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State Space Techniques In Approximation Theory With Applications To Design Of Recursive Optimal Estimators

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SECTION 8

GENERALIZATIONS ON DERIVATIVES OF BASE FITTING FUNCTIONS AND TRANSITION MATRICES

The first derivative of the Monomial Base is given by Eq. (22) or Eq. (26) as

$$\langle \dot{t} = \langle t\; V \tag{1}$$

where

$$V = \sum_{j=1}^{d-1} j\; E_{j(j+1)} \tag{2}$$

or

$$\underset{d\times d}{V} = \begin{bmatrix} 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 2 & 0 & & & \\ 0 & 0 & 0 & 3 & 0 & & \\ & & & & \ddots & & \\ & & & & & & d-1 \\ & & & & & & 0 \end{bmatrix} \tag{3}$$

For an arbitrary base we have

$$\langle \dot{f}(t) = \langle f(t)\; V_f \tag{4}$$

Where most cases considered in this report the velocity matrix V_f will be a constant. When it is not a constant, higher derivatives will yield

$$\langle \ddot{f}(t) = \langle \dot{f}(t) V_f + \langle f(t) \dot{V}_f \tag{5}$$

or using Eq. (4) in Eq. (5)

$$\langle \ddot{f}(t) = \langle f(t) \left[V_f^2 + \dot{V}_f \right] \tag{6}$$

with similar expressions for higher terms. When V_f is a constant the higher derivative becomes

$$\begin{bmatrix} \langle f(t) \\ \langle \dot{f}(t) \\ \langle \ddot{f}(t) \\ \vdots \\ f^{(i)}(t) \\ \vdots \\ f^{(d-1)}(t) \end{bmatrix} = \begin{bmatrix} f(t) \\ f(t)\; V_f \\ f\; V_f^2 \\ \cdot \\ f\; V_f^i \\ \cdot \\ f\; V_f^{d-1} \end{bmatrix} \tag{7}$$

These higher derivatives occur in state-variable approximations, for example

$$x(t) = \langle f(t)\, a \tag{8}$$

and the higher derivatives are

$$\begin{bmatrix} x(t) \\ \ddot{x}(t) \\ \cdot \\ \cdot \\ \cdot \\ x^{(d-1)}(t) \end{bmatrix} = \begin{bmatrix} f(t) \\ \dot{f}^{(t)} \\ \cdot \\ \cdot \\ \cdot \\ f^{(d-1)}(t) \end{bmatrix} a \rangle \tag{9}$$

Eq. (4) has the solution

$$\langle f(t+to) = \langle f(to)\ e^{V_f(t-to)} \tag{10}$$

where the transition matrix

$$\Phi_{ff}(t+to,\ to) = e^{V_f(t-to)} = I + V_f(t-to) + V_f^2\frac{(t-to)^2}{2!}\ . \ . \ . \tag{11}$$

or for small transitions $t\text{-}to = \Delta t$

$$\Phi_{ff}(t+\Delta t,\ t) = I + V_f\Delta t + V_f^2\frac{\Delta t^2}{2!} + \ . \ . \ . \tag{12}$$

and for linear-terms only

$$\Phi_{ff}(t+\Delta t,\ t) = I + V_f\Delta t \tag{13}$$

Returning to the monomial base of Eq. (1) and Eq. (3) case we have designating

$$\langle t \equiv \langle m(t) \tag{15}$$

$$\langle \dot{m} = \langle m(t)\ V \tag{16}$$

and

$$\langle m(t) = \quad m(o)\, e^{Vt} = \langle m(o)\ \Phi_m(t,o) \tag{17}$$

where

$$\Phi_t(t,o) = \Phi_m(t,o) = B(t) = D^{-1}(t)\ B(1)\ D(t) \tag{18}$$

and the t-binomial matrix B(t) is given by Eq. (158) appendix (A) as

$$e^{Vt} = B(t) = \begin{bmatrix} 1 & t & t^2 & t^3 & \cdots \\ 0 & 1 & 2t & 3t^2 & \\ 0 & 0 & 1 & 3t & \\ & & 0 & 1 & \\ & & & 0 & \\ & & & \cdot & \\ & & & \cdot & \end{bmatrix} = \Phi_t \tag{19}$$

we also have

$$\begin{aligned} \Phi_m(t) &= B(t) = e^{Vt} \\ \dot{\Phi}_m(t) &= \dot{B}(t) = V e^{Vt} = V\Phi_m(t) \\ \ddot{\Phi}_m(t) &= \ddot{B}(t) = V^2 e^{Vt} = V^2\Phi_m(t) \\ &\vdots \\ \Phi_m^{(d-1)}(t) &= B^{(d-1)}(t) = V^{d-1} e^{Vt} = V^{d-1}\Phi_m(t) \end{aligned} \tag{20}$$

where the nilpotent condition holds

$$V_m^d = 0_{dxd} \tag{21}$$

Consider the Poisson polynomials defined by Eq. (134)

$$\langle p(t)| = \langle t|B^{-1} = \langle m(t)|B^{-1} \tag{22}$$

Taking the first derivative

$$\langle \dot{p}(t)| = \langle tV|B^{-1} \tag{23}$$

or by Eq. (22) in Eq. (23)

$$\langle \dot{p}(t)| = \langle p(t)|B\,V\,B^{-1} \tag{24}$$

or

$$\langle \dot{p}(t)| = \langle p(t)|V_p \tag{25}$$

hence the velocity matrices are similar

$$V_p = \mathbb{F} V \mathbb{F}^{-1} \tag{26}$$

When the diagonal factorial matrix of Eq. (25) Section (B) is used in Eq. (26) are obtained

$$V_p = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & & & \\ 0 & 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & 0 & 1 & 0 & & & & \\ 0 & 0 & 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & & \\ & & \cdot & & & & & & \\ & & \cdot & & & & & & \\ & & \cdot & & & & & & \\ & & & & & & & 0 & 1 \\ 0 & 0 & 0 & & & & & 0 & 0 \end{bmatrix} \quad S_{uo} \tag{27}$$

Which is the shift-matrix of Eq. (32) Section (4).

Eq. (27) is the companion matrix for the dynamical system described by the d-th derivative constant.

$$x^{(d)}(t) = \text{constant} \tag{28}$$

The transition matrix for the Poisson base is

$$\phi_p(t) = e^{V_p t} = e^{\mathbb{F} V \mathbb{F}^{-1} t} = \mathbb{F}\, e^{Vt}\, \mathbb{F}^{-1} \tag{29}$$

or by Eq. (19)

$$\Phi_p(t) = \mathbb{F}\; B(t)\; \mathbb{F}^{-1} \tag{30}$$

Utilizing the diagonal matrix of integer factorials and its inverse in Eq. (30) one obtains

$$\phi_p(t) = \begin{bmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \\ & & & 0 \\ & & & \cdot \\ & & & \cdot \\ & & & \cdot \end{bmatrix} \tag{31}$$

which is the Taylor-Series state transition matrix of Eq. (177) Section (4).

For linear terms in Δt the transition matrix from t to $t + \Delta t$ is by Eq. (13) or Eq. (31).

$$\Phi_p(t+\Delta,\ t) \cong \begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & \Delta t & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (32)$$

and for $\Delta t = 1$

$$\Phi_p(t+1,\ t) \cong \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (33)$$

Equation (33) has as inverse

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (34)$$

The expression of Eq. (29) is easy to show, for by Eq. (17)

$$\langle m(t) = \langle m(o)\, e^{Vt} \qquad (35)$$

and by Eq. (22)

$$\langle p(t)\, \mathcal{F} = \langle p(0)\, \mathcal{F}\, e^{Vt} \qquad (36)$$

or

$$\langle p(t) = \langle p(0)\, \mathcal{F}\, e^{Vt} \mathcal{F}^{-1} \qquad (37)$$

and by Eq. (25)

$$\langle p(t) = \langle p(0)\, e^{V_p t} \qquad (38)$$

hence

$$V_p = \mathcal{F}\, V\, \mathcal{F}^{-1} \qquad (39)$$

and

$$\Phi_p(t) = \mathcal{F}\, \Phi\, \mathcal{F}^{-1} \qquad (40)$$

Consider the dyadic product of the monomial base under differentiation

$$\frac{d}{dt}\ [m(t)\rangle\langle m(t)] = \dot{m}\rangle\langle m + m\rangle\langle \dot{m} \qquad (41)$$

By Eq. (16)

$$\langle \dot{m} = \langle mV \tag{42}$$

and transposing

$$\dot{m}\rangle = V^T m\rangle \tag{43}$$

On the (0,1) interval under inner-product Eq. (39) and (40) yield

$$\int_0^1 m\rangle\langle \dot{m} = \int_0^1 m\rangle\langle mdtV = H_{iee} V \tag{44}$$

and

$$\int_0^1 \dot{m}\rangle\langle mdt = V^T \int_0^1 m\rangle\langle mdt = V^T H_{iee} \tag{45}$$

or

$$\int_0^1 \frac{d}{dt}(m\rangle\langle m)dt = H_{iu}V + V^T H_{iee} \tag{46}$$

Consider the states in the monomial base given by Eq. (150) Section (4)

$$x(t)\langle d\rangle = T_u(t)a\langle d\rangle \tag{47}$$

where by Eq. (151) and Eq. (75) Section (4)

$$T_v(t) = \begin{bmatrix} \langle t \\ \langle tV \\ \langle tV^2 \\ \cdot \\ \cdot \\ \cdot \\ \langle tV^{d-1} \end{bmatrix} \tag{48}$$

and for 5 x 5 case we have

$$T_v(t) = \begin{bmatrix} 1 & t & t^2 & t^3 & t^4 \\ 0 & 1 & 2t & 3t^2 & 4t^3 \\ 0 & 0 & 2 & 2\cdot 3t & 2\cdot 3\cdot 4t \\ 0 & 0 & 0 & 0 & 2\cdot 3\cdot 4 \end{bmatrix} \tag{49}$$

