-2}{n+1} & \frac{2(n-2)}{n+1} & \frac{3(n-2)}{n+1} & \frac{3(n-3)}{n+1} & \\ \frac{n-3}{n+1} & \frac{2(n-3)}{n+1} & \frac{3(n-3)}{n+1} & \frac{4(n-3)}{n+1} & \\ \vdots & & & & \\ \frac{1}{n+1} & & & & \frac{n}{n+1} \end{bmatrix} \tag{55}$$

Westlake gives the inverses of the above matrix in reference (90) page 140 as

$$(U^TU)^{-1} = M^{-1}_{U^TU} = [m^*_{ij}] \tag{56}$$

where

$$m^*_{ij} = \begin{pmatrix} \frac{i(n-j+1)}{n+1} & \text{for } i \leq j \\ m^*_{ji} & \text{for } i > j \end{pmatrix} \tag{57}$$

Note that the Frank matrix can be written as

$$(U^TU)^{-1} = \begin{bmatrix} \langle 1 \\ 2\langle 1 - \langle e_1 \\ 3\langle 1 - 2\langle e_1 - \langle e_2 \\ 4\langle 1 - 3\langle e_1 - 2\langle e_2 - \langle e_3 \\ \vdots \\ \langle c \end{bmatrix} \tag{58}$$

For example the 4 x 4 case one has by Eq. (55)

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{6}{5} & \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{bmatrix} \tag{59}$$

Ortega in reference (65) gives the eigenvalues of U^TU as

$$\lambda_k = 2 - 2 \cos \frac{k\Pi}{n+1}, \; k = 1 \; . \; . \; . \; n \tag{60}$$

with the corresponding eigenvectors

$$\left(\sin \frac{k\Pi}{n+1}, \; \sin \frac{2k\Pi}{n+1}, \; . \; . \; . \; \sin \frac{nk\Pi}{n+1}\right)^T, \; k = 1 \; . \; . \; . \; n \tag{61}$$

For example the 3 x 3 case yields

$$\det \begin{pmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{pmatrix} = (\lambda^2 - 4\lambda + 2)(2 - \lambda) \tag{62}$$

$$= -(\lambda - 2 + \sqrt{2})(\lambda - 2 - \sqrt{2})(\lambda - 2) \tag{63}$$

also one obtains

$$\det(U^TU - \lambda I) = - \lambda + 6\lambda^2 - 10\lambda + 4 = 0 \tag{64}$$

or

$$\lambda^3 = (4, \; -10, \; 6) \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix} \tag{65}$$

The <u>companion</u> matrix is

$$A_c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -10 & 6 \end{pmatrix} \tag{66}$$

The three eigenvalues of Eq. (63) are

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 2-\sqrt{2} & 0 & 0 \\ 0 & 2+\sqrt{2} & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{67}$$

The three eigenvectors of Eq. (61) are for example

$$\begin{pmatrix} \sin \frac{\Pi}{4} \\ \sin \frac{2\Pi}{4} \\ \sin \frac{3\Pi}{4} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{pmatrix} = b\rangle_1 \tag{68}$$

and the matrix eigenverctors is

$$B = [b\rangle_1,\ b\rangle_2,\ b\rangle_3] = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & -1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \tag{69}$$

Since U^TU is symmetric its eighnvectors are orthogonal, that is

$$BB^T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{70}$$

thus the unit magnitude orthogonal eighnvectors are

$$B_u = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix} \tag{71}$$

and

$$B_u^{-1} = B_u^T \tag{72}$$

and we have

$$(U^TU)B = B\Lambda \tag{73}$$

or

$$B^{-1}(U^TU)B = \Lambda \tag{74}$$

The Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2-\sqrt{2} & 2+\sqrt{2} & 2 \\ 6-4\sqrt{2} & 6+4\sqrt{2} & 4 \end{pmatrix} \tag{75}$$

relates the companion matrix via the relation.

$$A_c = V(U^TU)V^{-1} \tag{76}$$

The inverse of U^TU for the 3 x 3 case by Eq. (55) is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}^{-1} = \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix} \tag{77}$$

By Eq. (73) we also have

$$(U^TU)^{-1} = B\Lambda^{-1}B^{-1} \tag{78}$$

Using Eq. (77) in Eq. (42) we have

$$\underset{3x4}{U^*} = \tfrac{1}{4} \begin{pmatrix} -3 & 1 & 1 & 1 \\ -2 & -2 & 2 & 2 \\ -1 & -1 & -1 & 3 \end{pmatrix} \tag{79}$$

and the orthogonal projector is

$$\underset{4x4}{UU^*} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \tfrac{1}{4} \tag{80}$$

$$= [4I - 1(4\rangle\langle 4)1]\tfrac{1}{4} \tag{81}$$

$$= \underset{4x4}{I} - \frac{(4\rangle\langle 4)1}{4} \tag{82}$$

$$= \tilde{P} = P_{UU^*} \tag{83}$$

the orthogonal conplement projector onto the 1(4 vector. Using Eq. (80) in Eq. (41)

$$\langle \Delta x^{(1)}(i)U^* = \langle x(i)UU^* = \langle x(i)\tilde{P} \tag{84}$$

or

$$\langle \Delta x^{(1)}(i) = \langle x(i) - \langle x(i)P \tag{85}$$

where

$$P = \frac{1\rangle\langle 1}{4} \tag{86}$$

Thus if one is given the three difference vectors one can solve for the x(i) up to the component lost along 1(4 .

For the (N+1)xN case it is obvious that N+1)1 lies in the null-space of U, that is

$$\langle N+1)1 \underset{(N+1)xN}{U} = \langle N)0 \tag{87}$$

or in open-form

$$(1,\ 1,\ 1\ .\ .\ .) \begin{pmatrix} -1 & 0 & & 0 \\ 1 & -1 & & \\ 0 & 1 & & \\ 0 & 0 & & \\ \cdot & \cdot & & -1 \\ \cdot & \cdot & & 1 \end{pmatrix} = \langle 0 \tag{88}$$

The rank of U satisfies

$$\rho \underset{(N+1)N}{U} \quad = \quad \underset{NxN}{\rho(U^TU)} = N \tag{89}$$

since by Eq. (55) it is invertible.

Hence Eq. (81) can be generalized to

$$\underset{(N+1)(N+1)}{UU^*} \quad = \quad \underset{(N+1)(N+1)}{I} \quad - \quad 1\xcancel{\frac{(N+1 \quad N+1)}{N+1}}1 \tag{90}$$

or

$$UU^* = \tilde{P} \tag{91}$$

By Eq. (74) and (76)

$$\underset{NxN}{U^TU} = B\Lambda B^{-1} = V^{-1}A_cV \tag{92}$$

or structure wise

$$UU^* = U(U^TU)^{-1}U^T \tag{93}$$

$$= UV^{-1}A_c^{-1}VU^T \tag{94}$$

$$= UB\Lambda^{-1}B^{-1}U^T \tag{95}$$

The inverse of a companion matrix is given by Brand reference (14) as, for example,

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -d & -c & -b & -a \end{pmatrix}^{-1} \quad = \quad \begin{pmatrix} -c/d & -b/d & -a/d & -1/d \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{96}$$

and

$$\det\,(A_c^{-1} - \lambda I) = \lambda^{\psi} + \frac{c}{d}\lambda^3 + \frac{b}{d}\lambda^2 + \frac{a}{d}\lambda + \frac{1}{d} \tag{97}$$

In conjunction with these hints of structural approaches one should consider the Lancos-Decomposition of U as

$$\underset{(N+1)xN}{U} \quad = \quad \underset{(N+1)xN}{U_\ell} \quad \underset{(NxN)}{\Lambda_\sigma} \quad \underset{(NxN)}{V_\ell^T} \tag{98}$$

where

$$U_\ell^TU_\ell = \underset{NxN}{T} = V_\ell^TV_\ell \tag{99}$$

then

$$\underset{N\text{x}N}{U^T U} = V_\ell \Lambda_\sigma V_\ell^T = B \Lambda B^{-1} \tag{100}$$

or in terms of the unit magnitude eigenvector matrix

$$U^T U = V_\ell \Lambda_\sigma^2 V_\ell^T = B_u \Lambda B_u^T \tag{101}$$

where

$$\Lambda_\sigma = \Lambda^{\frac{1}{2}} \tag{102}$$

and positive square-roots are taken, hence

$$V_\ell = B_u \tag{103}$$

We have also

$$\underset{(N+1)(N+1)}{UU^T} = V_\ell \Lambda_\sigma^2 V_\ell^T \tag{104}$$

and

$$(UU^T)^* = V_\ell^T \Lambda_\sigma^{-2} V_\ell^T \tag{105}$$

and an alternate expression for U^* of Eq. (42) is

$$\underset{N(N+1)}{U^*} = \underset{N\text{x}(N+1)}{U^T} \; \underset{(N+1)(N+1)}{(UU^T)^*} \tag{106}$$

Further structural properties can be developed but will not be pursued further here.

The outer-Grammian matrix for the 4 x 4 case of Eq. (106) is

$$\underset{4\text{x}4}{UU^T} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \tag{107}$$

Finally by Eq. (55) in the general case of Eq. (42)

$$\underset{N(N+1)}{U^*} = (U^T U)^{-1} U^T \tag{108}$$

we have

$$\underset{Nx(N+1)}{U^*} = \frac{1}{N+1} \begin{bmatrix} N & N-1 & N-2 & N-3 & \dots & 1 \\ N-1 & 2(N-1) & 2(N-2) & 2(N-3) & & \\ N-2 & 2(N-2) & 3(N-2) & 3(N-3) & & \\ N-3 & 2(N-3) & 3(N-3) & 4(N-3) & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 1 & \dots & & & \dots & N \end{bmatrix} \tag{109}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & \dots & & & & 0 \\ 0 & -1 & 1 & 0 & \dots & & & & \\ 0 & 0 & -1 & 1 & 0 & \dots & & & \\ 0 & 0 & 0 & -1 & 1 & 0 & \dots & & \\ & & & & & & & & \\ & & & & & 0 & -1 & 1 & 0 \\ & & & & & & 0 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{N+1} \begin{bmatrix} -N & 1 & 1 & 1 & \dots \\ -(N-1) & -(N-1) & 2 & 2 & \\ -(N-2) & -(N-2) & -(N-2) & 3 & \\ -(N-3) & -(N-3) & -(N-3) & -(N-3) & \end{bmatrix} \tag{110}$$

and

$$\underset{(N+1)(N+1)}{UU^*} = \frac{1}{(N+1)} \begin{bmatrix} N & -1 & -1 & -1 & \dots & & -1 \\ -1 & N & -1 & -1 & & & \\ -1 & -1 & N & -1 & & & \\ -1 & -1 & -1 & N & & & \\ & & & & & & -1 \\ -1 & & & & & -1 & N \end{bmatrix} \tag{111}$$

$$= \frac{1}{(N+1)} [(N+1)I - 1\rangle\langle 1]$$

$$= I - \frac{1\rangle\langle 1}{N+1} = \widetilde{P}$$

Strong in reference (82) gives the LDU decomposition of the finite difference matrix as

$$\begin{bmatrix} 2 & -1 & 0 & & \\ -1 & 2 & -1 & & \\ 0 & -1 & 2 & & \\ & & & & 0 \\ & & & & -1 \\ & & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & & & & \\ -1/2 & 1 & & \bigcirc & & \\ 0 & -2/3 & 1 & & & \\ & & & & & 0 \\ 0 & & & 0 & \frac{1-n}{n} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & & & & \\ & 3/2 & & & \\ & & 4/3 & & \\ & & & \ddots & \\ & & & & \frac{n+1}{n} \end{bmatrix} \qquad \begin{bmatrix} 1 & -1/2 & & & \\ & 1 & -2/3 & & \\ & & 1 & \ddots & \\ & & & \ddots & \frac{1-n}{n} \\ & & & & 1 \end{bmatrix}$$

with det = $2 \left(\frac{3}{2}\right)\left(\frac{4}{3}\right) \cdot \cdot \cdot \left(\frac{n+1}{n}\right) = n+1$ product of pivots

c. All Combinations of First Differences

This portion and the portion on weighted differences is added as a generalization but will not be used extensively in this report. In many applications such as taking first differences of radar measurement vectors from n-stations, or sight-line measurement vectors from optical instruments, etc., one finds need for such analytics.

Consider a "dyadic product on a differencing relation".