The matrix $T_v(t)$ can be expressed in terms of known factors as

$$T_v(t) = \mathfrak{F}\, B(t) \tag{50}$$

and hence the inverse is:

$$T_v^{-1}(t) = B(-t)\, \mathfrak{F}^{-1} \tag{51}$$

Taking the derivative of Eq. (47)

$$\dot{x}(t)\rangle = \dot{T}_v(t)\, a\rangle \tag{52}$$

and

$$\dot{T}_v(t) = T_v(t)V \tag{53}$$

and Eq. (53) in Eq. (52) yields.

$$\dot{x}(t)\rangle = T_v(t)V\, a\rangle \tag{54}$$

Solving Eq. (47) for the constant vector $a\rangle$

$$a\rangle = T_v^{-1}(t)\, x(t)\rangle \tag{55}$$

and using Eq. (55) in Eq. (54)

$$\dot{x}(t)\rangle = T_v(t)\, V\, T_v^{-1}(t)\, x(t)\rangle \tag{56}$$

or

$$\dot{x}(t)\rangle = V_x\, x(t)\rangle \tag{57}$$

where

$$V_x = T_v(t)V\, T_v^{-1}(t) \tag{58}$$

Using Eq. (50) and Eq. (51) in Eq. (58)

$$V_x = A\, B(t)\, V\, B^{-1}(t)\, A^{-1} \tag{59}$$

Consider next the inner-similarity transformation of Eq. (59), namely

$$B(t)VB^{-1}(t) = V(t) \tag{60}$$

Eq. (60) can be shown for the 3 x 3 case as

$$\begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -t & t^2 \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \tag{61}$$

Solving Eq. (57) for the states

$$x(t)\rangle = e^{V_x t}\, x(o)\rangle \tag{62}$$

or using Eq. (59) and (60) in Eq. (62)

$$x(t)\rangle = e^{\not{P}\, V\, \not{P}^{-1}}\, x(o)\rangle \tag{63}$$

or

$$x(t)\rangle = \Phi_x(t,o)\; x(o)\rangle \tag{64}$$

By the analogy of Eq. (29)

$$\Phi_x(t,o) = e^{\not{P}\, V\, \not{P}^{-1} t} = \not{P}\, e^{Vt}\, \not{P}^{-1} \tag{65}$$

$$\Phi_x(t,o) = \not{P}\, \Phi_t\, \not{P}^{-1} = \not{P}\, B(t)\, \not{P}^{-1} \tag{66}$$

Note that the V_x velocity matrix of Eq. (63) is the same as the Poisson velocity matrix of Eq. (27) and is

$$V_x = \not{P} V \not{P}^{-1} = \begin{bmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & 0 & \\ & & \cdot & & & \\ & & \cdot & & & \\ & & \cdot & & & \\ 0 & & 0 & & 0 & 1 \\ 0 & & 0 & & & 0 \end{bmatrix} \tag{67}$$

Eq. (67) is obvious for if one sets

$$x^{(d)}(t) = x^{(d)}(o) \tag{68}$$

one obtains the state equation

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ \cdot \\ \cdot \\ \dot{x}^{(d)}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & 0 & & \\ & \vdots & & & & \\ 0 & 0 & & & 0 & 1 \\ 0 & 0 & & & 0 & 0 \end{bmatrix} x\rangle + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \dot{x}^{(d)}(t) \end{bmatrix} \tag{69}$$

Velocities for Gram Orthogonal Polynomial Base

The orthogonal polynomials on the interval (0,1) are given by Eq. (336) Section (1) for the 3 x 3 case as

$$\langle g = \langle t \begin{bmatrix} 1 & -1/2 & 1/6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \langle tBg \tag{70}$$

Taking the time derivative

$$\langle \dot{g} = \langle \dot{t}B_g = \langle tVB_g = \langle gB_g^{-1}VB_g \tag{71}$$

or

$$V_g = B_g^{-1} V B_g \tag{72}$$

The general terms for B_g as functions of the binomial coefficients etc. can be obtained for Section (1) to use in Eq. (71) or Eq. (72).

The orthogonal polynomials on the (0,1) interval are given by Eq. (337) Section (1) as

$$\langle s = \langle t \begin{bmatrix} 1 & -\sqrt{3} & \sqrt{5} \\ 0 & 2\sqrt{3} & -6\sqrt{5} \\ 0 & 0 & 6\sqrt{5} \end{bmatrix} = \langle t B_s \tag{73}$$

and

$$\langle \dot{s} = \langle s \, V_s \tag{74}$$

where

$$V_s = B_s^{-1} \, V \, B_s \tag{75}$$

=

The inverse B_s^{-1} is given by Eq. (360) Section (1) hence

$$V_s = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & \sqrt{3}/6 & \sqrt{3}/6 \\ 0 & 0 & \sqrt{5}/30 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\sqrt{3} & \sqrt{5} \\ 0 & 2\sqrt{3} & -6\sqrt{5} \\ 0 & 0 & 6\sqrt{5} \end{bmatrix} \tag{76}$$

or

$$V_s = \begin{bmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{3}\sqrt{5} \\ 0 & 0 & 0 \end{bmatrix} \tag{77}$$

One also has

$$\langle s = \langle g \, M_{gg}^{-\frac{1}{2}} = \langle t \, B_g M_{gg}^{-\frac{1}{2}} \tag{78}$$

hence

$$\langle \dot{s} = \langle s V_s \tag{79}$$

where

$$V_s = M_{gg}^{\frac{1}{2}} \, B_g^{-1} V B_g M_{11}^{-\frac{1}{2}} \tag{80}$$

The solution is

$$\langle s(t) = \langle s(o) \, e^{V_s t} \tag{81}$$

or

$$\langle s(t) = \langle s(o) \, B_s^{-1} e^{Vt} B_s \tag{82}$$

or

$$\langle s(t) = \langle s(o)\, B_s^{-1}\, B(t)\, B_s \tag{83}$$

where the t-binomial matrix is given by Eq. (19). Similar expressions are obvious for $\langle g\ (t)$.

Time Derivatives of the Legendre Polynomials (Modified). The classical Legendre polynomials or the interval (-1,1) are given by Eq. (209) Section (2) and for the unit upper triangular case called the Modified Legendre polynomials of Eq. (205) Section (2) we have for the 3 x 3 case.

$$\langle g = \langle t \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \langle tB_g \tag{84}$$

The ensuing analytical terms are exactly the same as for the Gram case except the matrix entrees are different. We obtain for example

$$V_g = B_g^{-1} V\, B_g \tag{85}$$

and by Eq. (84) and Eq. (22) Section (4)

$$V_g = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{86}$$

or

$$V_g = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = V_m \tag{87}$$

a surprising result which may not be valid for the d x d case, since

$$\langle g(t) = \langle g(o) B_g^{-1} e^{Vt} B_g \tag{88}$$

or by Eq. (85)

$$\langle g(t) = \langle g(o)\, e^{Vt} = \langle g(o) B(t) \tag{89}$$

The 3 x 3 case is valid for

$$B_g^{-1} B(t)\, B_g = \underset{3x3}{B(t)}$$

that is

$$\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B(t) \tag{90}$$

a few further observations will be made with respect to the derivatives of the monomial base. By Eq. (24) Section (B)

$$j^{(i)} = j(j-1)(j-2) \; . \; . \; . \; (j-i+1) \tag{91}$$

or

$$j^{(i)} = i! \begin{bmatrix} j \\ i \end{bmatrix} \tag{92}$$

and

$$\frac{d^i}{dt^i}(t^j) = j^{(i)} \; t^{j-i} = i! \begin{bmatrix} j \\ i \end{bmatrix} t^{j-i} \tag{93}$$

The equations of Eq. (45)

$$\begin{bmatrix} \lessdot t \\ \frac{d}{dt} \lessdot t \\ \frac{d^2}{dt^2} \lessdot t \\ \vdots \\ \frac{d^i}{dt^i} \lessdot t \\ \vdots \\ \frac{d^{d-1}}{dt^{d-1}} \lessdot t \end{bmatrix} = \begin{bmatrix} \lessdot t \\ \lessdot tV \\ \lessdot tV^2 \\ \vdots \\ \lessdot tV^i \\ \vdots \\ \lessdot tV^{d-1} \end{bmatrix} = T_v^{(t)} \tag{94}$$

have as i, j th element i, j = 0, 1, 2, . . . d-1 by Eq. (90)

$$T_v(t) = \begin{bmatrix} \frac{d^i(t^j)}{dt^i} \end{bmatrix} = \begin{bmatrix} j^{(i)} t^{j-i} \end{bmatrix} \tag{95}$$

For the nomomial base

$$\lessdot m(t) = (1, \; t, \; t^2 \; . \; . \; . \; t^{d-1} \tag{96}$$

and

$$\lessdot \dot{m}(t) = \lessdot m(t) \; V \tag{97}$$

we obtain elementwise

$$\boxed{\dot{m}_j = j m_{j-1}} \qquad (98)$$

$j = 0, 1, 2, \ldots, d-1$

In order to adequately carry on the direction here one needs a lot of properties and theorems about nilpotent matrix, full rank-upper-triangular, strictly upper-triangular relations under full rank upper triangular similarity etc. These topics as applied lead into spectral aspects of the matrices herein used also; but will not be pursued further here since the prime purpose is to obtain the time domain filter of interest in trajectory estimation. A later paper on frequency response should be done to tie-in the time domain with the frequency domain aspects of optimal estimation.

Scott in reference (80) (1964) says that recently (M. L. Boas 1963) a formula for the derivatives of Legendre polynomials was derived, and that the purpose of his paper (Scotts) was to obtain an analogous formula for the derivatives of Tchebychef polynomials of the second kind by making use of properties of Gegenbauer polynomials.

Morrison on p. 238 of his text (ref 61) states that the discrete Legendre polynomials do not easily permit differentiation, that any attempt to differentiat them completely fractures their structure and all order is thereafter irretrievable lost.