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix} \textcircled{\Delta} \; (x_0, x_1, x_2 \; . \; . \; . \; x_N) = \begin{bmatrix} \Delta x_{00} & \Delta x_{01} & . \; . \; . & \Delta x_{0N} \\ \Delta x_{10} & \Delta x_{11} & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \Delta x_{N0} & \Delta x_{NN} & & \end{bmatrix} \tag{112}$$

where

$$\Delta x_{in} = \begin{pmatrix} x_n - x_i & i \neq n \\ 0 & i = n \end{pmatrix} \tag{113}$$

There are $(N+1)^2$ elements in the matrix of Eq. (112), eliminating the N+1 diagonal zero elements and using the symmetry

$$\Delta x_{in} = -\Delta x_{ni}$$

we have

$$\frac{(N+1)^2 - (N+1)}{z} = \frac{(N+1)[N+1 - 1]}{z} = \frac{N(N+1)}{z} = M \tag{114}$$

For example for N+1 = 4

$$M = 4\frac{(3)}{2} = 6 \tag{115}$$

or six differences of the 4 variables. The difference matrix yields

$$(x_0,\ x_1,\ x_2,\ x_3)\begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} = \langle xU$$

$$= (\Delta x_{01},\ \Delta x_{02},\ \Delta x_{03},\ \Delta x_{12},\ \Delta x_{13},\ \Delta x_{23}) \qquad (116)$$

The inner-grammian is

$$UU^T = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} = 4I - 1\rangle\langle 1 \qquad (117)$$

or

$$UU^T = 4(I - \frac{1\rangle\langle 1}{4}) = 4\underset{4x4}{\tilde{P}} \qquad (118)$$

By Eq. (118) we see that U has rank equal $\tilde{P}$ or

$$\underset{4x6}{\rho(U)} = \underset{4x4}{\rho(UU^T)} = \rho\tilde{P} = 3 \qquad (119)$$

The psuedo inverse of Eq. (118) is

$$(UU^T)^* = \tfrac{1}{4}\tilde{P} \qquad (120)$$

since

$$(UU^T)(UU^T)^* = (UU^T)^*(UU^T) = \tilde{P} \qquad (121)$$

and

$$\underset{6x4}{U^*} = U^T(UU^T)^* \qquad (122)$$

$$U^* = \underset{6x4}{U^T}(\tfrac{1}{4})\underset{4x4}{\tilde{P}} = \tfrac{1}{4}U^T \qquad (123)$$

since

$$U^T\tilde{P} = U^T \qquad (124)$$

The inner-projector is

$$UU^* = UU^T(\tfrac{1}{4}) \qquad (125)$$

and by Eq. (118) in Eq. (125)

$$UU^* = \underset{4x4}{\tilde{P}} = I - \frac{1\rangle\langle 1}{4} \qquad (126)$$

The outer-projector is by Eq. (122)

$$\underset{6x4}{U^*}\underset{4x6}{U} = U^T(UU^T)^*U \tag{127}$$

and by Eq. (121)

$$\underset{6x6}{U^*U} = \underset{6x4}{U^T}\ \underset{4x4}{\tilde{P}}\ \underset{4x6}{U} = U^T(I - \frac{1 \rangle\langle 1}{4})U \tag{128}$$

or a congruent transformation on the inner-projector.

As an example of what some of these matrices look like, the outer Grammian is

$$\underset{6x4}{U^T}\underset{4x6}{U} = \begin{bmatrix} 2 & 1 & 1 & -1 & -1 & 0 \\ 1 & 2 & 1 & 1 & 0 & -1 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ -1 & 1 & 0 & 2 & 1 & -1 \\ -1 & 0 & 1 & +1 & 2 & 1 \\ 0 & -1 & 1 & -1 & 1 & 2 \end{bmatrix} \tag{129}$$

The psuedo inverse is

$$U^* = \tfrac{1}{4} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{130}$$

and the outer projector is

$$\underset{6x6}{U^*U} = \begin{bmatrix} 2 & 1 & 1 & -1 & -1 & 0 \\ 1 & 2 & 1 & 1 & 0 & -1 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ -1 & 1 & 0 & 2 & 1 & -1 \\ -1 & 0 & 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 & 1 & 2 \end{bmatrix} \tag{131}$$

If the x_i of Eq. (116) are three dimensional row vectors, then

$$(\underset{1}{\langle 3)x}, \underset{2}{\langle 3)x}, \underset{3}{\langle 3)x}, \underset{4}{\langle 3)x}) \underset{12x18}{U} = \underset{1x18}{\langle \Delta x} \tag{132}$$

where the 12 x 18 matrix is

$$\underset{12x18}{U} = \begin{bmatrix} -I & -I & -I & 0 & 0 & 0 \\ I & 0 & 0 & -I & -I & 0 \\ 0 & I & 0 & I & 0 & -I \\ 0 & 0 & I & 0 & I & I \end{bmatrix} \tag{133}$$

and the entries are 3 x 3. The corresponding outer-projector becomes

$$\underset{18x18}{U^*U} = \begin{bmatrix} 2I & I & I & -I & -I & 0 \\ I & 2I & I & I & 0 & -I \\ I & I & 2I & 0 & I & I \\ -I & I & 0 & 2I & I & -I \\ -I & 0 & I & I & 2I & I \\ 0 & -I & I & -I & I & 2I \end{bmatrix} \tag{134}$$

By inspection it can be seen that U of Eq. (116) has full rank factors.

$$\underset{4x6}{U} = \underset{4 \text{ x } 3}{\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}} \quad \underset{3 \text{ x } 6}{\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}} \tag{135}$$

Express U in terms of the full rank factor and

$$\underset{4x6}{U} = \underset{4x3}{A} \; \underset{3x6}{B} = \underset{4x3}{A} \; \underset{3x6}{B} \tag{136}$$

and the psuedo inverse is the commuted product of the psuedo inverse or

$$\underset{6x4}{U^*} = \underset{6x3}{B^*} \; \underset{3x4}{A^*} \tag{137}$$

and the two associated projectors are

$$\underset{4x4}{UU^*} = ABB^*A^* = AA^* \tag{138}$$

since

$$BB^* = A^*A = \underset{3x3}{I} \tag{139}$$

and the commute of Eq. (138) is

$$\underset{6x6}{U^*U} = B^*B \tag{140}$$

Since the matrix A of Eq. (138) is Eq. (23), by Eq. (111)

$$\underset{4x4}{UU^*} = I - \frac{1\rangle\langle 1}{\langle 11\rangle} = \tilde{P} \tag{141}$$

The 6 x 6 projector is given by Eq. (140) and the B^* matrix is

$$\underset{6x3}{B^*} = B^T(BB^T)^{-1} \tag{142}$$

where

$$\underset{3x3}{BB^T} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \tag{143}$$

$$(BB^T)^{-1} = \frac{1}{16} \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} \tag{144}$$

and using Eq. (144) in Eq. (142)

$$\underset{6x3}{B^*} = \begin{bmatrix} 8 & -4 & 0 \\ 4 & 4 & -4 \\ 4 & 0 & 4 \\ -4 & 8 & -4 \\ -4 & 4 & 4 \\ 0 & -4 & 8 \end{bmatrix} \quad \frac{1}{16} \tag{145}$$

and the projector of Eq. (140) is

$$\underset{6x6}{U^*U} = \frac{1}{16} \begin{bmatrix} \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix} & \begin{pmatrix} -4 & -4 & 0 \\ 4 & 0 & -4 \\ 0 & 4 & 4 \end{pmatrix} \\ & \\ \begin{pmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \\ 0 & -4 & 4 \end{pmatrix} & \begin{pmatrix} 8 & 4 & -4 \\ 4 & 8 & 4 \\ -4 & 4 & 8 \end{pmatrix} \end{bmatrix} \tag{146}$$

One can obtain the eigeuvalues, eigeuvectors and vandermonde matrices associated with the full rank matrix of Eq. (143) as was done with the matrix of Eq. (59) to obtain more structure; but it will not be pursued further here.

One interesting representation of the rank N+1 projector of Eq. (111) is

$$\underset{(N+1)(N+1)}{UU^*} = \frac{1}{N+1} \begin{bmatrix} (N+1) \ I^{\Delta} - \angle 1 \end{bmatrix} \quad 1 \qquad (147)$$

$$= \left(\frac{1}{N+1}\right) \begin{bmatrix} N \ {}^{1}\!\angle e \ - \angle 1 \\ N \ {}^{2}\!\angle e \ - \angle 1 \\ \cdot \\ \cdot \\ \cdot \\ N \ {}^{N}\!\angle e \ - \angle 1 \end{bmatrix} \qquad (148)$$

where

$$\angle 1 = \angle N+1) \ 1 = \underset{(N+1)x1}{(1, 1, \ldots 1)} \qquad (149)$$

It can be shown, though it will not be derived here that Eq. (128) generalizes to maximum combinations of Eq. (114) where

$$M = \frac{N(N+1)}{2}$$

that

$$\underset{MxM}{U^*U} = \underset{MxN}{U^T} \ \underset{NxN}{(I - 1} \ \underset{N}{\underline{(N \times N)}} \ 1) \ \underset{NxM}{U \ (N)} \qquad (150)$$

The applications of some of these matrices to obtain position solutions by minimizing the sums of the squares of the magnitudes of all combinations of first differences is given in ref (92).

d. Weighted Difference - Matrices

Consider next the weighted differences of Eq. (21) (with the index shifted to 1, 2, ... n), that is

$$(x_1, x_2, x_3, \ldots x_n) \underset{Mx(M+1)}{\begin{bmatrix} -w_{11} & 0 & 0 & \ldots & 0 \\ w_{21} & -w_{22} & 0 & & 0 \\ 0 & w_{32} & -w_{33} & & \cdot \\ 0 & 0 & w_{43} & & \cdot \\ \cdot & \cdot & 0 & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & w_{M-1,\ M-1} \\ 0 & 0 & 0 & & w_{M,\ M-1} \end{bmatrix}}$$

$$= \langle M-1)\ \Delta\ x_w \tag{151}$$

where

$$\langle M-1)\ \Delta x_w = (w_{21}x_2-w_{11}x_1,\ w_{32}x_3-w_{32}x_2,\ \dots\ w_M x_N - w_{N-1}x_{N-1}) \tag{152}$$

as an example one can obtain divided differences if

$$\begin{aligned} &w_{21} = w_{11} = 1/h_1 \\ &\cdot \\ &\cdot \\ &\cdot \\ &w_{M,\ M-1} = w_{M-1,\ M-1} = 1/h_{n-1} \end{aligned} \tag{153}$$

and Eq. (152) becomes

$$\langle M-1)\ \Delta x_w = \frac{x_2-x_1}{h_1}\ ,\ \frac{x_3-x_2}{h_2}\ ,\ \dots\ \frac{x_n-x_{n-1}}{h_{n-1}} \tag{154}$$

Consider the minimum set (4x3) case for the special constraint as shown below, that is

$$w_{21} = w_{22}$$

$$w_{32} = w_{33}$$

$$(x_1,\ x_2,\ x_3,\ x_4) \begin{bmatrix} -w_1 & 0 & 0 \\ w_2 & -w_2 & 0 \\ 0 & w_3 & -w_3 \\ 0 & 0 & w_4 \end{bmatrix} = \langle 4)\ xU_w = \langle 3)\ \Delta x \tag{155}$$

The 4x3 weighted difference matrix can be written as

$$\underset{4x3}{U_w} = \underset{4x4}{W}\ \underset{4x3}{U} \tag{156}$$

where

$$\underset{4x4}{W} = \begin{bmatrix} w_1 & 0 & 0 & 0 \\ 0 & w_2 & 0 & 0 \\ 0 & 0 & w_3 & 0 \\ 0 & 0 & 0 & w_4 \end{bmatrix} \tag{157}$$

and the psuedo-inverse is

$$\underset{3\text{x}4}{U_w^*} = \underset{3\text{x}3}{(U_w^T U_w)^{-1}} U_w^T \tag{158}$$

and

$$U_w^* = U_w^T \underset{4\text{x}4}{(U_w U_w^T)^*} \tag{159}$$

The inner-Grammian of Eq. (156) is

$$U_w^T U_w = \begin{bmatrix} -w_1 & w_2 & 0 & 0 \\ 0 & -w_2 & w_3 & 0 \\ 0 & 0 & -w_3 & w_4 \end{bmatrix} \begin{bmatrix} -w_1 & 0 & 0 \\ w_2 & -w_2 & 0 \\ 0 & w_3 & -w_3 \\ 0 & 0 & w_4 \end{bmatrix} \tag{160}$$

$$= \begin{bmatrix} w_1^2 + w_2^2 & -w_2^2 & 0 \\ -w_2^2 & w_2^2 + w_3^2 & -w_3^2 \\ 0 & -w_3^2 & w_3^2 + w_4^2 \end{bmatrix}$$

One can obtain the inverse, eigheuvalues, eigeuvectors, companion matrix etc. for the full rank case of Eq. (157) for different conditions on (4) w; however, it will not be done here.

The 4x4 outer Grammian is

$$\underset{4\text{x}4}{U_w U_w^T} = W\ UU^T\ W \tag{161}$$

and by Eq. (107)

$$U_w U_w^T = W \ = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad W \tag{162}$$

$$U_w U_w^T = \begin{bmatrix} w_1^2 & -w_1 w_2 & 0 & 0 \\ -w_1 w_2 & 2w_2^2 & -w_2 w_3 & 0 \\ 0 & -w_2 w_3 & 2w_3^2 & -w_3 w_4 \\ 0 & 0 & -w_3 w_4 & w_4^2 \end{bmatrix} \tag{163}$$

Consider next the maximum combinations of four things, or the 4x6 case, one has

$$\langle 4) \times \underset{4x6}{U_w} = \langle 6) \Delta x_w \tag{164}$$

where

$$\underset{4x6}{U_w} = \underset{4x4}{W} \; \underset{4x6}{U} \tag{165}$$

Using the full rank 3 factors for U of Eq. (135)

$$\underset{4x6}{U_w} = \underset{4x4}{W} \; \underset{4x3}{A} \; \underset{3x6}{B} = \underset{4x3}{A_w} \; \underset{3x6}{B} \tag{166}$$

$$\underset{6x4}{U_w^*} = B^* A_w^* \tag{167}$$

and

$$\underset{4x4}{U_w U_w^*} = \underset{4x3}{A_w A_w^*} \tag{168}$$

where

$$\underset{4x4}{W} \; \underset{4x3}{U} = \underset{4x3}{A_w} = U_w \tag{169}$$

is given by Eq. (156) and

$$\underset{3x4}{U_w^*} = \underset{3x3}{(U_w^T U_w)^{-2}} U_w^T \tag{170}$$

hence one needs to invert Eq. (160) to obtain U_w^*.

The elements of the matrix of Eq. (165) are

$$U_w = \begin{pmatrix} w_1 & 0 & 0 & 0 & -1 \\ 0 & w_2 & 0 & 0 & 1 \\ 0 & 0 & w_3 & 0 & 0 \\ 0 & 0 & 0 & w_4 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \tag{171}$$

or

$$U_w = \begin{bmatrix} -w_1 & -w_1 & -w_1 & 0 & 0 & 0 \\ w_2 & 0 & 0 & -w_2 & -w_2 & 0 \\ 0 & w_3 & 0 & w_3 & 0 & -w_3 \\ 0 & 0 & w_4 & 0 & w_4 & w_4 \end{bmatrix} \tag{172}$$

Corresponding to the difference relations

$$(x_1\ x_2\ x_3\ x_4)\begin{pmatrix} -w_1 & -w_1 & -w_1 & 0 & 0 & 0 \\ w_2 & 0 & 0 & -w_2 & -w_2 & 0 \\ 0 & w_3 & 0 & w_3 & 0 & -w_3 \\ 0 & 0 & w_4 & 0 & w_4 & w_4 \end{pmatrix}$$

$$= (w_2x_2-w_1x_1,\ w_3x_3-w_1x_1,\ w_4x_4-w_1x_1,\ w_3x_3-w_2x_2,$$
$$w_4x_4-w_2x_2,\ w_4x_4-w_3x_3). \tag{173}$$

The inner-Grammian is

$$\underset{4x4}{U_w\,U_w^{\,T}} = \begin{bmatrix} 3w_1^2 & -w_1w_2 & -w_1w_3 & -w_1w_4 \\ -w_1w_2 & 3w_2^2 & -w_2w_3 & -w_2w_4 \\ -w_3w_1 & -w_3w_2 & 3w_3^2 & -w_3w_4 \\ -w_4w_1 & -w_4w_2 & -w_4w_3 & 3w_4^2 \end{bmatrix} \tag{174}$$

$$= W\left|\underset{4x4}{U^TU}\right| W = 4W^2-W>\!\!<W \tag{175}$$

and by Eq. (118)

$$U_wU_w^{\,T} = W\left[4(I - 1\ (4\underset{4}{>\!\!<}4)\ \underset{-}{1}\right] W \tag{176}$$

The outer Grammian is

$$\underset{6x6}{U_w^T\,U_w} =$$

$$= \left[\begin{pmatrix} w_1^2 + w_2^2 & w_1^2 & w_1^2 \\ w_1^2 & w_1^2 + w_3^2 & w_1^2 \\ w_1^2 & w_1^2 & w_1^2 + w_4^2 \end{pmatrix}\begin{pmatrix} w_2^2 & -w_2^2 & 0 \\ w_3^2 & 0 & -w_3^2 \\ 0 & w_4^2 & w_4^2 \end{pmatrix}\right] \tag{177}$$

$$\left[\begin{pmatrix} -w_2^2 & w_3^2 & 0 \\ -w_2^2 & 0 & w_4^2 \\ 0 & -w_3^2 & w_4^2 \end{pmatrix}\begin{pmatrix} w_2^2 + w_3^2 & w_2^2 & -w_3^3 \\ w_2^2 & w_2^2 + w_4^2 & w_4^2 \\ -w_3^2 & w_4^2 & w_3^2 + w_4^2 \end{pmatrix}\right]$$

A second weighted difference matrix (as an example) for the minimal set is

$$\begin{bmatrix} -w_1 & w_2 & 0 \\ 0 & -w_2 & w_3 \end{bmatrix} \quad \begin{bmatrix} -w_1 & w_2 & 0 & 0 \\ 0 & -w_2 & w_3 & 0 \\ 0 & 0 & -w_3 & w_4 \end{bmatrix}$$

$$= \begin{bmatrix} w_1^2 & -w_1w_2-w_2^2 & w_2w_3 & 0 \\ 0 & -w_2^2-w_2w_3 & -w_2w_3-w_3^2 & w_3w_4 \end{bmatrix} \tag{178}$$

e. <u>Kth Forward Differences From Back of Span to Front</u>

The following considerations of first, second, third, and kth differences will be restricted to unweighted and only the minimum set of differences (not all combinations of differences). For example, consider the five point span of data or variables as column vectors.

$$\begin{pmatrix} \Delta^{(1)}\ x\ (i,\ 0) \\ \Delta^{(1)}\ x\ (i,\ 1) \\ \Delta^{(1)}\ x\ (i,\ 2) \\ \Delta^{(1)}\ x\ (i,\ 3) \end{pmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{pmatrix} x(i,\ 0) \\ x(i,\ 1) \\ x(i,\ 2) \\ x(i,\ 3) \\ x(i,\ 4) \end{pmatrix} \tag{179}$$

or

$$\Delta^1\ x\ (i)\langle 4 \rangle = \underset{4x5}{U^T}\ x\ (i)\langle 5 \rangle \tag{180}$$

The second difference vector is

$$\begin{pmatrix} \Delta^2\ x(i,\ 0) \\ \Delta^2\ x(i,\ 1) \\ \Delta^2\ x(i,\ 2) \end{pmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{pmatrix} \Delta^1\ x(i,\ 0) \\ \Delta^1\ x(i,\ 1) \\ \Delta^1\ x(1,\ 2) \\ \Delta^1\ x(i,\ 3) \end{pmatrix} \tag{181}$$

or

$$\Delta^2\ x(i)\ \langle 3 \rangle = \underset{3x4}{U^T}\ \underset{4x5}{U^T}\ x(i)\ \langle 5 \rangle \tag{182}$$

where the distinction in the difference matrices is indicated by the size. The matrix product of Eq. (182) is

$$\begin{pmatrix} \Delta^2\ x(i,\ 0) \\ \Delta^2\ x(i,\ 1) \\ \Delta^2\ x(i,\ 2) \end{pmatrix} = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x(i,\ 0) \\ x(i,\ 1) \\ x(i,\ 2) \\ x(i,\ 3) \\ x(i,\ 4) \end{bmatrix} \tag{183}$$

The third difference yeilds

$$\begin{pmatrix} \Delta^3\ x(i,\ 0) \\ \Delta^3\ x(i,\ 1) \end{pmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{pmatrix} \Delta^2\ x(i,\ 0) \\ \Delta^2\ x(i,\ 1) \\ \Delta^2\ x(i,\ 2) \end{pmatrix} \tag{184}$$

or in terms of the x(i)

$$\begin{pmatrix} \Delta^3\ x(i,\ 0) \\ \Delta^3\ x(i,\ 1) \end{pmatrix} = \begin{bmatrix} -1 & 3 & -3 & 1 & 0 \\ 0 & -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} x(i,\ 0) \\ x(i,\ 1) \\ x(i,\ 2) \\ x(i,\ 3) \\ x(i,\ 4) \end{bmatrix} \tag{185}$$

The fourth and final difference is

$$\Delta^4\ x(i,\ 0) = (-1,\ 1) \begin{pmatrix} \Delta^3\ x(i,\ 0) \\ \Delta^3\ x(i,\ 1) \end{pmatrix} \tag{186}$$

or

$$\Delta^4\ x(i,\ 0) = (1,\ -4,\ 6,\ -4,\ 1) \begin{pmatrix} x(i,\ 0) \\ x(i,\ 1) \\ x(i,\ 2) \\ x(i,\ 3) \\ x(i,\ 4) \end{pmatrix} \tag{187}$$

Packaging the previous equations at the initial part n=0, and

$$\Delta^{(0)}\ x(i,\ 0) = x(i,\ 0) \tag{188}$$

we obtain

$$\begin{pmatrix} \Delta^{(0)}\ x(i, 0) \\ \Delta^{(1)}\ x(i, 0) \\ \Delta^{(2)}\ x(i, 0) \\ \Delta^{(3)}\ x(i, 0) \\ \Delta^{(4)}\ x(i, 0) \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \begin{pmatrix} x(i, 0) \\ x(i, 1) \\ x(i, 2) \\ x(i, 3) \\ x(i, 4) \end{pmatrix} \tag{189}$$

Observe that the matrix of Eq. (189) is the transpose of the inverse of the binomial matrix of Eq. (35) appendix (A), or

$$\Delta x(i, 0)\,(5\rangle_s = B^{-T}\, x(i)\,(5\rangle \tag{190}$$

where the vector of Eq. (189) is the discrete-analog of the continuous state vector of Eq. ( ) sec ( ) that is

$$\begin{bmatrix} x(i, 0) \\ \Delta x(i, 0) \\ \Delta^{(2)} x(i, 0) \\ \Delta^{(3)} x(i, 0) \\ \Delta^{(4)} x(i, 0) \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \\ \dddot{x}\ (t) \\ x^{(4)}(t) \end{bmatrix}$$

It is obvious that the n^th discrete difference package is

$$\begin{pmatrix} x(i, 0) \\ \Delta x(i, 0) \\ \Delta^{(2)} x(i, 0) \\ \cdot \\ \cdot \\ \cdot \\ \Delta^{(M)} x(i, 0) \end{pmatrix} = B^{-T} \begin{pmatrix} x(i,0) \\ x(i, 1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x(i, M) \end{pmatrix} \tag{191}$$

Multiply Eq. (190) by B^T, hence

$$x(i)\,(5\rangle = B^T\ \Delta x(i, 0)\ (5\rangle_{\Delta 2} \tag{192}$$

or

$$\begin{pmatrix} x(i, 0) \\ x(i, 1) \\ x(i, 2) \\ x(i, 3) \\ x(i, 4) \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \begin{pmatrix} x(i, 0) \\ \Delta x(i, 0) \\ \Delta^{(2)} x(i, 0) \\ \Delta^{(3)} x(i, 0) \\ \Delta^{(4)} x(i, 0) \end{pmatrix} \tag{193}$$

An extension of the vectors of Figure (1) yields the second differences geometry

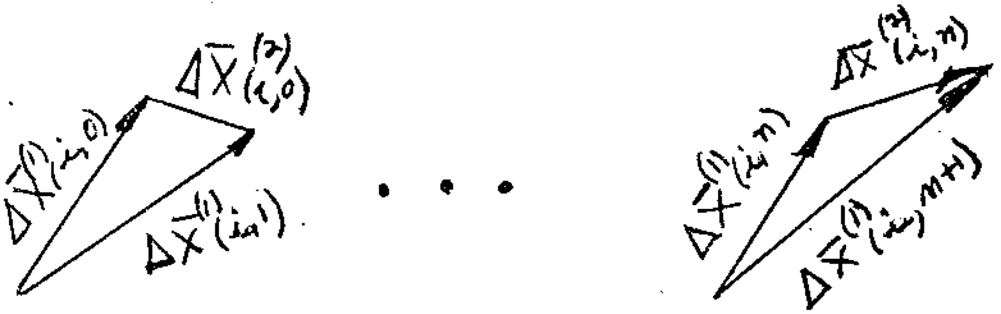

Figure (2) Second Forward Difference Vectors that is

$$\Delta\bar{x}^{(2)}(i,\ n) = \Delta\bar{x}^{(1)}(i,\ n+1) - \Delta\bar{x}^{(1)}(i,\ n) \tag{194}$$

One can construct similar geometry for the higher differences.

If we package Eq. (179), Eq. (181), Eq. (184) and Eq. (187) as a row of column vectors, we obtain

$$\begin{bmatrix} x(i,\ 0) & \Delta^{1}x(i,\ 0) & \Delta^{2}x(i,\ 0) & \Delta^{3}x(i,\ 0) & \Delta^{4}x(i,\ 0) \\ x(i,\ 1) & \Delta^{1}x(i,\ 1) & \Delta^{2}x(i,\ 1) & \Delta^{3}x(i,\ 1) & 0 \\ x(i,\ 2) & \Delta^{1}x(i,\ 2) & \Delta^{2}x(i,\ 2) & 0 & 0 \\ x(i,\ 3) & \Delta^{1}x(i,\ 3) & 0 & 0 & 0 \\ x(i,\ 4) & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} Ix(i)\rangle, & \underset{4x5}{U}\, x\rangle, & \underset{3x4}{U}\,\underset{4x5}{U}\, x(i)\rangle, & \underset{2x3}{U}\,\underset{3x4}{U}\,\underset{4x5}{U}\, x(i)\rangle, & \underset{1x2}{U}\,\underset{2x3}{U}\,\underset{3x4}{U}\,\underset{4x5}{U}\, x(i)\rangle \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix} \tag{195}$$

Note that the first row vector of Eq. (195) is the transpose of the discrete state vector of Eq. (188).