Time Derivatives of the Exponentially Weighted Base and Lagrange Polynomials

The exponentially weighted base is given by

$$\langle t_e = e^{-at/2} \langle t \qquad (99)$$

and the derivative is

$$\frac{d}{dt} \langle t_e = \frac{-a}{2} e^{-at/2} \langle t + e^{-at/2} \dot{\langle} t \qquad (100)$$

$$= e^{-at/2} \langle t [V - a\ I]$$

Let

$$b = \frac{a}{2} \qquad (101)$$

then

$$\langle t_e = e^{-bt} \langle t \qquad (102)$$

and

$$\langle \dot{t}_e = \langle t_e(V - bI) \qquad (103)$$

$$\langle \ddot{t}_e = \langle t_e(V - bI)^2 = \langle t_e V_e^2 \qquad (104)$$

with (for the 4 x 4 case)

$$V_e = \begin{bmatrix} -b & 1 & 0 & 0 \\ 0 & -b & 2 & 0 \\ 0 & 0 & -b & 3 \\ 0 & 0 & 0 & -b \end{bmatrix} \qquad (105)$$

For the exponentially weighted Poisson polynomials, that is

$$\langle p = \langle t\, R^{-1} \qquad (106)$$

and

$$\langle p_e = \langle p\, e^{-bt} \qquad (107)$$

$$\langle \dot{p}_e = \langle p_e V_{pe} \qquad (108)$$

with

$$V_{pe} = \begin{bmatrix} -b & 1 & 0 & 0 \\ 0 & -b & 1 & 0 \\ 0 & 0 & -b & 1 \\ 0 & 0 & 0 & -b \end{bmatrix} \qquad (109)$$

Equation (109) is a well known inverse for

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}^{-1} = \begin{bmatrix} \lambda^{-1} & -\lambda^{-2} & \lambda^{-3} & -\lambda^{-4} \\ 0 & \lambda^{-1} & -\lambda^{-2} & \lambda^{-3} \\ 0 & 0 & \lambda^{-1} & -\lambda^{-2} \\ 0 & 0 & 0 & \lambda^{-1} \end{bmatrix} \qquad (110)$$

and for $\lambda = -b$

$$\begin{bmatrix} -b & 1 & 0 & 0 \\ 0 & -b & 1 & 0 \\ 0 & 0 & -b & 1 \\ 0 & 0 & 0 & -b \end{bmatrix}^{-1} = \begin{bmatrix} -b^{-1} & -b^{-2} & -b^{-3} & -b^{-4} \\ 0 & -b^{-1} & -b^{-2} & -b^{-3} \\ 0 & 0 & -b^{-1} & -b^{-2} \\ 0 & 0 & 0 & -b^{-1} \end{bmatrix} \qquad (111)$$

Since there are so many versions of the orthogonal and orthonomal polynomials, the classical and the modified ones that only two examples will be given here. Consider Eq. (54) which relates the classical Lagrange polynomials to the monomial base.

$$\langle \ell(t) = \langle t D(a^n)\mathbb{R}^{-1}R\mathbb{R} = \langle t T_{t\ell} \tag{112}$$

with derivative

$$\langle \dot{\ell} = \langle t V_t D(a^n)\mathbb{R}^{-1}R\mathbb{R} \tag{113}$$

or the derivative in the monomial base is given by Eq. (113) and in the $\langle \ell$ base by Eq. (112) in Eq. (113)

$$\langle \dot{\ell} = \langle \ell T_{t\ell}^{-1} V_t D(a^n)\mathbb{R}^{-1}R\mathbb{R} \tag{114}$$

or

$$\langle \dot{\ell} = \langle \ell T_{t\ell}^{-1} V T_{t\ell} = \langle \ell V_\ell \tag{115}$$

Powers of V_1 yield

$$V_\ell^d = T_{t\ell}^{-1}\, V_t^d\, T_{t\ell} \tag{116}$$

and since

$$V^d = 0 \tag{117}$$

then

$$V_\ell^d = 0 \tag{118}$$

hence V_ℓ is nilpotent.

Consider next the Lagrange functions of Eq. (55) Section (2)

$$\begin{aligned} \langle \ell_f(t) &= e^{-at/2} \langle t D(a^n)\mathbb{R}^{-1}R\mathbb{R} \\ &= \langle t_e T_{ef} \end{aligned} \tag{119}$$

and by Eq.

$$\langle \dot{\ell}_f = \langle t_e V_e T_{ef} = \langle \ell_f T_{ef}^{-1}\, V_e T_{ef} \tag{120}$$

Since V_e by Eq. (105) is full rank, the powers of

$$V_{\ell f}^d = T_{ef}^{-1}\, V_e^d\, T_{ef} \tag{121}$$

is also full rank.

The trigonometric set

$$\langle f(t) = (\cos \omega_o t,\ \sin \omega_o t,\ \cos 2\omega_o t,\ \sin 2\omega_o t) \tag{122}$$

has a velocity matrix given by

$$\langle \dot{f}(t) = \langle f(t) \begin{bmatrix} 0 & \omega_o & 0 & 0 \\ \omega_o & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\omega_o \\ 0 & 0 & -2\omega_o & 0 \end{bmatrix} \tag{123}$$

or

$$V_\omega = \begin{bmatrix} J\omega_o & 0 \\ 0 & 2\omega_o J \end{bmatrix} \tag{124}$$

where

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{125}$$

and

$$J^4 = I \tag{126}$$

The base transition matrix is given by Eq. (10), Eq. (11) and Eq. (12)

$$\langle f(t+\Delta t) = \langle f(t)\, \Phi(\Delta t) \tag{127}$$

For

$$\Delta t = 1ho$$

that is one step or

$$n\Delta t = nho$$

n steps, we can use the discrete relations of Section (5).

From Eq. (123) Section (5) for i = o

$$\langle t(n) = \langle nT_u(\beta)D(ho) \tag{128}$$

and for i = 1

$$\langle t(n+1) = \langle nT_u(1)T_u(\beta)D(ho) \tag{129}$$

where by Eq. (38) Section (5) for 6 x 6

$$T_u(1) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & 1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{130}$$

Note that $T_u(1)$ commutes with $T_u(\beta)D(ho)$ hence Eq. (129) becomes

$$\langle t(n+1) = \langle t(n)\ T_u(1) = \langle t(n)\Phi_{tt} \tag{131}$$

with the transition matrix

$$\Phi_{tt} = T_u(1) = \Phi_{tt}(n+1,\ n) \tag{132}$$

In a similar manner for updating k points

$$\Phi_{tt}(n+k,\ n) = T_u(k) \tag{133}$$

Consider the case given by Eq. (99)

$$\langle t_e = e^{-at/2} \langle t(n) \tag{134}$$

with

$$t = ho(\beta+n) = t(n) \qquad \beta = \frac{to}{ho} \tag{135}$$

$$t(n+1) = ho(\beta+n+1) \tag{136}$$

and

$$e^{\frac{-at(n+1)}{2}} = e^{\frac{ho(\beta+n)+ho}{2}}$$

Using Eq. (131) and Eq. (136) in Eq. (99) at $t(n+1)$

$$\langle t_e(n+1) = e^{\frac{ho}{2}} \langle t_e(n)\ T_u(1) \tag{137}$$

or

$$\langle t_e(n+1) = \langle t_e(n)\Phi_e \tag{138}$$

with

$$\Phi_e = e^{\frac{ho}{2}}\ T_u(1) \tag{139}$$

and for k data points ahead

$$\langle t_e(n+k) = \langle t_e(n)\ \Phi_e(k) \tag{140}$$

$$\Phi_e(n+k,\ n) = e^{\frac{kho}{2}}\ T_u(k) \tag{141}$$

Note that the transition matrix of Eq. (105)

$$\Phi_e(t,o) = e^{(V-bI)t} = e^{-bt} e^{Vt} \tag{142}$$

Eq. (110) is true because the matrices V and bI commute.

The transition matrix for the monomial base by Eq. (18) is

$$\Phi_e(t,o) = e^{-bt} D^{-1}(t) B(1) D(t) \tag{143}$$

For the exponentially weighted case into the infinite past with zero measurements there one should use the front of the span to the back of span indexing with the back span point going to negative infinity. By Eq. (172) and Eq. (173) Section (5).

$$e^{\frac{aho(\beta+1)}{2}} e^{\frac{-ahoj}{2}} = e^{\frac{\alpha o}{2}} \theta^{\gamma/2} \tag{144}$$

and by Eq. (176) Section (5)

$$\langle f_e(\ell,\gamma) = e^{\alpha o/2} \theta^{\gamma/2} \langle \gamma T_u(\ell) T_u(\beta) D(ho) \tag{145}$$

For μ data point advance of the front of the span, the time becomes by Eq. (170) Section (5)

$$t(\ell+\mu,\ \gamma) = ho + (\beta+\ell+\mu) - ho\gamma \tag{146}$$

and Eq. (112) becomes

$$e^{\frac{aho(\beta+\mu+\ell)}{2}} e^{\frac{-aho\gamma}{2}} = e^{\frac{\alpha o}{2}} \theta^{\frac{\alpha}{2}} \theta^{\frac{-\mu}{2}} \tag{147}$$

The base by Eq. (113) becomes

$$\langle f_e(\ell+\mu,\ \gamma) = e^{\frac{\alpha o}{2}} \theta^{\frac{\alpha}{2}} \theta^{\frac{-\mu}{2}} \langle \gamma T_u(1) T_u(\mu) T_u(\beta) D(ho) \tag{148}$$

and due to the commutitivity of the T_u (upper triangular) matrices in this particular case) one has

$$\langle f_e(\ell+\mu,\ \gamma) = \theta^{-\mu/2} \langle f_e(\ell,\gamma)\ T_u(\mu) \tag{149}$$

$$= \langle f_e(\ell,\gamma)\ \Phi_e$$

where

$$\Phi_e = \theta^{-\mu/2} T_u(\mu) \tag{150}$$

and for one point forward advance one has

$$\Phi_e(\ell+1,\ \gamma) = \theta^{-\frac{1}{2}} T_u(1) \tag{151}$$

and for the 3 x 3 case the exponentially weighted polynomial fitting function transition relation is

$$\angle f_e(\ell+1, \gamma) = \theta^{-\frac{1}{2}} \angle f_e(\ell,\gamma) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad (152)$$

FITTING FUNCTIONS STATE EQUATIONS

The state equations for

$$x(t) = \langle f(t)\ a\rangle \tag{1}$$

are obtained in this section. The derivative of Eq. (1) is

$$\dot{x} = \langle \dot{f}\ a\rangle \tag{2}$$

where

$$\langle \dot{f} = \langle f\ V_f \tag{3}$$

or

$$\dot{x} = \langle f\ V_f a\rangle \tag{4}$$

The second derivative is

$$\ddot{x} = \langle \dot{f}\ V_f\ a\rangle + \langle f\ \dot{V}_f\ a\rangle \tag{5}$$