Returning to Eq. (188), if the fourth difference vector is zero for all i, that is

$$\Delta^{4}x(i,\ 0) \ \hat{\equiv}\ 0 = (1,\ -4,\ 6,\ -4,\ 1)\ x(i)\ (5\rangle \tag{196}$$

for $\forall i = 0,\ 1,\ 2,\ \ldots$

or

$$x(i,\ 4) \overset{i}{\equiv} (-1,\ 4,\ -6,\ 4) \begin{pmatrix} x(i,\ 0) \\ x(i,\ 1) \\ x(i,\ 2) \\ x(i,\ 3) \end{pmatrix} \overset{i}{\equiv} \ cxw\ \left(4\right\rangle \tag{197}$$

Using the dependence relation of Eq. (197) we obtain

$$\begin{pmatrix} x(i,\ 0) \\ x(i,\ 1) \\ x(i,\ 2) \\ x(i,\ 3) \\ x(i,\ 4) \end{pmatrix} = \begin{pmatrix} \underset{4x4}{I} \\ \left\langle 4\right)\ c \end{pmatrix}_{5x4} \begin{pmatrix} x(i,\ 0) \\ x(i,\ 1) \\ x(i,\ 2) \\ x(i,\ 3) \end{pmatrix} = \underset{5x4}{F_c}\ x(i)\ \left(4\right\rangle \tag{198}$$

Using Eq. (198) in Eq. (188)

$$\begin{pmatrix} x(i,\ 0 \\ \Delta x(i,\ 0) \\ \Delta^2 x(i,\ 0) \\ \Delta^3 x(i,\ 0) \\ \Delta^4 x(i,\ 0) \end{pmatrix} = \ B^{-T}\ F_c\ x(i)\ \left(4\right\rangle \tag{199}$$

where

$$\underset{5x5}{B^{-T}}\ \underset{5x4}{F_c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{200}$$

hence for the condition of Eq. (195), we need only consider the 4x4 case, that is

$$\begin{pmatrix} x(i,\ 0) \\ \Delta^1 x(i,\ 0) \\ \Delta^2 x(i,\ 0) \\ \Delta^3 x(i,\ 0) \end{pmatrix} = \underset{4x4}{B^{-T}} \begin{pmatrix} x(i,\ 0) \\ x(i,\ 1) \\ x(i,\ 2) \\ x(i,\ 3) \end{pmatrix} \tag{201}$$

or

$$\Delta x(i)\ \left(4\right\rangle_s = \underset{4x4}{B^{-T}}\ x(i)\ \left(4\right\rangle \quad . \tag{202}$$

The general connection of Eq. (13) between successive spans for the 5x5 case is

$$\begin{bmatrix} x(i+1,\ 0) \\ x(i+1,\ 1) \\ x(i+1,\ 2) \\ x(i+1,\ 3) \\ x(i+1,\ 4) \end{bmatrix} = S_{uo} \begin{pmatrix} x(i,\ 0) \\ x(i,\ 1) \\ x(i,\ 2) \\ x(i,\ 3) \\ x(i,\ 4) \end{pmatrix} + \mathbf{e}(5\rangle_5 \;\; x(i+1,\ 4)\ . \tag{203}$$

By Eq. (197) we have for i+1

$$x(i+1,\ 4) = (-1,\ 4,\ -6,\ 4) \begin{pmatrix} x(i+1,\ 0) \\ x(i+1,\ 1) \\ x(i+1,\ 2) \\ x(i+1,\ 3) \end{pmatrix} = \langle c\ x(i+1)\ (4\rangle \tag{204}$$

Eq. (204) can be rewritten as

$$\begin{aligned} x(i+1,\ 4) &= (0,\ -1,\ 4,\ -6,\ 4)\ x(i)\ (5\rangle \\ &= \langle 5c\ x(i)\ (5\rangle \end{aligned} \tag{205}$$

Using Eq. (205) in Eq. (203)

$$x(i+1)(5\rangle = S_{uo}\ x(i)(5\rangle + \mathbf{e}(5\rangle_5\ \langle 5)\ c\ x(i)\ (5\rangle \tag{206}$$

$$= \left[S_{uo} + \mathbf{e}(5\rangle_5\ \langle 5)\ c\right]\ x(i)\ (5\rangle \tag{207}$$

or

$$x(i+1)(5\rangle = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 4 & -6 & 4 \end{bmatrix} x(i)(5\rangle \tag{208}$$

which is clearly singular. The full rank transition matrix can be obtained from Eq. (197) where

$$x(i+1,\ 3) = x(i,\ 4) = (-1,\ 4,\ -6,\ 4) \begin{pmatrix} x(i,\ 0) \\ x(i,\ 1) \\ x(i,\ 2) \\ x(i,\ 3) \end{pmatrix} \tag{209}$$

and

$$\begin{pmatrix} x(i+1,\ 0) \\ x(i+1,\ 1) \\ x(i+1,\ 2) \\ x(i+1,\ 3) \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 4 & -6 & 4 \end{bmatrix} \begin{pmatrix} x(i,\ 0) \\ x(i,\ 1) \\ x(i,\ 2) \\ x(i,\ 3) \end{pmatrix} \tag{210}$$

or

$$x(i+1)\ (4\rangle \quad = \quad \Phi_{xx}\ x(i)\ (4\rangle \tag{211}$$

Note that

$$\Phi_{xx}^{2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 4 & -6 & 4 \\ -4 & 15 & -20 & 10 \end{bmatrix} \tag{212}$$

we can obtain a transition matrix on the discreet states by Eq. (199) (for the assumptions of Eq. (197)) as

$$\begin{pmatrix} x(i,\ 0) \\ \Delta x(i,\ 0) \\ \Delta^{2}x(i,\ 0) \\ \Delta^{3}x(i,\ 0) \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \quad x(i)\rangle \tag{213}$$

and

$$\begin{pmatrix} x(i+1,\ 0) \\ \Delta x(i+1,\ 0) \\ \Delta^{2}x(i+1,\ 0) \\ \Delta^{3}x(i+1,\ 0) \end{pmatrix} = \underset{4x4}{B^{-T}} \quad x(i+1)(4\rangle \tag{214}$$

Using Eq. (210) and the inverse of Eq. (213) in Eq. (214)

$$\Delta x(i+1)(4\rangle_{s} \quad = B^{-T}\ \Phi_{xx}\ B^{T}\ \Delta x(i)(4\rangle_{s} \tag{215}$$

$$= \Phi_{\Delta\Delta}\Delta x(i)(4\rangle_{s} \tag{216}$$

where

$$\Phi_{\Delta\Delta} = B^{-T}\ \Phi_{xx}B^{T} \tag{217}$$

or in open form the transition-matrix is

$$\Phi_{\Delta\Delta} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{218}$$

and

$$\Phi_{\Delta\Delta}^{2} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{219}$$

Discrete Differences of the Fitting Functions

When the variables x(i, n) are expressed in the form

$$x(i,\ n) = f(i,\ n)a\rangle \qquad (220)$$

One obtains for example

$$Ux(i)(5\rangle = U \begin{bmatrix} f(i,\ 0) \\ f(i,\ 1) \\ f(i,\ 2) \\ \cdot \\ \cdot \\ \cdot \\ f(i,\ n) \end{bmatrix} a\rangle \qquad (221)$$

or

$$Ux(i)\rangle = \begin{bmatrix} f(i,\ 1) & -\ f(i,\ 0) \\ f(i,\ 2) & -\ f(i,\ 1) \\ \cdot & \\ \cdot & \\ \cdot & \\ f(i,\ n+1) & -\ f(i,\ n) \\ \cdot & \\ \cdot & \\ \cdot & \\ f(i,\ n) & -\ f(i,\ n-1) \end{bmatrix} a\rangle \qquad (222)$$

etc for higher differences.

The general terms by Eq. (201) for Forward Differences

$$\Delta x\rangle_s = \begin{pmatrix} x(i,\ 0) \\ \Delta x(i,\ 0) \\ \Delta^2 x(i,\ 0) \\ \cdot \\ \cdot \\ \cdot \\ \Delta^n x(i,\ 0) \\ \cdot \\ \cdot \\ \cdot \\ \Delta^N x(i,\ 0) \end{pmatrix} = B^{-T} \begin{pmatrix} x(i,\ 0) \\ x(i,\ 1) \\ x(i,\ 2) \\ \cdot \\ \cdot \\ \cdot \\ x(i,\ n) \\ \cdot \\ \cdot \\ \cdot \\ x(i,\ N) \end{pmatrix} = B^{-T}\ x(i)(N+1\rangle \qquad (223)$$

or

$$\begin{bmatrix} x(i,\ 0) \\ \Delta x(i,\ 0) \\ \Delta^2 x(i,\ 0) \\ \cdot \\ \cdot \\ \cdot \\ \Delta^N x(i,\ 0) \end{bmatrix} = \begin{pmatrix} \binom{0}{0}(-1)^0 & 0 & & & \\ \binom{1}{0}(-1)^0, & \binom{1}{1}(-1), & 0 & & \\ \binom{2}{0}(-1)^0, & \binom{2}{1}(-1), & \binom{2}{2}(-1)^2\ , & 0 & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \binom{N}{0}(-1)^0, & \binom{N}{1}(-1)^1 & ,\binom{N}{2}(-1)^2\ , & \ldots\binom{N}{N} & -1^N \end{pmatrix} \begin{bmatrix} x(i,\ 0) \\ x(i,\ 1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x(i,\ N) \end{bmatrix} \qquad (224)$$

and by Eq. (62) appendix (A) the general term is

$$\Delta x^n(i,\ 0) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x_{n-k} \qquad (225)$$

also

$$x(i)\rangle \ = B^T \ \Delta x(i)\rangle_s \qquad (226)$$

and the general term is

$$x(i,\ n) = x(i,\ 0) + n\Delta x(i,\ 0) + \frac{n(n-1)}{2:}\Delta^2 x(i,\ 0) + \ \ldots\ + \Delta^n x(i,\ 0) \qquad (227)$$

$$x(i,\ n) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k x(i,\ 0) \qquad (228)$$

By Eq. (53) appendix (B)

$$\rlap{|}F B = S_b\ C_E^T \qquad (229)$$

$$B = \rlap{|}F^{-2}\ S_b\ C_E^T \qquad (230)$$

$$B^{-1} = C_E^{-T}\ s_b^{-1}\ \rlap{|}F \qquad (231)$$

$$B^{-T} = \rlap{|}F\ S_b^{-T}\ C_E^{-1} \qquad (232)$$

and using Eq. (232) in Eq. (223)

$$\Delta x(i,\ 0)(N+1\rangle_s \ = \ \rlap{|}F\ S_b^{-T}\ C_E^{-1}\ x(i)(N+1\rangle \qquad (233)$$

Consider the following example where the matrices are 4x4, and the polynomial is of degree 3

$$x(i,\ n) = (1,\ n,\ n^2,\ n^3)\ T_u(i)\ T_u(\beta)D(ho)a\rangle \qquad (234)$$

or

$$x(i,\ n) = \langle n\ \ T(i,\ \beta,\ h)\ a\rangle$$

$$x(i,\ n) = \langle 4)\ n\ a^{c}\ (4\rangle \tag{235}$$

and for 4 points

$$x(i)(4\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 2^{2} & 2^{3} \\ 1 & 3 & 3^{2} & 3^{3} \end{bmatrix}\ a^{c}\rangle\ \ = \underset{4x4}{C_{E}}\ a^{c}\rangle \tag{236}$$

Using Eq. (236) in Eq. (233)

$$\Delta x(i,\ 0)(4\rangle_{s}\ \ = \rlap{$\dagger$}F\ S_{b}^{-T}\ C_{E}^{-2}\ C_{E}\ a^{i}\rangle \tag{237}$$

$$= \rlap{$\dagger$}F\ S_{B}^{-T}\ a^{c}\rangle \tag{238}$$

where

$$\rlap{$\dagger$}F S_{b}^{-T} = B^{-T}\ C_{E}\ \ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{bmatrix} \tag{239}$$

If the discrete states are desired at i+n, then by Eq. (224)

$$\begin{bmatrix} x(i+n,\ 0) \\ \Delta^{1}x(i+n,\ 0) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \Delta^{N}x(i+n,\ 0) \end{bmatrix} = \begin{bmatrix} x(i,\ n) \\ \Delta^{1}x(i,\ n) \\ \Delta^{2}x(i,\ n) \\ \cdot \\ \cdot \\ \cdot \\ \Delta^{N}x(i,\ n) \end{bmatrix} = B^{-T} \begin{bmatrix} x(i+n,\ 0) \\ x(i+n,\ 1) \\ x(i+n,\ 2) \\ \cdot \\ \cdot \\ \cdot \\ x(i+n,\ N) \end{bmatrix} \tag{240}$$

and the n^{th} term of Eq. (240) is

$$\Delta^{n}\ x(i,\ n) = \sum_{k=0}^{n} \binom{n}{j}\ (-1)^{j}\ x_{i+n-j} \tag{241}$$

f. K^{th} Backward Differences From Front of Span to Back

A number of authors use the notation and distinction between the forward-difference operator

$$\Delta f_{n} = f_{n+1} - f_{n} \tag{242}$$

and the backward-difference operator

$$\nabla f_n = f_n - f_{n-1} \tag{243}$$

for ∇ read "Nabla" and Δ read "Delta".