By Eq. (5) we see we need

$$\langle \dot{f} = \langle f V \tag{6}$$

and

$$\langle \ddot{f} = \langle \dot{f} V + \langle f\ \dot{V} \tag{7}$$

$$= \langle f\ \ [V^2 + \dot{V}] \tag{8}$$

The third derivative is

$$\langle \dddot{f} = \langle \dot{f}\ [V_2 + \dot{V}] + \langle f\ [2V\dot{V} + \ddot{V}]$$

or

$$\langle \dddot{f} = \langle f\ [V^3 + 3V\dot{V} + \ddot{V}] \tag{9}$$

For the many cases of fitting functions discussed in these pages

$$\dot{V} = 0 \tag{10}$$

hence Eq. (8) becomes

$$\langle \ddot{f} = \langle f\ V^2 \tag{11}$$

$$\langle \ddot{f} = \langle f\ V^3$$

$$\langle f^{d-1} = \langle f\ V^{d-1}$$

$$\langle f^{d} = \langle f\ V^{d}$$

and for those fitting functions with nil-potent V

$$V^d = 0 \tag{12}$$

The state vector is for condition of Eq. (11)

$$\begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \\ \cdot \\ \cdot \\ \cdot \\ x^{d-1} \end{pmatrix} = \begin{pmatrix} \langle f\ (t) \\ \langle f\ V \\ \langle f\ V^2 \\ \cdot \\ \cdot \\ \cdot \\ \langle f\ V^{d-1} \end{pmatrix} a\rangle = F_v(t) a\rangle \tag{13}$$

Eq. (13) can also be written as

$$x\rangle_s \quad \begin{bmatrix} \langle a \\ \langle a\ V^T \\ \langle a\ (V^T)^2 \\ \cdot \\ \cdot \\ \langle a\ (V^T)^{d-1} \end{bmatrix} f(t)\rangle \tag{14}$$

$$= T_v(a)\ f\ (t)\rangle \tag{15}$$

The derivative of the state vector is

$$\dot{x}\rangle_s = F_v V a\rangle = F_v(a)\ V^T\ f(t)\rangle \tag{16}$$

Inverting Eq. (13) and Eq. (15)

$$a\rangle = F_v(t)^{-1}\ x\rangle_s \tag{17}$$

and

$$f(t)\rangle = F_v(a)^{-1}\ x\rangle_s \tag{18}$$

Using Eq. (17) and Eq. (18) in Eq. (16)

$$\dot{x}\rangle_s = F_v(t)\ V_f\ F_v^{-1}\ x\rangle_s \tag{19}$$

and

$$\dot{x}\rangle_s = F_v(a)\ V_f^T\ F_v^{-1}(a)\ x\rangle_s \tag{20}$$

or

$$\dot{x}\rangle_s \quad F_x\ x\rangle_s \tag{21}$$

with

$$F_x = F_v(t)\ V_f\ F_v^{-1}(t) = F_v(a)\ V_f^T\ F_v^{-1}(a) \tag{22}$$

For the monomial base (polynomials)

$$F_x = S_{uo} = \begin{pmatrix} 0 & 1 & 0 & 0 & . & . & . \\ 0 & 0 & 1 & 0 & . & . & . \\ 0 & 0 & 0 & 1 & . & . & . \\ 0 & 0 & . & . & 0 & . & . \end{pmatrix} \tag{23}$$

For the trigonometric base of Eq. (122) sec (8) and Eq. (123) sec (8) for the 4 x 4 case

$$F_x = F_w V_w F_w^{-1} \tag{24}$$

or

$$F_x = \begin{bmatrix} \frac{4}{3} J - \frac{2}{3} I & \frac{1}{3} J - \frac{2}{3} I \\ -\frac{4}{3} J + \frac{8}{3} I & -\frac{1}{3} J + \frac{8}{3} I \end{bmatrix} \tag{25}$$

$$\frac{2}{3} \begin{pmatrix} -I & -I \\ 4I & 4I \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 4J & J \\ -4J & -J \end{pmatrix} \tag{26}$$

where the nomalization of the W_o factor has been assumed.

The fitting function transition matrix of Eq. (6) is

$$\langle f(t) = \langle f(0)\, e^{V_f t} \tag{27}$$

or

$$\langle f(t) = \langle f(0)[I + V_f t + V_f \frac{t^2}{2!} + . . .] \tag{28}$$

where

$$\Phi_f\ (t,0) = [I + V_f t + . . .] = e^{V_f t} \tag{29}$$

which is a Taylor series expansion.

The state transition matrix by Eq. (20) is

$$x(t)\rangle_s = e^{F_v(a)\, V_f^T\, F_v^{-1}(a)\, t}\ x(0)\rangle_s \tag{30}$$

or

$$x(t)\rangle_s = e^{F_v(t)\, V_f\, F_v^{-1}(t)\, t}\ x(0)\rangle_s \tag{31}$$

with

$$\Phi_x\ (t,0) = e^{F_v V_f F_v^{-1} t} \tag{32}$$

$$\Phi_x = F_v \, \Phi_f \, F_v^{-1} \tag{33}$$

Consider the states given by Eq. (14)

$$x(t)\rangle_s = T_v(a) \; f(t)\rangle \tag{34}$$

and at $t + \Delta t$

$$x(t + \Delta t)\rangle_s = T_v(a) \; f(t + \Delta t)\rangle \tag{35}$$

if

$$\langle\dot{f}(t) = \langle f(t) V_f \tag{36}$$

for constant V_f matrices

$$\langle f(t) = \langle f(t) \, e^{V_f(t-to)} \tag{37}$$

or

$$\langle f(t) = \langle f(to)[I + V_f(t-to) + V_f^2 \, \frac{(t-to)^2}{2!} + \ldots] \tag{38}$$

if we set

$$\Delta t = t - to \tag{39}$$

or

$$t = to + \Delta t \tag{40}$$

In terms of the transition matrix

$$\langle f(t) = \langle f(to) \; \Phi_f(t,to) \tag{41}$$

and Eq. (35) becomes

$$x(to + \Delta t)\rangle_s = T_v(a) \; \Phi_f^T(t + \Delta t, t_o) \; f \; (t_o)\rangle \tag{42}$$

or for any t

$$x(t + \Delta t)\rangle_s = T_v(a) \; \Phi_f^T(t + \Delta t, t) \; f(t)\rangle \tag{43}$$

Inverting Eq. (34)

$$f(t)\rangle = T_v^{-1}(a) \; x(t)\rangle_s \tag{44}$$

and using Eq. (44) in Eq. (43)

$$x(t + \Delta t)\rangle_s = T_v(a) \; \Phi_f^T(t + \Delta t, t) \; T_v^{-1}(a) \; x(t)\rangle_s \tag{45}$$

or

$$x(t + \Delta t)\rangle_s = \Phi_x(t + \Delta t, t) \; x(t)\rangle_s \tag{46}$$

with state transition matrix given by

$$\Phi_x(t + \Delta t,t) = T_v(a)\ \Phi_f^T(t + \Delta t,t)\ T_v^{-1}(a) \tag{47}$$

For the 3 x 3 case the matrix V is by Eq. (22) for the second degree polynomial monomial base as

$$V^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \tag{48}$$

and

$$\langle a\ V^T = (a_0,\ a_1,\ a_2) \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = (a_1,\ 2a_2,\ 0\ ) \tag{49}$$

and

$$\langle a(V^T)^2 = (a_1,\ 2a_2,\ 0) \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = (2a_2,\ 0,0) \tag{50}$$

hence

$$T_v(a) = \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & 2a_2 & 0 \\ 2a_2 & 0 & 0 \end{bmatrix} \tag{51}$$

One can compute $T_v^{-1}(a)$ and then compute by Eq. (47) to obtain $\Phi_x(t + \Delta t,t)$; however, this route is tedious. By Eq. (19)

$$\dot{x}(t)\rangle_s = F_v(t)\ V_f\ F_v^{-1}\ x(t)\rangle_s \tag{52}$$

and for the polynomial case considered by Eq. (220) sec (2)

$$S_{uo} = T_v(a)\ V_f\ T_v^{-1}(a) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{53}$$

Also Eq. (52) has solution

$$x(t)\rangle_s = e^{F_v V_f F_v^{-1} t}\ x(o)\rangle_s = \Phi_x x(o)\rangle_s \tag{54}$$

where

$$\Phi_x = e^{F_v V_f F_v^{-1} t} = F_v\ e^{V_f t}\ F_v^{-1} \tag{55}$$

with

$$e^{V_f t} = \Phi_f \tag{56}$$

The state transition matrix is given by Eq. (160) sec (4) as

$$\Phi_x(t + \Delta t, t) = \begin{bmatrix} 1 & \Delta t & \frac{\Delta t^2}{2!} \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix} \tag{57}$$

The base transition matrix is

$$\Phi_f = I + V_f \Delta t + V_f^2 \frac{\Delta t^2}{2!} \tag{58}$$

or

$$\Phi_f = \begin{pmatrix} 1 & \Delta t & \Delta t^2 \\ 0 & 1 & 2\Delta t \\ 0 & 0 & 1 \end{pmatrix} \tag{59}$$

Note the difference in the two transition matrices for this case.

As an example consider the monomial base of degree 2

$$\langle t = (1,\ t,\ t^2) \tag{60}$$

and

$$\langle \dot{t} = t\, V_t \tag{61}$$

with

$$\langle t(t) = t(0)\, e^{V_t t} = \langle t(o)\ \Phi_t(t,o) \tag{62}$$

with

$$\Phi_t(t,o) = \left(I + V_t t + V_t \frac{t^2}{2!}\right) \tag{63}$$

or

$$\Phi_t(t,o) = \begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix} = B(t) \tag{64}$$

Where B(t) implies the binomial coefficient matrix. By Eq. (53)

$$\dot{x}(t)\rangle_s = S_{uo}\ x(t)\rangle_s = T_v(a)\ V_t\ T_v^{-1}(a)\ x(t)\rangle_s \tag{65}$$

or

$$x(t)\rangle_s = \Phi_x(t,o)\ x(o)\rangle_s$$

with

$$\Phi_x(t,o) = e^{S_{uo} t} = I + S_{uo} t + \frac{S_{uo}^2 t^2}{2!} \tag{66}$$

or

$$\Phi_x(t,o) = \begin{pmatrix} 1 & t & \frac{t^2}{2!} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \tag{67}$$

Note that the coefficients of Eq. (64) are the binomial coefficients and those of Eq. (67) are the Taylor series expansion coefficients.