If we consider the linear convolved equation of Eq. (30) where the real-time point x(i, N) is at the top of the span, we obtain

$$\begin{bmatrix} \nabla x(i, N) \\ \nabla x(i, N-1) \\ \cdot \\ \cdot \\ \cdot \\ \nabla x(i, n) \\ \cdot \\ \cdot \\ \cdot \\ \nabla x(i, 0) \end{bmatrix} = \underset{N\ x(N+1)}{\begin{bmatrix} 1 & -1 & 0 & \dots\dots\dots & \\ 0 & 1 & -1 & 0 & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & & & 1 & -1 \end{bmatrix}^{T}} \begin{bmatrix} x(i, N) \\ x(i, N-1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x(i, 1) \\ x(i, 0) \end{bmatrix} \tag{244}$$

$$\begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & \dots & & 1 & -1 \end{pmatrix} \equiv U_b^T = -U^T \tag{245}$$

The matrix U_b generates different transition matrices, etc.

Consider now the front-to-back-span of Eq. (1) but with the span front time point at the top, that is

$$x(\ell)(N+1\rangle = \begin{pmatrix} x(\ell, 0) \\ x(\ell, -1) \\ x(\ell, -2) \\ \cdot \\ \cdot \\ \cdot \\ x(\ell, -N) \end{pmatrix} \tag{246}$$

and the backward differencing operator of Eq. (245) operating on Eq. (246) transposed

$$\langle N+1)\ x(\ell)\ U_b = \langle N)\ \nabla x(\ell) \tag{247}$$

where for the 4 point case the column vectors are

$$\begin{pmatrix} \nabla x(\ell-0) \\ \nabla x(\ell-1) \\ \nabla x(\ell-2) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x(\ell-0) \\ x(\ell-1) \\ x(\ell-2) \\ x(\ell-3) \end{pmatrix} \tag{248}$$

The second backward differences are

$$\begin{pmatrix} \nabla^2 x(\ell-0) \\ \nabla^2 x(\ell-1) \end{pmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{pmatrix} \nabla x(\ell-0) \\ \nabla x(\ell-1) \\ \nabla x(\ell-2) \end{pmatrix} \tag{249}$$

or by Eq. (248) in Eq. (249)

$$\begin{pmatrix} \nabla^2 x(\ell-0) \\ \nabla^2 x(\ell-1) \end{pmatrix} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{pmatrix} x(\ell-0) \\ x(\ell-1) \\ x(\ell-2) \\ x(\ell-3) \end{pmatrix} \tag{250}$$

The third backward difference is

$$\nabla^3 x(\ell-0) = (1,\ -3,\ 3,\ -1)\ x(\ell)\ (4\rangle \tag{251}$$

Packaging the "discrete states"

$$\begin{pmatrix} \nabla x^0(\ell-0) \\ \nabla^1 x(\ell-0) \\ \nabla^2 x(\ell-10) \\ \nabla^3 x(\ell-10) \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 3 & -1 \end{bmatrix} \begin{pmatrix} x(\ell-10) \\ x(\ell-1) \\ x(\ell-2) \\ x(\ell-3) \end{pmatrix} \tag{252}$$

which is the transpose of the Rutishauser matrix of Eq. (76) appendix (A). hence the general case is

$$\nabla x(\ell)\ (N+1\rangle_s = R^T\ x(\ell)\ (N+1\rangle \tag{253}$$

Since by Eq. (94), appendix (A) the Rutishauser matrix is its own inverse

$$x(\ell)\ (N+1\rangle = R^T\ \nabla x(\ell)\ (N+1\rangle_s \tag{254}$$

If x(ℓ, j) is a third degree polynomial then by Eq. (196)

$$\nabla^4\ x(\ell-0) \overset{\ell}{\equiv} 0 = (1,\ -4,\ 6,\ -4,\ 1) \begin{pmatrix} x(\ell-0) \\ x(\ell-1) \\ x(\ell-2) \\ x(\ell-3) \end{pmatrix} \tag{255}$$

for all ℓ.

Successive spans of 4 data points are connected via

$$\begin{pmatrix} x(\ell+1-0) \\ x(\ell+1-1) \\ x(\ell+1-2) \\ x(\ell+1-3) \end{pmatrix} = \begin{pmatrix} x(\ell+1) \\ x(\ell) \\ x(\ell-1) \\ x(\ell-2) \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x(\ell-0) \\ x(\ell-1) \\ x(\ell-2) \\ x(\ell-3) \end{pmatrix} + \mathbf{e}\rangle^1\ x(\ell+1) \tag{256}$$

If we advance Eq. (255) one step to ℓ+1, we have

$$\nabla^4 x(\ell+1,\ 0) = 0 = (1,\ -4,\ 6,\ -4,\ 1) \begin{pmatrix} x(\ell+1-0) \\ x(\ell+1-1) \\ x(\ell+1-2) \\ x(\ell+1-3) \end{pmatrix} \tag{257}$$

or

$$x(\ell+1) = (4,\ -6,\ 4,\ -1) \begin{pmatrix} x(\ell-0) \\ x(\ell-1) \\ x(\ell-2) \end{pmatrix} \tag{258}$$

Using Eq. (258) in Eq. (256)

$$\begin{pmatrix} x(\ell+1) \\ x(\ell-0) \\ x(\ell-1) \\ x(\ell-2) \end{pmatrix} = \begin{bmatrix} 4 & -6 & 4 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x(\ell-0) \\ x(\ell-1) \\ x(\ell-2) \\ x(\ell-3) \end{pmatrix} \tag{259}$$

or

$$x(\ell+1)\ (4\Big\rangle = \Phi_{xx_b}\ x(\ell)\ (4\Big\rangle \tag{260}$$

The discrete-state relation of Eq. (253) holds for the $(\ell+1)^{th}$ span, hence

$$\nabla x(\ell+1)\Big\rangle_s = R^T\ x(\ell+1)\Big\rangle \tag{261}$$

and by Eq. (259) and (254)

$$\nabla x(\ell+1)\Big\rangle_s = R^T\ \Phi_{xxb}\ R^T\ \nabla x(\ell)\Big\rangle_s \tag{262}$$

or

$$\nabla x(\ell+1)\Big\rangle_s = \Phi_{\nabla\nabla}\ \nabla x(\ell)\Big\rangle_s \tag{263}$$

where

$$\Phi_{\nabla\nabla} = R^T\ \Phi_{xxb}\ R^T \tag{264}$$

or

$$\Phi_{\nabla\nabla} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{265}$$

It is of interest to observe that one can also obtain the transition matrices in a different way. For example at ℓ+1, we have

$$\begin{pmatrix} x(\ell+1,\ 0) \\ x(\ell+1,\ -1) \\ x(\ell+1,\ -2) \\ x(\ell+1,\ -3) \end{pmatrix} = {}_{4x4}\Gamma\ T_u(\ell+1)T_u(\beta)D(ho)a\rangle \tag{266}$$

$$x(\ell+1)(4\rangle = \Gamma T_u(1)T_u(\ell)T_u(\beta)D(ho)a\rangle \tag{267}$$

and at ℓ we have

$$x(\ell)\ \ (4\rangle = \Gamma T_u(\ell)T_u(\beta)D(ho)a\rangle \tag{268}$$

or

$$\Gamma^{-1}\ x(\ell)\ \ (4\rangle = T_u(\ell)T_u(\beta)D(ho)a\rangle \tag{269}$$

or using Eq. (269) in Eq. (266)

$$x(\ell+1)\ \ (4\rangle = \Gamma T_u(1)\ \ {}^{-1}\ x(\ell)\ \ (4\rangle \tag{270}$$

or

$$\Phi_{xxb} = \Gamma B \Gamma^{-1} \tag{271}$$

The relationships between the states and the variables over the span can be packaged for all the forward difference case via Eq. (213)

$$\begin{bmatrix} x(i,\ 0) & x(i+1,\ 0) & \cdots & x(i+N,\ 0) \\ \Delta x(i,\ 0) & \Delta x(i+1,\ 0) & \cdots & \Delta x(i+N,\ 0) \\ \Delta^2 x(i,\ 0) & \Delta^2 x(i+1,\ 0) & \cdots & \Delta^2 x(i+N,\ 0) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \Delta^N x(i,\ 0), & \Delta^N x(i+1,\ 0) & \cdots & \Delta^N x(i+N,\ 0) \end{bmatrix}$$

$$= B^{-T} \begin{bmatrix} x(i,\ 0) & x(i+1,\ 0) & \cdots & x(i+N,\ 0) \\ x(i,\ 1) & x(i+1,\ 1) & & x(i+N,\ 1) \\ x(i,\ 2) & x(i+1,\ 2) & & x(i+N,\ 2) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x(i,\ N) & x(i+1,\ N) & \cdots & x(i+N,\ N) \end{bmatrix} \tag{272}$$

A similar relation holds for Eq. (261) connected via R^T.

g. Discrete Differencing of Newton Polynomials

The row vector of x(i, n) given by Eq. (235) is

$$x(i,\ 0),\ x(i,\ 1)\ x(i,\ 2)\ \ldots\ x(i,\ n)\ \ldots\ x(i,\ N) = \langle N+1)\ x(i) \qquad (273)$$

or for d=N (the polynomial degree)

$$\langle N+1)\ x(i) = \langle N+1)\ a^c\ [n\rangle_0\ ,\ n\rangle_1\ ,\ n\rangle_2\ ,\ \ldots\ n\rangle_n\ \ldots\ n\rangle_N]$$

$$= \langle a^c \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 2 & 3 & n & n+1 & & N \\ . & 1 & 2^2 & 3^2 & n^2 & (n+1)^2 & & N^2 \\ . & . & 2^3 & 3^3 & n^3 & (n+1)^3 & & N^3 \\ . & . & . & . & . & . & & . \\ . & . & . & . & . & . & & . \\ . & . & . & . & . & . & & . \\ . & . & 2^n & 3^n & n^n & (n+1)^n & & N^n \\ . & . & . & . & . & . & & . \\ . & . & . & . & . & . & & . \\ . & . & . & . & . & . & & . \\ 0 & 1 & 2^N & 3^N & n^N & (n+1)^N & & N^N \end{bmatrix} \qquad (274)$$

operating an Eq. (274) with U

$$\langle x(i)U = \langle \Delta^{(1)} x(i) = \langle z^i\ C_E^T\ U \qquad (275)$$

where C_E^T is given by Eq. (51) appendix (B), and is

$$N^T = C_E^T \qquad (276)$$

The matrix product of Eq. (275) is

$$N^T U = \begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 \\ 0 & 1 & 2 & 3 & & n & (n+1) & & N \\ . & . & 2^2 & 3^2 & & n^2 & (n+1)^2 & & N^2 \\ . & . & . & . & & . & . & & . \\ . & . & . & . & & . & . & & . \\ 0 & 1 & 2^N & 3^N & & n^N & (n+1)^N & & N^N \end{bmatrix}$$

$$U = [n\rangle_1 - n\rangle_0\ ,\ \ldots\ n\rangle_{n+1} - n\rangle_n\ \ldots]$$

$$= \begin{bmatrix} 1-1 & 1-1 & 1-1 & \dots & 1-1 & \dots \\ 1 & 2-1 & 3-2 & . & n+1-n & . \\ . & 2^2-1 & 3^2-2^2 & . & (n+1)^2-n^2 & . \\ . & 2^3-1 & . & . & . & . \\ . & 2^n-1 & . & . & . & . \\ . & 2^N-1 & 3^N-2^N & \dots & (n+1)^N-n^N & \dots \end{bmatrix} \tag{277}$$

The transformation to the backward factorial functions of Eq. (44) appendix (B) is

$$\begin{bmatrix} n^{(0)} \\ n^{(1)} \\ n^{(2)} \\ . \\ . \\ . \\ n^{(n)} \\ . \\ . \\ . \\ n^{(N)} \end{bmatrix} = S_b \begin{bmatrix} n^0 \\ n^1 \\ n^2 \\ . \\ . \\ . \\ n^n \\ . \\ . \\ . \\ n^N \end{bmatrix} \tag{278}$$

or

$$n^{(\ )}\rangle = S_b\ n\rangle_n \tag{279}$$

or

$$n\rangle_n = S_b^{-1}\ n^{(\ )}\rangle \tag{280}$$

Using Eq. (280) in Eq. (235) transposed

$$x(i,\ n) = \langle a^i n\rangle_n = \langle a^i\ S_B^{-1}\ n^{(\ )}\rangle = \langle gn^{(\ )}\rangle \tag{281}$$

where

$$\langle g = \langle a^i\ S_b^{-1} \tag{282}$$

Using Eq. (281) in the row-vector of Eq. (274)

$$\langle x(i) = \langle g\ [n^{(\ )}\rangle_0\ ,\ n^{(\ )}\rangle_1\ ,\ n^{(\ )}\rangle_2\ \dots\ n^{(\ )}\rangle_n\ \dots\ n^{(\ )}\rangle_N] \tag{283}$$

or in open form

$$(x(i,\ 0),\ x(i,\ 1),\ x(i,\ 2),\ \dots\ x(i,\ n)\ \dots\ x(i,\ N)$$

$$= \langle g \begin{bmatrix} 0^{(0)} & 1^{(0)} & 2^{(0)} & \cdots & n^{(0)} & (n+1)^{(0)} & \cdots & N^{(0)} \\ 0^{(1)} & 1^{(1)} & 2^{(1)} & & n^{(1)} & (n+1)^{(1)} & & N^{(1)} \\ 0^{(2)} & 1^{(2)} & 2^{(2)} & & n^{(2)} & (n+1)^{(2)} & & N^{(2)} \\ \cdot & \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & & \cdot \\ 0^{(N)} & 1^{(N)} & 2^{(N)} & & n^{(N)} & (n+1)^{(N)} & & N^{(N)} \end{bmatrix} \tag{284}$$

By Eq. (29) appendix (B), we have

$$\langle x(i) = \langle g \; S^{(\ )} \tag{285}$$

where $S^{(\ )}$ is the matrix of backward factorial functions.