If we have the Poisson base fitting functions

$$\langle t_p = \langle t \,|\!|^{-1} \tag{68}$$

then

$$\langle \dot{t}_p = \langle t_p \,|\!|\, V_t \,|\!|^{-1} = \langle t_p S_{uo} \tag{69}$$

and

$$\langle t_p(t) = \langle t_p(o)\ \Phi_p(t,o) \tag{70}$$

with

$$\Phi_p(t,\ ) = e^{S_{uo} t} \tag{71}$$

which is the same as $\Phi_x(t,o)$ of Eq. (66)

Likewise, the states of the Poisson functions become

$$\dot{x}(t)\rangle_{sp} = V_t\ x(t)\rangle_{sp} \tag{72}$$

and

$$x(t)\rangle_{sp} = e^{V_t t} x(o)\rangle_{sp} = B(t)x(o)\rangle_{sp} \tag{73}$$

If we make a change of variable in the time axis from front of data back

$$t = t_f - t''' \tag{74}$$

then

$$dt = -\,dt''' \tag{75}$$

and Eq. (59) becomes

$$\Phi(t''' + \Delta t''',\ t''') = \begin{pmatrix} 1 & -\Delta t''' & \frac{(\Delta t''')^2}{2!} \\ 0 & 1 & -\Delta t''' \\ 0 & 0 & 1 \end{pmatrix} \tag{76}$$

Consider as an example now the exponentially weighted polynomial case. The base transition matrix is given by Eq. (107) sec (2) as

$$\Phi_e = e^{ho/2}\, T_u(1) \tag{77}$$

for one-step advance.

By Eq. (105) sec (2) for the 3 x 3 case

$$V_e = V_t - bI \tag{78}$$

and the states by Eq. 20

$$\dot{x}(t)\rangle\!\rangle_s = F_v(a)[V_t - bI]F_v^{-1}(a)\ x(t)\rangle\!\rangle_s \tag{79}$$

By Eq. (110) and Eq. (111) sec ( )

$$\Phi_x(t,o) = F_v(a)\ \Phi_e(t,o)F_v^{-1}(a) \tag{80}$$

$$\Phi_x(t,o) = F_v(a)\ D^{-1}(t)\ B(1)\ D(t)\ F_v^{-1}(a)\ e^{-bt} \tag{81}$$

The matrix

$$F_v(a) = \begin{pmatrix} \langle a \\ \langle aV_e^T \\ \langle a(V_e^T)^2 \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 \\ -ba_0 + a_1 & -ba_1 + 2a_2 & -ba^2 \\ b^2a_o - 2ba_1 & B^2a_1 - 4ba_2 & b^2a_2 \end{pmatrix} \tag{82}$$

Eq. (82) is seen to be quite difficult to invert.

SECTION 10
GENERAL RELATIONS FOR PARAMETER ESTIMATION

This section considers discreet estimation for the observational system.

$$z(t)(m\rangle_j = y(t)(m\rangle_j + u(t)(m\rangle_j \quad (1)$$

$z(t)(m\rangle$ is an m-dimensional time varying measurement vector, $y(t)(m\rangle$ is the signal and $u(t)(m\rangle$ is the additive measurement noise; the sequence $j=o,\ldots j_{max}$ where j_{max} will represent a countably infinite number and is to be thought of as the discreet population space; or the j^{th} experiment. The indices to be tagged onto t (the sample space variable) will correspond to the double indices for a fixed span, forward, central and backward. The following sections will consider the recursive cases corresponding to infinite memory filters and fixed memory or moving fixed span filters.

The signal sequence $y(t)(m\rangle$ will be considered for a number of cases some of which are stochastic, some deterministic.

For example if

$$y(t)(m\rangle_j = \underset{mxp}{H(t)}\, x(t)(p\rangle_j \quad (2)$$

then the p-dimensional state vector is stochastic, and may be the output of a linear dynamical system. The <u>parameter estimation</u> problem for the special case where the degree of the polynomial or the nature of the fitting functions is known, that is

$$x(t)(p\rangle = \underset{pxd}{B}\, f(t)(d\rangle \quad (3)$$

where the signal is exactly expressible as a linear comination of d fitting-functions (deterministic with no - approximations) is considered first. In this section the scalar measurement case will be considered, that is m=1

$$z(t)_j = \langle p)h(t)\, \underset{pxd}{B}\, f(t)(d\rangle \quad (4)$$

with the additional constraint

$$\langle d)h(t) = (1,0,0,\ldots 0) \quad (5)$$

In a later section the case will be considered where the measurement is scalar position x(t) and scalar velocity $\dot{x}(t)$, that is

$$\underset{2xp}{H(t)} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \end{pmatrix} \tag{6}$$

and

$$y(2\rangle = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} . \tag{7}$$

The case for this section thus is

$$z_j(t) = x_j(t) + u_j(t) \tag{8}$$

or

$$z_j(t) = \langle d) f(t)\, a(d\rangle + u_j(t) \tag{9}$$

that is

$$x(t) = f_o(t)\, a_o + f_1(t)\, a_1 + \ldots + f_{a-1}(t)\, a_{d-1} \tag{10}$$

at the end of this section some examples will be given for the indices on time

$$\langle d) f(i,n),\ \langle f(k,m),\ \langle f\{\gamma, j\} \tag{11}$$

but for the most part the number of observations will be k, that is a k-vector of measurements

$$\langle k)_j z = (z_o, z_2, \ldots z_{k-1})_j \tag{12}$$

and by Eq (9) for the sample column vector

$$z(k|_j\rangle = \underset{kxd}{F}\, a(d\rangle + v(k\rangle_j = \underset{kxd}{F}\, \hat{a}(d\rangle_j + \tilde{z}(k\rangle_j \tag{13}$$

where

$$\underset{kxd}{F} = \begin{bmatrix} \langle d) f(o) \\ \vdots \\ \langle d) f(k-1) \end{bmatrix} \tag{14}$$

and the vector $\hat{a}(d\rangle_j$ is the estimate of the parameter vector $a\rangle$ for the j^{th} test. Note that the vector $a\rangle$ is assumed the same each test.

For d ≤ k and the rank of F equal to d, that is

$$\underset{dxk}{\rho}\, F = d \tag{15}$$

then there exists many rank d matrices W of size dxk such that Eq (13) becomes

$$w\ z(k)\rangle_j = L_o\ a\rangle + wv\rangle_j = L_o\ \hat{a}\rangle_j + W\ \tilde{z}\rangle_j \tag{16}$$

where

$$w\ \tilde{z}\rangle_j = 0\rangle \tag{17}$$

and

$$\underset{dxd}{L_o} = \underset{dxd}{wF} \tag{18}$$

is square, full rank (hence invertable) and a constant matrix.

Multiplying Eq (16) by L_o^{-1}

$$(wF)^{-1}\ w\ z\rangle_j = a\rangle + L_o^{-1}\ w\ v\rangle = \hat{a}\rangle_j \tag{19}$$

Using Eq (19) in Eq (13)

$$z\rangle_j = F(wF)^{-1}\ wz\rangle_j + \tilde{z}\rangle_j \tag{20}$$

or

$$z\rangle_j = \hat{z}\rangle_j + \tilde{z}\rangle_j \tag{21}$$

where

$$\hat{z}\rangle_j = F(wF)^{-1}\ w\ z\rangle_j \tag{22}$$

Note that the matrix of Eq (22) is a projector (idempotent index 2) for

$$[F(wF)^{-1}\ w][F(wF)^{-1}\ w] = F(wF)^{-1}\ w \tag{23}$$

also by Eq (20) solving for $\tilde{z}\rangle_j$

$$\tilde{z}\rangle_j = [I - F(wF)^{-1}\ w]\ z\rangle_j \tag{24}$$

where the matrix $I - F(wF)^{-1}\ w$ is also idempotent index 2.

If the matrix L_o of Eq (18) equals I, that is

$$Fw = L_o = I \tag{25}$$

then the projector of Eq (22) and Eq (23) becomes

$$\underset{kxk}{P_{Fw} = Fw} = P_{Fw}^2 \tag{26}$$

and the oblique compliment projector of Eq (24) becomes

$$\tilde{P}_{Fw} = \underset{jxj}{I} - P_{Fw} = \tilde{P}_{Fw}^2 \tag{27}$$

and

$$P_{Fw} \tilde{P}_{Fw} = 0 \tag{28}$$

By Eq (22) and Eq (25)

$$\hat{z}\rangle_j = Fw\ z\rangle_j = P_{Fw}\ z\rangle_j \tag{29}$$

and by Eq (24)

$$\tilde{z}\rangle_j = \tilde{P}_{Fw}\ z\rangle_j \tag{30}$$

By Eq (19) and Eq (25)

$$w\ z\rangle_j = a\rangle + w\ v\rangle_j = \hat{a}\rangle_j \tag{31}$$

For the special case of the weighting matrix w equal to the psuedo-inverse of F, one obtains the unweighted solution, that is

$$\underset{dxk}{w} = \underset{dxk}{F^*} = (F^T F)^{-1} F^T \tag{32}$$

and by Eq (19)

$$\hat{a}\rangle = F^* z\rangle_j = (F^T F)^{-1} F^T z\rangle \tag{33}$$

which is the standard "normal" equation for the parameter.

The estimate of the measurement vector is by Eq (29)

$$\hat{z}\rangle = P_{FF^*} z\rangle \tag{34}$$

where the symmetric projector

$$P_{FF^*} = (FF^*) = F(F^T F)^{-1} F \tag{35}$$

and the vector $\tilde{z}\rangle$ of Eq (30) is orthogonal to the vector of Eq (34). The special cases of Eq (32) will be considered after a number of other results are obtained.

For the weighting case using Eq (25) in Eq (19)

$$w\ z\rangle_j = a\rangle + w\ v\rangle_j = \hat{a}\rangle_j \tag{36}$$

Partitioning the weighting matrix w into its row space dxk

$$\underset{dxk}{w} = [w(d\rangle_o, w(d\rangle_1 \ldots w(d\rangle_{k-1}] \tag{37}$$

and using Eq (37) in Eq (36)

$$\hat{a}(d\rangle_j = w(d\rangle_o z_o + \ldots + w(d\rangle_{k-1} z_{k-1} \tag{38}$$

or the d-dimensional parameter vector is a linear combination of the weighting vectors. Note that the weighting vectors are to be determined in some optimal sense such that they are the same each experiment, that is they are not to be a function of j in some "future sequence"; but will be a function of some "a prior" past sequence of tests.

The error in the estimate of the function-parameter by Eq (36) is

$$a\rangle - \hat{a}\rangle_j = \tilde{a}\rangle_j = -w\ z\rangle_j \tag{39}$$

If the population sequence j of Eq (13) is packaged as a row of column vectors we have

$$\underset{}{[z(k\rangle_1,\ z(k\rangle_2\ .\ .\ .\ z(k\rangle_j]} = \underset{k\times j}{Z} = \underset{k\times d}{F}\ a(d\rangle\langle j)\ 1 + \underset{k\times j}{V} \tag{40}$$

$$\underset{k\times j}{Z} = \underset{k\times d}{F}\ \underset{d\times j}{\hat{A}} + \tilde{Z}\ . \tag{41}$$

Consider next the package of measurement noise vectors of Eq (40)

$$\underset{k\times j}{V} = [v(k\rangle_1,\ .\ .\ .\ v(k\rangle_j] = \begin{bmatrix} \langle j)\ v(0) \\ \langle j)\ v(1) \\ \vdots \\ \langle j)\ v(k-1) \end{bmatrix} \tag{42}$$

Equation (42) considers j column vectors in k-space and k row vectors in j-space.

The mean (or arithmetic average) of the j column vectors in k space of Eq (42) is

$$\underset{k\times j}{V}\ 1^*(j\rangle = \frac{v(k\rangle_1 + .\ .\ .\ v(k\rangle_j}{j_{max}} = \mu\ (k\rangle_v \tag{43}$$

where

$$1^*(j\rangle = \frac{1(j\rangle}{j_{max}} = \frac{1(j\rangle}{\langle 1\ 1\rangle}\ . \tag{44}$$

If the k-dimensional mean vector of Eq (43) is the zero vector in k-space then o(k⟩ is the point of symmetry with respect to the sequence of j vectors in k-space
The j-dimensional row space interpretation of Eq (43) is

$$\underset{k\times j\ \ j\times 1}{V\ 1^*(j\rangle} = \begin{bmatrix} \langle j)v(o) & 1^*(j\rangle \\ \langle j)v(1) & 1^*(j\rangle \\ \vdots & \\ \langle j)v(k-1) & 1^*(j\rangle \end{bmatrix}_{k\times 1} \tag{45}$$

$$V\, 1^*(j\rangle = [\mu(o), \mu(2) \ . \ . \ . \ \mu(k-1)] \tag{46}$$

$$V\, 1^*(j\rangle = \mu(k\rangle_v \tag{47}$$

By Eq (45) we see a column vector of inner-products and if the $\mu(k\rangle_v$ vector is zero in k-space, then all k of the row vectors in j-space are perpendicular to the $1^*(j\rangle$.

The mean of the k row vectors in j-space is

$$\langle k)1^* \underset{k\times j}{V} = \langle k)1^* \begin{bmatrix} \langle j)v(o) \\ . \\ . \\ . \\ \langle j)v(k-1) \end{bmatrix} = \langle j)m_v \tag{48}$$

with a corresponding inner-product interpretation.

In j-space for the k vectors

$$\langle j)v(k) = \langle j)m_v + \langle j)\tilde{v}_{km} \tag{49}$$

or package-wise

$$\underset{k\times j}{V} = 1(k\rangle \langle j)m_v + \tilde{V}_m \tag{50}$$

By Eq (48) in Eq (50)

$$\underset{k\times j}{V} = 1(k\rangle \langle k)1^* V + \tilde{V}_m \tag{51}$$

or

$$\underset{k\times j}{\tilde{V}_m} = (I - \underset{k\times k}{1(k\rangle\langle k)1^*})\underset{k\times j}{V} \tag{52}$$

which is projection in k-space.

The most useful projection notion is in j-space. Consider the j column vectors in k-dimensional space of Eq (42), which by vector addition may be decomposed as

$$v(k\rangle_j = \mu(k\rangle_v + \tilde{v}(k\rangle_j \tag{53}$$

or package-wise

$$\underset{k\times j}{V} = \mu(k\rangle_v \langle j)\ 1 + \underset{k\times j}{\tilde{V}} \tag{54}$$

Using Eq (47) in Eq (54)

$$V = V_{k\times j} \, 1^*_{k\times j}(j\rangle \langle j)1 + \tilde{V} \tag{55}$$

or

$$\tilde{V} = V_{k\times j} (I - 1^*(j\rangle \langle j) 1)_{j\times j} \tag{56}$$

or

$$\tilde{V} = V \tilde{P}_{11} \tag{57}$$

also by Eq (55)

$$V = \hat{V} + \tilde{V} \tag{58}$$

where

$$\hat{V} = V P_{11} \tag{59}$$

and the rank-one projector in j-space is

$$\underset{j\times j}{P_{11}} = \frac{1(j\rangle \langle j)1}{\langle j)1 \quad 1(j\rangle} \tag{60}$$

The vector partitioned into the j-dimensional row vectors are

$$\begin{bmatrix} \langle j)v(o) \\ \langle j)v(1) \\ \cdot \\ \cdot \\ \cdot \\ \langle j)v(k-1) \end{bmatrix} = \begin{bmatrix} \langle j)\hat{v}(o) \\ \langle j)\hat{v}(1) \\ \cdot \\ \cdot \\ \cdot \\ \langle j)\hat{v}(k-1) \end{bmatrix} + \begin{bmatrix} \langle j)\tilde{v}(o) \\ \langle j)\tilde{v}(1) \\ \cdot \\ \cdot \\ \cdot \\ \langle j)\tilde{v}(k-1) \end{bmatrix} \tag{61}$$

as shown in Fig (1)

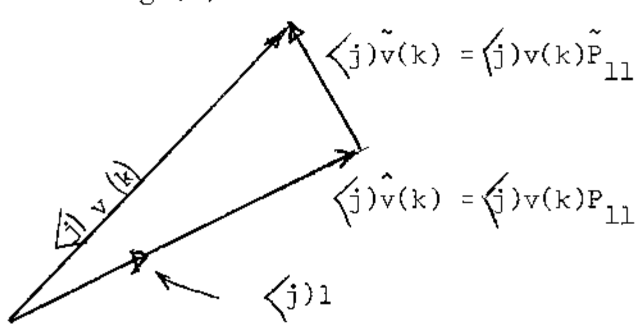

FIG (1) ORTHOGONAL PROJECTION OF NOISE VECTORS.

Thus we see if one has a zero mean noise case that all vectors are perpendicular to the $\langle j)1$ vector as shown in Fig (2)

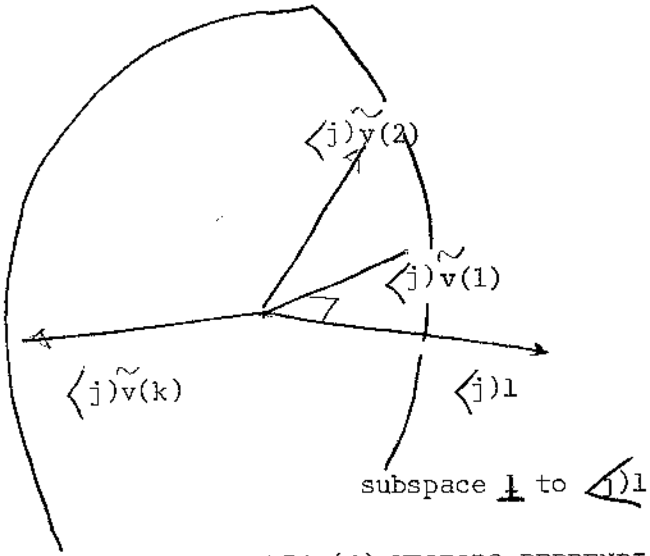

FIG (2) VECTORS PERPENDICULAR TO ⟨j) 1

If all k-vectors in j-space are mutually perpendicular one has a diagonal matrix

$$V\,V^T = \begin{bmatrix} \langle v_o\, v_o \rangle & o & \cdots & o \\ o & & & o \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ o & \cdots & o & \langle v_{k-1}\, v_{k-1} \rangle \end{bmatrix} \quad . \tag{62}$$

We can now define the variance matrix of the measurement noise as

$$\underset{kxk}{R} = \lim_{j_m \to \infty} \left\{ \underset{kxj}{\tilde{V}^T}\ \underset{jxk}{\tilde{V}} \quad \frac{1}{j_{max}} \right\} \tag{63}$$

$$= \lim_{j_m \to \infty} \left[\sum_{j=1}^{j_{max}} \tilde{v}(k\rangle_j \langle k)\tilde{v}^j \right] \frac{1}{j_{max}}$$

since

$$(\tilde{v}(k\rangle_1, \ldots \tilde{v}(k\rangle_j \begin{bmatrix} {}^1\langle k)\tilde{v} \\ \cdot \\ \cdot \\ \cdot \\ {}^j\langle k)\tilde{v} \end{bmatrix} = \sum_{j=1}^{j_{max}} \tilde{v}(k\rangle_j \, {}^j\langle k)\tilde{v} \tag{64}$$

The serial time correlations are given by

$$R = \begin{bmatrix} r(o,o) & r(o,1) \; . \; . \; . & r(o,k^{-1}) \\ . & & . \\ . & & . \\ . & & . \\ r(k^{-1},0) \; . \; . \; . \; . \; . \; . \; . \; . & & r(k^{-1},k^{-1}) \end{bmatrix} \tag{65}$$

The standard definition of the noise variance matrix of Eq (63) is

$$R = \xi \left\{ [v(k\rangle_j - \mu(k\rangle_v] \, [{}_j\langle k)v - {}_v\langle k)\mu \,] \right\} \tag{66}$$

Return now to Eq (19) where we have placed the constraint on

$$\underset{dxk}{w} \, \tilde{z}(k\rangle_j = o(d\rangle \tag{67}$$

for all tests j.