The first difference of Eq. (283) is

$$\langle x(i)U = \langle g \; S^{(\ )} \; U \tag{286}$$

The matrix product is

$$\underset{(N+1)(N+1)}{S^{(\ )}}\underset{(N+1)\mathrm{x}N}{U} = \begin{bmatrix} 0^{0} - 1^{(0)} & 2^{(0)} - 1^{(0)} & (n+1)^{(0)} - n^{(0)} & \cdots \\ 0 - 1^{(1)} & 2^{(1)} - 1^{(1)} & (n+1)^{(1)} - n^{(1)} & \\ 0 & 2^{(2)} - 0 & \cdot & \\ \cdot & 0 & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & (n+1)^{(n)} - n^{(n)} & \\ \cdot & \cdot & (n+1)^{(n)} - 0 & \\ 0 & 0 & 0 & \end{bmatrix} \tag{287}$$

We can write Eq. (286) as

$$\Delta^{(1)} \langle x(i) = \Delta^{(1)}\langle x(i) = \langle x(i)U \tag{288}$$

$$= \langle g \; \Delta^{(1)} \; S^{(\ )} \tag{289}$$

where the columns are

$$\Delta^{(1)} \; S^{(\ )} = [\Delta^{(1)} \; n^{(\ )}\rangle_0 \;\; , \; \Delta^{(1)} \; n^{(\ )}\rangle_1 \;\;\; , \; \ldots \; \Delta^{(1)} \; m^{(\ )}\rangle_N] \tag{290}$$

The n^{th} element of the n^{th} column of Eq. (217) or Eq. (290) is

$$\Delta^{(1)} \; n^{(n)} = (n+1)^{(n)} - n^{(n)} = nn^{(n-1)} \tag{291}$$

or the i-j^{th} element is

$$\Delta^{(1)}\; i^{(j)} = (i+1)^{(j)} - i^{(j)} = j\; i^{(j-1)} \tag{292}$$

which is the discreet analog of

$$\frac{dt^{j}}{dt} = jt^{j-1} \tag{293}$$

The first difference of the i^{th} column vector as shown in Eq. (287) can be written as

$$\Delta^{(1)}\begin{bmatrix} i^{(0)}\\ i^{(1)}\\ i^{(2)}\\ i^{(3)}\\ \cdot\\ \cdot\\ \cdot\\ i^{(j)}\\ \cdot\\ \cdot\\ \cdot\\ i^{(N)} \end{bmatrix} = \begin{bmatrix} 0\\ 1\\ 2i^{(1)}\\ 3i^{(2)}\\ \cdot\\ \cdot\\ \cdot\\ ji^{(j-1)}\\ \cdot\\ \cdot\\ \cdot\\ Ni^{(N-1)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & & i^{(0)}\\ 1 & 0 & 0 & 0 & & 0 & & i^{(1)}\\ 0 & 2 & 0 & 0 & & 0 & & i^{(2)}\\ 0 & 0 & 3 & 0 & & 0 & & i^{(3)}\\ \cdot & & & & & & & \cdot\\ \cdot & & & & & & & \cdot\\ \cdot & & & & & & & \cdot\\ & & & & & & & \\ \cdot & & & & & & & \cdot\\ \cdot & & & & & & & \cdot\\ \cdot & & & & & & & \cdot\\ 0 & & & & & N & 0 & i^{(N)} \end{bmatrix} \tag{294}$$

$$(N+1)x(N+1)$$

or

$$\Delta^{(1)} \;\; i^{(\;)}\rangle = V^{T}\, i^{(\;)}\rangle \tag{295}$$

which is the discreet analog of Eq. (21) section (4)

$$\frac{dt\rangle}{dt} = V^{T}t\rangle \tag{296}$$

we thus have

$$\begin{array}{l} \Delta^{(2)}\; i^{(\;)}\rangle = (V^{T})^{2}\; i^{(\;)}\rangle\\ \Delta^{(3)}\; i^{(\;)}\rangle = (V^{T})^{3}\; i^{(\;)}\rangle\\ \cdot\\ \cdot\\ \cdot\\ \Delta^{(j)}\; i^{(\;)}\rangle = (V^{T})^{j}\; i^{(\;)}\rangle \end{array} \tag{297}$$

using Eq. (297) in the Newton polynomial expression of Eq. (281) and packaging the discreet states as a column vector

$$\begin{bmatrix} x(i,\ n) \\ \Delta^{(1)}x(i,\ n) \\ \Delta^{(2)}x(i,\ n) \\ \Delta^{(3)}x(i,\ n) \\ \cdot \\ \cdot \\ \cdot \\ \Delta^{(N)}x(i,\ n) \end{bmatrix} = \begin{bmatrix} \langle g \ i^{(\ )}\rangle \\ \langle g \ \Delta^{(1)} i^{(\ )}\rangle \\ \langle g \ \Delta^{(2)} i^{(\ )}\rangle \\ \langle g \ \Delta^{(3)} i^{(\ )}\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle g \ \Delta^{(N)} i^{(\ )}\rangle \end{bmatrix} \qquad (298)$$

$$\Delta x(i,\ n)\rangle_s = \begin{bmatrix} \langle g \\ \langle g V^T \\ \langle g (V^T)^2 \\ \langle g (V^T)^3 \\ \cdot \\ \cdot \\ \cdot \\ \langle g (V^T)^N \end{bmatrix} \qquad i^{(\ )}\rangle \qquad (299)$$

$$\Delta x(i,\ n)\rangle_s = \begin{bmatrix} \langle i^{(\ )} \\ \langle i^{(\ )}V \\ \langle i^{(\ )}V^2 \\ \langle i^{(\ )}V^3 \\ \cdot \\ \cdot \\ \cdot \\ \langle i^{(\ )}V^N \end{bmatrix} \qquad g\rangle \qquad (300)$$

The forward difference state vector is given by Eq. (223) as

$$\Delta x\rangle_s = B^{-T}\ x(i)\ (N+1\rangle \qquad (301)$$

By Eq. (36) appendix (B) we have

$$B^{-T} = F\ \left[S^{(\ )}\right]^{-T}, \qquad (302)$$

or Eq. (302) in Eq. (301) yields

$$\Delta x(i,\ n)\rangle_s = F\ \left[S^{(\ )}\right]^{-T}\ x(i)\ (N+1\rangle \qquad (303)$$

Transposing Eq. (285)

$$x(i)\rangle = S^{(\ )T}\ g\rangle \qquad (304)$$

and Eq. (304) in Eq. (303) yields

$$\Delta x(i,\ n)\rangle_s \quad = F\ g\rangle \tag{305}$$

Using the transpose of Eq. (282) in Eq. (305)

$$\Delta x(i,\ n)\rangle_s \quad = FS_b^{-T}\ a^i\rangle \tag{306}$$

where

$$a^i\rangle \quad = T_u(i)T_u(\beta)D(ho)a\rangle \tag{307}$$

Using Eq. (307) in Eq. (306)

$$\Delta x(i,\ n)\rangle_s \quad = FS_b^{-T}\ T_u(i)T_u(\beta)D(ho)a\rangle \tag{308}$$

Update to i+1

$$\Delta x(i+1,\ n)\rangle_s \quad = FS_b^{-T}T_u(1)T_u(i)T_u(\beta)D(ho)a\rangle \tag{309}$$

By Eq. (308)

$$S_b^T\ F^{-1}\ \Delta x(i,\ n)\rangle_s \quad = T_u(i)T_u(\beta)D(ho)a\rangle \tag{310}$$

and using Eq. (310) in Eq. (309)

$$\Delta x(i+1,\ n)\rangle_s \quad = FS_b^{-T}\ B(1)S_b^T\ F^{-1}\ \Delta x(i,\ n)\rangle_s \tag{311}$$

or the state transition matrix

$$\Phi_{\Delta\Delta}(i+1,\ i) = FS_b^{-T}\ B(1)S_b^T\ F^{-1} \tag{312}$$

Many other relations can be obtained but will not be pursued further here. One can consider the analog between the continuous monomial base and the discrete "Newton base" for computational efficiency.

Section 7

DIVIDED DIFFERENCES AND INTERPOLATING POLYNOMIALS

This section derives some classical interpolating polynomials in state space format with some interesting matrix relations.

a. LaGrange Interpolating Polynomials

Consider a polynomial in time

$$f(t) = \langle t \; a \rangle \qquad (1)$$

and the n+1 equi distance points on the time axis $(t_o, t_1, t_2, \ldots t_n)$. Define the polynomial

$$\Psi(t) = \prod_{i=o}^{n}(t-t_i) \qquad (2)$$

or

$$\Psi(t) = (t-t_o)(t-t_1)(t-t_2). \; . \; .(t-t_n) \qquad (3)$$

Consider next the n+1 polynomials generated by the index i, that is

$$\Psi_o(t) = t-t_o = (-t_o, 1) \begin{bmatrix} 1 \\ t \end{bmatrix} \qquad (4)$$

$$\Psi_1(t) = (t-t_o)(t-t_1) = t^2 - t_1 t - t_o t + t_1 t_o \qquad (5)$$

$$= (t_1 t_o, -(t_o+t_1), 1) \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix} \qquad (6)$$

$$\Psi_2(t) = (t-t_o)(t-t_1)(t-t_2)$$

or

$$\Psi_2 = [-t_o t_1 t_2, \; t_o t_1 + t_o t_2 + t_1 t_2, \; -(t_o+t_1+t_2)] \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} \qquad (7)$$

and

$$\Psi_3(t) = (t-t_o)(t-t_1)(t-t_2)(t-t_3) \qquad (8)$$

or

$$\Psi_3 = \langle 5)t \begin{bmatrix} t_0t_1t_2t_3 \\ -(t_0t_1t_2+t_0t_1t_3+t_1t_2t_3) \\ t_0t_1+t_0t_2+t_0t_3+t_1t_2+t_1t_3+t_2t_3 \\ -(t_0+t_1+t_2+t_3) \\ 1 \end{bmatrix} \tag{9}$$

where

$$(1,t,t^2,t^3,t^4) = \langle 5)t \tag{10}$$

Define the row-tuple of polynomials

$$\langle 5)\theta(t) = (\theta_0,\theta_1,\theta_2,\theta_3,\theta_4) = (1,\Psi_0,\Psi_1,\Psi_2,\Psi_3) \tag{11}$$

By Eq (4) through Eq (10) we have

$$\langle 5)\ t\ B = \langle 5)\ \theta\ (t) \tag{12}$$

where the constant matrix B is the matrix of coordinates of the $\langle\theta$ base in the monomial base $\langle t$, and for the 5x5 example case here looks like

$$\theta(t)\rangle = B^T t\rangle \tag{13}$$

with

$$\begin{bmatrix} 1 \\ t-t_0 \\ (t-t_0)(t-t_1) \\ (t-t_0)(t-t_1)(t-t_2) \\ (t-t_0)(t-t_1)(t-t_2)(t-t_3) \end{bmatrix} = B^T \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{bmatrix} = \theta(t)(5\rangle \tag{14}$$

B =

$$\begin{bmatrix} 1 & -t_0 & t_0t_1 & -t_0t_1t_2 & t_0t_1t_2t_3 \\ 0 & 1 & -(t_0+t_1) & (t_0t_1+t_0t_2+t_1t_2) & -(t_0t_1t_2+t_0t_1t_3+t_0t_2t_3+t_1t_2t_3) \\ 0 & 0 & 1 & -(t_0+t_1+t_2) & t_0t_1+t_0t_2+t_0t_3+t_1t_2+t_1t_3+t_2t_3 \\ 0 & 0 & 0 & 1 & -(t_0+t_1+t_2+t_3) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{15}$$

One could take t_i in Eq (15) to be the integers and obtain some interesting matrix relations.