We can multiply Eq (40) by w

$$wZ = a(d\rangle \langle j)1 + wV = \underset{dxj}{\hat{A}} \tag{68}$$

where

$$w\tilde{Z} = \underset{dxj}{o} \tag{69}$$

Define

$$\underset{dxj}{A} = a(d\rangle \langle j)1 \tag{70}$$

then by Eq (68)

$$A - \hat{A} = \tilde{A} = -wV = -w[\mu(k\rangle_v \langle j)1 + \tilde{V}] \tag{71}$$

$$Z = FA + V = FwZ + \tilde{Z} \tag{72}$$

or

$$\underset{kxj}{\tilde{Z}} = \underset{kxk}{(I - Fw)}\underset{kxk}{Z} = \underset{kxj}{\tilde{P}_{Fw}Z} \tag{73}$$

and by Eq (72)

$$\hat{Z} = FwZ = P_{Fw}Z \tag{74}$$

The variance of the estimate of the parameters can now be defined by Eq (71) as

$$\underset{dxd}{\ddagger_{\tilde{a}\tilde{a}}} = \lim_{j\to\infty} \left[\tilde{A}\,\tilde{A}^T \frac{1}{j_{max}} \right]$$

$$\ddagger_{\tilde{a}\tilde{a}} = \lim_{j\to\infty} \left[\sum_{j=1}^{j_{max}} \tilde{a}(d\rangle_j \langle {}^{j} d)\tilde{a} \frac{1}{j_{max}} \right]$$

$$\ddagger_{\tilde{a}\tilde{a}} = \xi\left[(\tilde{a}(d\rangle \langle d)\tilde{a})\right] \tag{75}$$

where ξ is the standard expected-value operator. In terms of R and w Eq (75) is

$$\ddagger_{\tilde{a}\tilde{a}} = \lim_{j\to\infty} \left(\frac{wVV^T w^T}{j} \right) \tag{76}$$

or

$$\ddagger_{\tilde{a}\tilde{a}} = w[\mu(k\rangle_v \langle k)\mu_v + R]\, w^T \tag{77}$$

since

$$\lim_{j_m\to\infty} \frac{VV^T}{j_{max}} = \lim \frac{(\mu(k\rangle\langle j)1 - \tilde{V})(1(j\rangle\langle k)\mu - \tilde{V}^T)}{j_{max}} \tag{78}$$

$$= \mu(k\rangle \langle k)\mu + R \quad . \tag{79}$$

If the noise mean is not known it is clear by Eq (68), Eq (71) and Eq (79) one can not seperate the constant noise mean vector from the constant parameter fitting the function vector. It will be assumed hence forth that the measurement noise mean is known and can be subtracted out. In the real world applications one may use Friedlands bias estimation technique to estimate the bias in the measurement noise and in the plant noise or function fitting mis-modelling.

Multiplying Eq (72) on the left by $\tilde{P}_{Fw}$

$$\tilde{P}_{Fw}\ z = \tilde{P}_{Fw}\ V = \tilde{Z} \tag{80}$$

since

$$\tilde{P}_{Fw}\ FA = o \tag{81}$$

since by Eq (28)

$$\tilde{P}_{Fw}\ F = o\ . \tag{82}$$

The variance of the estimate of the measurements by Eq (80) can be written as

$$\ddagger_{\tilde{a}\tilde{a}} = \lim_{j_{max} \to \infty} (\tilde{Z}\tilde{Z}^T \frac{1}{j_{max}}) \tag{83}$$

or

$$\ddagger_{\tilde{a}\tilde{a}} = \tilde{P}_{Fw}(\ \mu(k\rangle_v \langle \overset{v}{k})\mu + R)\ \tilde{P}_{Fw}^T$$

$$= \tilde{P}_{Fw}\ R\ \tilde{P}_{Fw}^T \tag{84}$$

since we assume $\mu(k\rangle_v \langle \overset{v}{k})\mu$ and the oblique projector is known; or equivalently assume the noise is zero mean.

The results of Eq (63), Eq (79) and Eq (84) can be compactly written.

$$\begin{bmatrix} \tilde{V} \\ kxj \\ \tilde{Z} \\ kxj \\ \tilde{A} \\ dxj \end{bmatrix} = \begin{bmatrix} I \\ kxk \\ \tilde{P}_{Fw} \\ \\ -w \end{bmatrix} \tilde{V} \tag{85}$$

which is a function of the single independent matrix $\tilde{V}$ (ignoring $\mu(k\rangle_v$)

Transposing Eq (85)

$$(\tilde{V}^T, \tilde{Z}^T, \tilde{A}^T) = \tilde{V}^T(I,\ \tilde{P}^T,\ -w^T) \tag{86}$$

and forming the outer product

$$\begin{bmatrix}\tilde{V}\\ \tilde{Z}\\ \tilde{A}\end{bmatrix}(\tilde{V}^T,\tilde{Z}^T,\tilde{A}^T) = \begin{bmatrix} I\\ \tilde{P}_{Fw}\\ -w\end{bmatrix} \tilde{V}\,\tilde{V}^T(I,\ \tilde{P}^T_{Fw},\ -w^T) \tag{87}$$

or

$$\begin{bmatrix}\tilde{V}\,\tilde{V}^T & \tilde{V}\,\tilde{Z}^T & \tilde{V}\,\tilde{A}^T\\ \tilde{Z}\,\tilde{V}^T & \tilde{Z}\,\tilde{Z}^T & \tilde{Z}\,\tilde{A}^T\\ \tilde{A}\,\tilde{V}^T & \tilde{A}\,\tilde{Z}^T & \tilde{A}\,\tilde{A}^T\end{bmatrix} =$$

$$\begin{bmatrix}\tilde{V}\,\tilde{V}^T & \tilde{V}\,\tilde{V}^T\,\tilde{P}_{Fw}^{\ T} & -\tilde{V}\,\tilde{V}^T\,w^T\\ \tilde{P}_{Fw}\,\tilde{V}\,\tilde{V}^T & \tilde{P}_{Fw}\,\tilde{V}\,\tilde{V}^T\,\tilde{P}_{Fw}^{\ T} & -\tilde{P}_{Fw}\,R\,w^T\\ -w\,\tilde{V}\,\tilde{V}^T & -w\,\tilde{V}\,\tilde{V}^T\,\tilde{P}_{Fw}^{\ T} & +\,w\,\tilde{V}\,\tilde{V}^T\,w^T\end{bmatrix} \tag{88}$$

and at the variance-covariance level

$$\begin{bmatrix}\Sigma_{\tilde{v}\tilde{v}} & \Sigma_{\tilde{v}\tilde{z}} & \Sigma_{\tilde{v}\tilde{a}}\\ \Sigma_{\tilde{z}\tilde{v}} & \Sigma_{\tilde{z}\tilde{z}} & \Sigma_{\tilde{z}\tilde{a}}\\ \Sigma_{\tilde{a}\tilde{v}} & \Sigma_{\tilde{a}\tilde{z}} & \Sigma_{\tilde{a}\tilde{a}}\end{bmatrix} =$$

$$\begin{bmatrix}R & R\,\tilde{P}_{Fw} & -R\,w^T\\ \tilde{P}_{Fw}\,R & \tilde{P}_{Fw}\,R\,\tilde{P}_{Fw}^{\ T} & -\tilde{P}_{Fw}\,R\,w^T\\ -w\,R & -w\,R\,\tilde{P}_{Fw}^{\ T} & w\,R\,w^T\end{bmatrix} \tag{89}$$

Note that Eq (85) is a j x (2k + d) matrix and a matrix of variance of Eq (87) and Eq (88) is a (2k + d) square matrix with rank given by Eq (87) as k when R is full rank equal to k.

For the special case where the parameter vector has dimension one, that is by Eq (10) and Eq (8)

$$z_j(k) = a_o + v_j(k) \tag{90}$$

or for d=1 and w a row vector

$$\underset{1\times d}{w} = \langle d)\, w \qquad (91)$$

and by Eq (14)

$$\underset{k\times 1}{F} = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} = 1(k\rangle \qquad (92)$$

$$\underset{(2k+d)(2k+d)}{\ddagger} = \begin{bmatrix} R & R\,\tilde{P}_{1w}^{T} & -R\,w \\ \tilde{P}_{1w}\,R & \tilde{P}_{1w}\,R\,\tilde{P}_{1w}^{T} & -\tilde{P}_{1w}\,R\,w \\ -\,w\,R & -\,w\,R\,\tilde{P}_{1w}^{T} & w\,R\,w \end{bmatrix} \qquad (93)$$

other expressions for the independent matrix R,

By Eq (40)

$$Z = F\,A + V = F\,\hat{A} + \tilde{Z} \qquad (94)$$

or

$$V = -F(\tilde{A}) + \tilde{Z} \qquad (95)$$

$$= (I, -F) \begin{bmatrix} \tilde{Z} \\ \tilde{A} \end{bmatrix}$$

and transposing

$$V^{T} = (\tilde{Z}^{T}, \tilde{A}^{T}) \begin{bmatrix} I \\ -F^{T} \end{bmatrix} \qquad (96)$$

The product

$$V\,V^{T} = (I, -F) \begin{bmatrix} \ddagger_{\tilde{z}\tilde{z}} & \ddagger_{\tilde{z}\tilde{a}} \\ \ddagger_{\tilde{a}\tilde{z}} & \ddagger_{\tilde{a}\tilde{a}} \end{bmatrix} \begin{bmatrix} I \\ -F^{T} \end{bmatrix} \qquad (97)$$

or

$$R = \Phi_{\tilde{z}\tilde{z}} - \Phi_{\tilde{z}\tilde{a}} F^T - F \Phi_{\tilde{a}\tilde{z}} + F \Phi_{\tilde{a}\tilde{a}} F^T \tag{98}$$

which interestingly "looks-like" an algebraic Riccatti equation in F.