Observe that the vectors $\theta\rangle$ are a base when B is the invertible, that is by proper selection of the points t_i. Note also that the vectors θ_i become zero as a function of t when $t = t_i$; thus they act like a base only between the t_i points.

Define a new set of polynomials

$$g_k(t) = \prod_{\substack{i=o \\ i\neq k}}^{n} (t-t_i) \qquad (16)$$

$$k = o,1,2,\ldots n$$

For example for n=3, these four polynomials are

$$\begin{aligned} g_o &= (t-t_1)(t-t_2)(t-t_3) \\ g_1 &= (t-t_o)(t-t_2)(t-t_3) \\ g_2 &= (t-t_o)(t-t_1)(t-t_3) \\ g_3(t) &= (t-t_o)(t-t_1)(t-t_2) \end{aligned} \qquad (17)$$

Clearly these polynomials can be written as

$$\begin{bmatrix} g_o(t) \\ g_1(t) \\ g_2(t) \\ g_3(t) \end{bmatrix} = \begin{bmatrix} (t-t_o)^{-1} \\ (t-t_1)^{-1} \\ (t-t_2)^{-1} \\ (t-t_3)^{-1} \end{bmatrix} \quad \Psi_3(t) \qquad (18)$$

These polynomials evaluated at the t_i points are different from zero, that is for distinct t_i, we have

$$g_k(t_k) = (t_k-t_o)\ \ldots\ (t_k-t_{k-1})(t_k-t_{k+1})\ \ldots\ (t_k-t_n) \neq 0 \qquad (19)$$

and for the four polynomials

$$\begin{aligned} g_o(t_o) &= (t_o-t_1)(t_o-t_2)(t_o-t_3) \\ g_1(t_1) &= (t_1-t_o)(t_1-t_2)(t_1-t_3) \\ g_2(t_2) &= (t_2-t_o)(t_2-t_1)(t_2-t_3) \\ g_3(t_3) &= (t_3-t_o)(t_3-t_1)(t_3-t_2) \end{aligned} \qquad (20)$$

Multiplying the terms of Eq (17), the connection between the polynomials and the monomial base is

$$\begin{bmatrix} g_o(t) \\ g_1(t) \\ g_2(t) \\ g_3(t) \end{bmatrix} = \begin{bmatrix} -t_1t_2t_3 & t_1t_2+t_1t_3+t_2t_3 & -(t_1+t_2+t_3) & 1 \\ -t_ot_2t_3 & t_ot_2+t_ot_3+t_2t_3 & -(t_o+t_2+t_3) & 1 \\ -t_ot_1t_3 & t_ot_1+t_ot_3+t_1t_3 & -(t_o+t_1+t_3) & 1 \\ -t_ot_1t_2 & t_ot_1+t_ot_2+t_1t_2 & -(t_o+t_1+t_2) & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} \qquad (21)$$

or

$$g(t)(4\rangle = G\ t(4\rangle \qquad (22)$$

and

$$t(4\rangle = G^{-1}\ g(t)(4\rangle \qquad (23)$$

The inverse G^{-1} is obtained later. Note as before that $g(t)$ forms a base except at the t_i points, since

$$g_k(t_i) = \begin{cases} g_k(t_k) \neq 0 & i = k \\ 0 & i \neq k \end{cases} \qquad (24)$$

Define now a new set of polynomials

$$\ell_k(t) = \frac{g_k(t)}{g_k(t_k)} = \prod_{\substack{i=o \\ i\neq k}}^{n} \frac{(t-t_i)}{(t_k-t_i)} \qquad (25)$$

The first four of these look like

$$\begin{bmatrix} \ell_o(t) \\ \ell_1(t) \\ \ell_2(t) \\ \ell_3(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{g_o(t_o)} & 0 & 0 & 0 \\ 0 & \frac{1}{g_1(t_1)} & 0 & 0 \\ 0 & 0 & \frac{1}{g_2(t_2)} & 0 \\ 0 & 0 & 0 & \frac{1}{g_3(t_3)} \end{bmatrix} \begin{bmatrix} g_o(t) \\ g_1(t) \\ g_2(t) \\ g_3(t) \end{bmatrix} \qquad (26)$$

or $\ell_k(t)$ looks like

$$\ell_o(t) = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_o-t_1)(t_o-t_2)(t_o-t_3)}$$

$$\ell_1(t) = \frac{(t-t_o)(t-t_2)(t-t_3)}{(t_1-t_o)(t_1-t_2)(t_1-t_3)}$$

$$\ell_2(t) = \frac{(t-t_o)(t-t_1)(t-t_3)}{(t_2-t_o)(t_2-t_1)(t_2-t_3)}$$

$$\ell_3(t) = \frac{(t-t_o)(t-t_1)(t-t_2)}{(t_3-t_o)(t_3-t_1)(t_3-t_2)}$$

Note that

$$\begin{bmatrix} \ell_o(t_o) \\ \ell_1(t_1) \\ \ell_2(t_2) \\ \ell_3(t_3) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{27}$$

or

$$\ell_k(t_i) = \frac{g_k(t_i)}{g_k(t_k)} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \tag{28}$$

Eq (26) can be written as

$$\ell(t)(4\rangle = D^{-1}(g_k)\ g(t)(4\rangle \tag{29}$$

Using Eq (22) in Eq (28)

$$\ell(t)(4\rangle = D^{-1}(g_k)\ G\ t(4\rangle \tag{30}$$

also

$$[\ell(t_o)\rangle\ , \ell(t_1)\rangle , \ell(t_2)\rangle , \ell(t_3)\rangle] = I_{4x4} \tag{31}$$

One can now obtain the derivatives and higher derivativei of the polynomials of Eq (30) as

$$\dot{\ell}(t)\rangle = D^{-1}(g_k)\ G\ V_t\ t\rangle \tag{32}$$

$$\vdots$$

$$\ell^{(d-1)}(t)\rangle = D^{-1}(g_k)\ G\ V_t^{d-1}\ t\rangle$$

etc.

The relation of Eq (30) provides a nice matrix-theoretic "goodie" for by Eq (31)

$$[\ell(t_o)\rangle\ ,\ \ell(t_1)\rangle ,\ \ell(t_2)\rangle\ ,\ \ell(t_3)\rangle\ ] = D^{-1}(g_k)\ G\ T = I \qquad (33)$$

where T is the <u>vandermonde matrix</u>

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ t_o & t_1 & t_2 & t_3 \\ t_o^2 & t_1^2 & t_2^2 & t_3^2 \\ t_o^3 & t_1^3 & t_2^3 & t_3^3 \end{bmatrix} \qquad (34)$$

or the inverse of the Vandermonde matrix is very loosely constructed to be by Eq (33)

$$T^{-1} = D^{-1}(g_k)\ G \qquad (35)$$

Likewise

$$T = G^{-1}D(g_k) \qquad (36)$$

and

$$TD^{-1}(g_k) = G^{-1} \qquad (37)$$

Hence the matrix of Eq (23) is easily obtained; the open form for the G matrix is given in Eq (21).

Consider now an approximating function

$$p(t) = \langle n+1)a\ \ell(t)(n+1\rangle \qquad (38)$$

where the fitting functions $\ell(t)\rangle$ are those of Eq (30), then an arbitrary function x(t) can be approximated by

$$\mathbf{x}(t) = p(t) + x_\varepsilon(t) \qquad (39)$$

or

$$x(t) = \langle a\quad \ell(t)\rangle + x_\varepsilon(t) \qquad (40)$$

where $x_\varepsilon(t)$ is the approximating error. The function evaluated at the t_i points is

$$[x(t_o),\ x(t_1)\ldots x(t_n) = \langle a\ /\ \ell(t_o)\rangle\ ,\ \ldots\ell(t_n)] + \langle x_\varepsilon \qquad (41)$$

or by equation (31) in Eq (41)

$$\langle n+1)x = \langle a\ I + \langle x_{\varepsilon} \qquad (42)$$

If the function values are known at the points t_i, apply the n+1 constraints

$$\langle n+1)x = \langle a \qquad (43)$$

or using Eq (43) in Eq (40)

$$x(t) = \langle n+1)x\ \ell(t)(n+1\rangle + x_{\varepsilon}(t) \qquad (44)$$

and by Eq (38)

$$p(t) = \langle n+1)x\ \ell(t)(n+1\rangle \qquad (45)$$

which is called the LaGrange form of the interpolation polynomial.

One can prove a standard theorem which states: Theorem: Given a real valued function x(t) and n+1 distinct points $t_o, t_1 \ldots t_n$, there exists exactly one polynomial of degree $\leq n$ which interpolates x(t) at $t_o, t_1, \ldots t_n$.

By Eq (18)

$$\Psi(t) = [(t-t_o),(t-t_1),\ldots(t-t_n)]\ g(t) \qquad (46)$$

$$= \sum_{i=1}^{m} (t-t_i)g_i(t) \qquad (47)$$

By Eq (2)

$$\Psi(t) = (t-t_o)(t-t_1)\ldots(t-t_n) \qquad (48)$$

and the derivative is

$$\Psi(t) = (t-t_1)\ldots(t-t_n) + (t-t_o)(t-t_2)\ldots+(t-t_o)(t-t_1)\ldots(t-t_{n-1})$$

or

$$\Psi(t) = g_o(t) + g_1(t) +\ldots g_n(t) \qquad (49)$$

$$\Psi(t) = \langle 1\ g(t)\rangle \qquad (50)$$

If we evaluated the derivative at the t_i points by Eq (24) we obtain

$$\begin{aligned} \dot{\Psi}(t_o) &= g_o(t_o) \\ \dot{\Psi}(t_1) &= g_1(t_1) \\ &\vdots \\ \dot{\Psi}(t_n) &= g_n(t_n) \end{aligned} \tag{51}$$

or

$$[\dot{\Psi}(t_o), \dot{\Psi}(t_1), \ . \ . \ . \ \dot{\Psi}(t_n)] = (g_o(t_o) \ldots g_n(t_n)) \tag{52}$$

Using Eq (51) in Eq (26) one obtains

$$\begin{bmatrix} \ell_o(t) \\ \ell_1(t) \\ \vdots \\ \ell_n(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\dot{\Psi}(t_o)} & & 0 \\ & \frac{1}{\dot{\Psi}(t_1)} & \\ & & \ddots \\ 0 & & \frac{1}{\dot{\Psi}(t_n)} \end{bmatrix} \begin{bmatrix} g_o(t) \\ g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix} \tag{53}$$

and by Eq (18) in Eq (53)

$$\begin{bmatrix} \ell_o(t) \\ \ell_1(t) \\ \vdots \\ \ell_n(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\dot{\Psi}(t_o)} & & 0 \\ & \frac{1}{\dot{\Psi}(t_1)} & \\ & & \ddots \\ 0 & & \frac{1}{\dot{\Psi}(t_n)} \end{bmatrix} \begin{bmatrix} (t-t_o)^{-1} \\ (t-t_1)^{-1} \\ \vdots \\ (t-t_n)^{-1} \end{bmatrix} \psi(t) \tag{54}$$

or

$$\begin{bmatrix} \ell_o(t) \\ \ell_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \ell_n(t) \end{bmatrix} = \begin{bmatrix} \dfrac{\Psi(t)}{\dot{\Psi}(t_o)(t-t_o)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\Psi(t)}{\dot{\Psi}(t_n)(t-t_n)} \end{bmatrix} \tag{55}$$

Using Eq (55) in Eq (45), the interpolating polynomial is seen to be

$$p(t) = [x(t_o), x(t_1) \; . \; . \; . \; x(t_n)] \begin{bmatrix} \dfrac{\Psi(t)}{\dot{\Psi}(t_o)(t-t_o)} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\Psi(t)}{\dot{\Psi}(t_n)(t-t_n)} \end{bmatrix} \tag{56}$$

or

$$p(t) = \sum_{i=o}^{n} x(t_i) \; \frac{\Psi(t)}{\dot{\Psi}(t_i)(t-t_i)} \tag{57}$$

By Eq (29)

$$\ell(t)\rangle = D^{-1}(g_k) \; g(t)\rangle \tag{58}$$

$$\langle 1 \; \ell(t)\rangle = \langle 1 \; D^{-1}(g_k) \; g(t)\rangle = 1 \tag{59}$$

for

$$\langle 1 \; \ell(t)\rangle = (g^{-1}(t_o), \; g^{-1}(t_1)\ldots g^{-1}(t_n) \begin{bmatrix} g_o(t) \\ g_1(t) \\ \cdot \\ \cdot \\ \cdot \\ g_n(t) \end{bmatrix} \tag{60}$$

b. Newton Form of Interpolating Polynomials and Generalized weighted Differences.

Consider some weighted differences of x(i), i=i,2,3,4

or

$$[x(o),\ x(1),\ x(2),\ x(3)] \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = [\Delta x(o),\ \Delta x(1),\ \Delta x(2)] \qquad (61)$$

where

$$[\Delta x(o),\ \Delta x(1),\ \Delta x(2)] = [x(1)-x(o),\ x(2)-x(1),\ x(3)-x(2)] \qquad (62)$$

If we now define the generalized weights of Eq (155) sec (6) as

$$\begin{aligned} (t_o-t_1)^{-1} &= h_{o1}^{-1} = w_{o1} \\ (t_o-t_2)^{-1} &= h_{o2}^{-1} = w_{o2} \\ (t_o-t_3)^{-1} &= h_{o3}^{-1} = w_{o3} \\ (t_1-t_2)^{-1} &= h_{12}^{-1} = w_{12} \\ (t_1-t_3)^{-1} &= h_{13}^{-1} = w_{13} \\ (t_2-t_3)^{-1} &= h_{23}^{-1} = w_{23} \end{aligned} \qquad (63)$$

By Eq (63) the weighted difference matrix is

$$[x(o),x(1),x(2),x(3)] \begin{bmatrix} -w_{o1} & 0 & 0 \\ w_{o1} & -w_{12} & 0 \\ 0 & w_{12} & -w_{23} \\ 0 & 0 & w_{23} \end{bmatrix} \qquad (64)$$

$$= [\frac{x(1)-x(o)}{t_1-t_o},\quad \frac{x(2)-x(1)}{t_2-t_1},\quad \frac{x(3)-x(2)}{t_3-t_2}]$$

Define the first divided differences of Eq (64) via the double index

$$\begin{bmatrix} \dfrac{x(1)-x(o)}{t_1-t_o} \\ \dfrac{x(2)-x(1)}{t_2-t_1} \\ \dfrac{x(3)-x(2)}{t_3-t_2} \end{bmatrix} \equiv \begin{bmatrix} x(o,1) \\ x(1,2) \\ x(2,3) \end{bmatrix} = \begin{bmatrix} \Delta_{ho}x(o) \\ \Delta_{h1}x(1) \\ \Delta_{h2}x(2) \end{bmatrix} \qquad (65)$$

Define the second divided differences as

$$[x(o,1),x(1,2),x(2,3)] \begin{bmatrix} -h_{o2}^{-1} & 0 \\ h_{o2}^{-1} & -h_{13}^{-1} \\ 0 & h_{13}^{-1} \end{bmatrix}$$

$$= \left[\frac{x(1,2)-x(o,1)}{t_2-t_o} , \quad \frac{x(2,3)-x(1,2)}{t_3-t_1}\right] \qquad (66)$$

$$= [x(o,1,2) \, , \, x(1,2,3)] \qquad (67)$$

The successive products, Eq (64) and Eq (66) is

$$(x_o,x_1,x_2,x_3) \begin{bmatrix} -h_{o1}^{-1} & 0 & 0 \\ h_{o1}^{-1} & -h_{12}^{-1} & 0 \\ 0 & h_{12}^{-1} & -h_{23}^{-1} \\ 0 & 0 & h_{23}^{-1} \end{bmatrix} \begin{bmatrix} -h_{o2}^{-1} & 0 \\ h_{o2}^{-1} & -h_{13}^{-1} \\ & \\ 0 & h_{13}^{-1} \end{bmatrix}$$

$$= (x_o,x_1,x_2,x_3) \begin{bmatrix} h_{o1}^{-1}\, h_{o2}^{-1} & 0 \\ -h_{o1}^{-1}\, h_{o2}^{-1} -h_{12}^{-1}\, h_{o2}^{-1} & h_{12}^{-1}\, h_{13}^{-1} \\ h_{12}^{-1}\, h_{o2}^{-1} & -h_{12}^{-1}\, h_{13}^{-1} -h_{23}^{-1}\, h_{13}^{-1} \\ 0 & h_{23}^{-1}\, h_{13}^{-1} \end{bmatrix} \qquad (68)$$

or

$$[x(o,1,2),x(1,2,3)] = \langle 4)x \underset{4x3}{H} \underset{3x2}{H} \quad [\Delta^2_{h_{2o}} x(o),\ \Delta^2_{h_{31}} x(1)] \tag{69}$$

The elements of Eq (68) can be simplified for example

$$-h_{o1}^{-1}\, h_{o2}^{-1}\, -h_{12}^{-1}\, h_{o2}^{-1} = -\frac{1}{(t_o-t_1)} \quad \frac{1}{(t_2-t_o)} \quad - \quad \frac{1}{(t_2-t_1)(t_2-t_o)}$$

$$= \quad - \quad \frac{1}{(t_1-t_o)(t_2-t_1)} \tag{70}$$

etc.

The matrix product of Eq (69) has elements

$$\underset{4x3}{H}\ \underset{3x2}{H} = \begin{bmatrix} \frac{1}{(t_1-t_o)(t_2-t_o)} & 0 \\ -\frac{1}{(t_1-t_o)(t_2-t_1)} & \frac{1}{(t_2-t_1)(t_3-t_1)} \\ \frac{1}{(t_2-t_o)(t_2-t_1)} & -\frac{1}{(t_2-t_1)(t_3-t_2)} \\ 0 & \frac{1}{(t_3-t_1)(t_3-t_2)} \end{bmatrix} \tag{71}$$

If we now take the third difference

$$\Delta^3_{h_{o3}} x(o) = \frac{x(o,1,2)-x(1,2,3)}{t_3-t_o} = \frac{\Delta^2_{h_{31}} x(1) - \Delta^2_{h_{2o}} x(o)}{t_3-t_o} \tag{72}$$

or

$$\Delta^3_{h_{o3}} x(o) = [x(o,1,2),x(1,2,3)] \begin{bmatrix} -\frac{1}{h_{o3}} \\ \frac{1}{h_{o3}} \end{bmatrix} \tag{73}$$

and successive matrix product yield

$$\underset{\substack{\text{4x3 3x2}\\ \text{2x1}}}{H\ H\ H} = \begin{bmatrix} \frac{1}{(t_o-t_1)(t_o-t_2)(t_o-t_3)} \\ \\ \frac{1}{(t_1-t_o)(t_1-t_2)(t_1-t_3)} \\ \\ \frac{1}{(t_2-t_o)(t_2-t_1)(t_2-t_3)} \\ \\ \frac{1}{(t_1-t_o)(t_1-t_2)(t_1-t_3)} \end{bmatrix} = \begin{bmatrix} g_o^{-1}(t_o) \\ \\ g_1^{-1}(t_1) \\ \\ g_2^{-1}(t_2) \\ \\ g_3^{-1}(t_3) \end{bmatrix} \qquad (74)$$

In a similar manner one can show with i=4 or five data points

$$\underset{\text{5x4 4x3 3x2 2x1}}{<5)x\ H\ H\ H\ H} = x(t_o,t_1,t_2,t_3,t_4) \qquad (75)$$

$$= \Delta^{(5)}_{h_{o4}} x(o)$$

where

$$\underset{\text{5x1}}{H} = \begin{bmatrix} \frac{1}{(t_o-t_1)(t_o-t_2)(t_o-t_3)(t_o-t_4)} \\ \frac{1}{(t_1-t_o)(t_1-t_2)(t_1-t_3)(t_1-t_4)} \\ \frac{1}{(t_2-t_o)(t_2-t_1)(t_2-t_3)(t_2-t_4)} \\ \frac{1}{(t_3-t_o)(t_3-t_1)(t_3-t_2)(t_3-t_4)} \\ \frac{1}{(t_4-t_o)(t_4-t_1)(t_4-t_2)(t_4-t_3)} \end{bmatrix} = \begin{bmatrix} \frac{1}{g_o(t_o)} \\ \frac{1}{g_1(t_1)} \\ \frac{1}{g_2(t_2)} \\ \frac{1}{g_3(t_3)} \\ \frac{1}{g_4(t_4)} \end{bmatrix} \qquad (76)$$

The general expressions

$$x(t_o,t_1,t_2,\ldots t_n) = \sum_{i=o}^{n} \frac{x(t_i)}{g_i(t_i)} \qquad (77)$$

The transformation matrix connecting the function values at the t_i time points to successively higher order differences, that is

$$\begin{bmatrix} x(t_o) \\ x(t_o,t_1) \\ x(t_o,t_1,t_2) \\ x(t_o,t_1,t_2,t_3) \\ x(t_o,t_1,t_2,t_3,t_4) \end{bmatrix} = \begin{bmatrix} x(t_o) \\ \Delta^{(1)}_{h_{1o}}(t_o) \\ \Delta^{(2)}_{h_{2o}}(t_o) \\ \Delta^{(3)}_{h_{3o}}(t_o) \\ \Delta^{(4)}_{h_{4o}}(t_o) \end{bmatrix} = L \begin{bmatrix} x(t_o) \\ x(t_1) \\ x(t_2) \\ x(t_3) \\ x(t_4) \end{bmatrix} \qquad (78)$$

where

$$L^T = \begin{bmatrix} 1 & \frac{1}{t_o-t_1} & \frac{1}{(t_o-t_1)(t_o-t_2)} & \frac{1}{(t_o-t_1)(t_o-t_2)(t_o-t_3)} & \frac{1}{(t_o-t_1)(t_o-t_2)(t_o-t_3)(t_o-t_4)} \\ 0 & \frac{1}{t_1-t_o} & \frac{1}{(t_1-t_o)(t_1-t_2)} & \frac{1}{(t_1-t_o)(t_1-t_2)(t_1-t_3)} & \frac{1}{(t_1-t_o)(t_1-t_2)(t_1-t_3)(t_1-t_4)} \\ 0 & 0 & \frac{1}{(t_2-t_o)(t_2-t_1)} & \frac{1}{(t_2-t_o)(t_2-t_1)(t_2-t_3)} & \frac{1}{(t_2-t_o)(t_2-t_1)(t_2-t_3)(t_2-t_4)} \\ 0 & 0 & 0 & \frac{1}{(t_3-t_o)(t_3-t_1)(t_3-t_2)} & \frac{1}{(t_3-t_o)(t_3-t_1)(t_3-t_2)(t_3-t_4)} \\ 0 & 0 & 0 & 0 & \frac{1}{(t_4-t_o)(t_4-t_1)(t_4-t_2)(t_4-t_3)} \end{bmatrix} \qquad (79)$$

or Eq (79) can be written as

$$\langle 5)\Delta^{(\ )}x(t_o) = \langle 5)\ x\ L^T \tag{80}$$

or

$$\langle 5)\Delta^{(\ )}x(t_o)L^{-T} = \langle 5)\ x \tag{81}$$

the row vector of function values. By Eq (45)

$$p(t) = \langle 5)\ x\ \ \ell(t)\ (5\rangle \tag{82}$$

and Eq (81) in Eq (82)

$$p(t) = \langle 5)\Delta^{(\ )}x(t_o)\ L^{-T}\,\ell(t)(5\rangle \tag{83}$$

For the 4x4 case it is easy to show for example that

$$\begin{bmatrix} 1 & \frac{1}{t_o-t_1} & \frac{1}{(t_o-t_1)(t_o-t_2)} & \frac{1}{(t_o-t_1)(t_o-t_2)(t_o-t_3)} \\ 0 & \frac{1}{t_1-t_o} & \frac{1}{(t_1-t_o)(t_1-t_2)} & \frac{1}{(t_1-t_o)(t_1-t_2)(t_1-t_3)} \\ 0 & 0 & \frac{1}{(t_2-t_o)(t_2-t_1)} & \frac{1}{(t_2-t_o)(t_2-t_1)(t_2-t_3)} \\ 0 & 0 & 0 & \frac{1}{(t_3-t_o)(t_3-t_1)(t_3-t_2)} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & (t_1-t_o) & (t_2-t_o) & (t_3-t_o) \\ 0 & 0 & (t_2-t_o)(t_2-t_1) & (t_3-t_o)(t_3-t_1) \\ 0 & 0 & 0 & (t_3-t_o)(t_3-t_1)(t_3-t_2) \end{bmatrix} \tag{84}$$

The matrix of Eq (84) is the row-tuple of the base polynomials of Eq (14) (that is the first 4)

$$\theta(t)(4\rangle = \begin{bmatrix} 1 \\ t-t_o \\ (t-t_o)(t-t_1) \\ (t-t_o)(t-t_1)(t-t_2) \end{bmatrix} \qquad (85)$$

or

$$[\theta(t_o)(4\rangle,\ \theta(t_1)(4\rangle,\ \theta(t_2)(4\rangle,\ \theta(t_3)(4\rangle] = \underset{4x4}{L^{-T}} \qquad (86)$$

The standard Newton-form of the polynomial of Eq (83) is

$$p(t) = \sum_{i=o}^{n} x(t_o,t_1,\ldots,t_i) \prod_{j=o}^{i-1} (t-t_j) \qquad (87)$$

It is believed that the foregoing state-space methods applied to classical interpolation, and extended to cubic-splines, B-splines, etc. will yield beautiful simplifications.