By Eq (89) we see that the variances and covariances of the measurements and the parameters are dependent on the measurement noise variance matrix R, the fitting function matrix F and the weighting matrix w. If one selects the fitting functions to be used, for example polynomials of a known degree, trigonometric etc, then F is determined. The matrix R is a characteristic of the measuring instrument and in the real world one attempts to estimate what it is; however for the purpose here it will be assumed known. The only other matrix of concern is the optimal weighting matrix w. A number of cases can be considered: let w imply weighted, $\tilde{w}$ (compliment meaning no weight) and like-wise for constrained parameter c and unconstrained parameter $\tilde{c}$, we thus have the four cases

$$\begin{bmatrix} \tilde{c} \\ c \end{bmatrix} (\tilde{w}, w) = \begin{bmatrix} \tilde{c}\,\tilde{w} & \tilde{c}w \\ c\,\tilde{w} & cw \end{bmatrix} \tag{99}$$

where the cases are

1. $\tilde{c}\ \tilde{w}$ - unconstrained - unweighted

2. $\tilde{c}\ w$ - unconstrained - weighted

3. $c\ \tilde{w}$ - constrained - unweighted

4. $c\ w$ - constrained - weighted

The first case is considered first in the following section.

Section 11 UNCONSTRAINED UNWEIGHTED PARAMETER ESTIMATION

The unweighted parameter estimation cases considered in this section use the W matrix to be the psuedo-inverse of F given by Eq. (32) sec (10) as

$$W = F^* = (F^T F)^{-1}\, F^T \tag{1}$$

with the measurement estimates given by Eq. (34) sec (10) as

$$\hat{z}\;(k\rangle_j = FF^*\, z\;(k\rangle_j \tag{2}$$

The parameter estimate is given by Eq. (33) sec (10) as

$$\hat{a}\;(d\rangle_j = F^*\, z\rangle_j \;=\; (F^T F)^{-1}\, F^T\, z\;(k\rangle_j \tag{3}$$

The projections and compliment projectors are

$$P_{FW} = FF^* = P^T_{FF^*} \tag{4}$$

and

$$\tilde{P}_{FW} = I - FF^* = \tilde{P}_{FF^*} = \tilde{P}^T_{FF^*} \tag{5}$$

The matrix of variances for this case becomes by Eq. (89) sec (10)

$$\sum_{(2k+d)(2k+d)} = \begin{bmatrix} R & \Sigma_{\tilde{v}\tilde{z}} & \Sigma_{\tilde{v}\tilde{a}} \\ \Sigma_{\tilde{z}\tilde{v}} & \Sigma_{\tilde{z}\tilde{z}} & \Sigma_{\tilde{z}\tilde{a}} \\ \Sigma_{\tilde{a}\tilde{v}} & \Sigma_{\tilde{a}\tilde{z}} & \Sigma_{\tilde{a}\tilde{a}} \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} R & R\tilde{P}^T_{FF^*} & -RF^{*T} \\ \tilde{P}_{FF^*}R & \tilde{P}_{FF^*}R\tilde{P}^T_{FF^*} & -\tilde{P}_{FF^*}RW^T \\ -F^*R & -F^*R\tilde{P}^T_{FF^*} & F^*RF^{*T} \end{bmatrix} \tag{7}$$

We shall look at the "normal equations" of Eq. (3) for some "often used" cases.

Case 1. Back to front (forward) Polynomials

The fitting functions are the power series or polynomials, that is

$$\langle f(t) = (1,\ t,\ t^2,\ \dots\ t^{d-1}) \tag{8}$$

and for the indexing given by Eq. (123) sec (5), we have

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$$\sphericalangle t(i,n) = \sphericalangle n\ T_u(i)T_u(\beta)D(ho) \tag{9}$$

If we assume batch-processing (not recursive spans) set i = 0 in Eq. (128) sec (5) and Eq. (9) becomes a function of n = 0, 1, 2 ... N

$$\sphericalangle t(n) = \sphericalangle n\ T_u(\beta)D(ho) \tag{10}$$

and for the second degree polynomial case by Eq. (34) sec (5)

$$T_u(\beta) = \begin{bmatrix} 1 & \beta & \beta^2 \\ 0 & 1 & 2\beta \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

where by Eq. (16) sec (5)

$$\beta = to/ho. \tag{12}$$

By Eq. (34) sec (5)

$$D(ho) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & ho & 0 \\ 0 & 0 & h^2o \end{bmatrix} \tag{13}$$

The F matrix of Eq. (14) sec (10) becomes that of Eq. (124) sec (5)

$$\underset{(N+1)x3}{T} = \begin{bmatrix} \sphericalangle t(o) \\ \sphericalangle t(1) \\ \cdot \\ \cdot \\ \cdot \\ \sphericalangle t(N) \end{bmatrix} = \underset{(N+1)x3}{N}\ T_u(\beta)D(ho) \tag{14}$$

The transpose of Eq. (14) is

$$\underset{3x(N+1)}{T^T} = \underset{3x3}{D(ho)}\ T_u^T(\beta)\ \underset{3x(N+1)}{N^T} \tag{15}$$

The matrix N^T is given by Eq. (50) sec (5) as

$$\underset{3x(N+1)}{N^T} = \begin{bmatrix} 1 & 1 & 1 & 1 \ldots 1 \\ 0 & 1 & 2 & 3 \ldots N \\ 0 & 1 & 2^2 & 3^2 \ldots N^2 \end{bmatrix} \tag{16}$$

The inverse of the discrete metric-matrix required in Eq. (1) is

$$(T^TT)^{-1} = \left[D(ho)T_u^T(\beta)N^TNT_u(\beta)D(ho)\right]^{-1} \tag{17}$$

$$(T^T T)^{-1} = D^{-1}(ho) T_u^{-1}(\beta)(N^T N)^{-1} T_u^{-T}(\beta) D^{-1}(ho) \tag{18}$$

The inverse $T_u^{-1}(\beta)$ is given by Eq. (149) sec (A) as

$$T_u^{-1}(\beta) = \begin{bmatrix} 1 & -\beta & \beta^2 \\ 0 & 1 & -2\beta \\ 0 & 0 & 1 \end{bmatrix} \tag{19}$$

The inverse $(N^T N)^{-1}$ is given by Eq. (118) sec (5) as

$$(N^T N)^{-1} = \frac{3}{(N+1)(N+2)(N+3)} \begin{bmatrix} 3N^2+3N+2 & -(12N+6) & 10 \\ -(12N+6) & \dfrac{4(2N+1)(8N-3)}{N(N-1)} & \dfrac{-60}{(N-1)} \\ 10 & \dfrac{-60}{(N-1)} & \dfrac{60}{N(N-1)} \end{bmatrix} \tag{20}$$

By Eq. (3), Eq. (15) and Eq. (18)

$$\hat{a}\; 3\rangle_j = D^{-1}(ho) T_u^{-1}(\beta)(N^T N)^{-1} N^T\; z\; N+1\rangle_j \tag{21}$$

Consider the term $N^T z\rangle$ of Eq. (21)

$$N^T z\; N+1\rangle = \begin{bmatrix} 1 & \langle N)\, 1 \\ 0 & \langle N)\, c \\ 0 & \langle N)\, c^2 \end{bmatrix} \begin{pmatrix} zo \\ z\; N\rangle \end{pmatrix} \tag{22}$$

$$= \begin{bmatrix} zo + \langle N)1 \; z\, N\rangle \\ \langle N)c \; z\, N\rangle \\ \langle N)c^2 \; z\, N\rangle \end{bmatrix} \tag{23}$$

where the counting number vector is

$$\langle N)\; c = (1, 2, 3, 4, \ldots N) \tag{24}$$

and the vector of squares of the counting numbers is

$$\langle N)\; c^2 = (1, 2^2, 3^2, \ldots N^2) \tag{25}$$

and

$$\langle N)\; 1 = (1, 1, 1, \ldots 1) \tag{26}$$

Eq. (23) can be written as

$$N^T \; z \; \langle N+1 \rangle = \begin{bmatrix} z(o) + \sum_{i=1}^{N} z(i) \\ \sum_{i=1}^{N} iz(i) \\ \sum_{i=1}^{N} i^2 z(i) \end{bmatrix} \tag{27}$$

Thus Eq. (27) used with Eq. (20) in Eq. (21) for arbitrary t_o and constant data sampling interval h_o yields the estimate of the parameters.

$$\hat{a} \; \langle 3 \rangle = \lambda T_{hou}^{-1} \begin{bmatrix} 3N^2+3N+2 & -(12N+6) & 10 \\ -(12N+6) & \frac{4(2N+1)(8N-3)}{N(N-1)} & \frac{-60}{N-1} \\ 10 & \frac{-60}{N-1} & \frac{60}{N(N-1)} \end{bmatrix} \begin{bmatrix} zo+\sum_{i=1}^{N} z(i) \\ \sum_{i=1}^{N} iz(i) \\ \sum_{i=1}^{N} i^2 z(i) \end{bmatrix} \tag{28}$$

where

$$\lambda = \frac{3\sigma_v}{(N+1)(N+2)(N+3)} \tag{29}$$

and

$$T_{hou}^{-1} = D^{-1}(ho)T_u^{-1}(\beta) \tag{30}$$

The variance of the estimate of the parameters is given by Eq. (7) as

$$\sum_{\tilde{a}\tilde{a}} = T^*RT^{*T} \tag{31}$$

The psuedo-inverse of T given by Eq. (14) is

$$\underset{3x(N+1)}{T^*} = D^{-1}(ho)T_u^{-1}(\beta)\underset{3xN+1}{N^*} \tag{32}$$

and the transpose is

$$T^{*T} = N^{*T}T_u^{-1}(\beta)D^{-1}(ho) \tag{33}$$

Using Eq. (32) and (33) in Eq. (31)

$$\underset{3x3}{\sum} \tilde{a}\tilde{a} = D^{-1}(ho)T_u^{-1}(\beta)N^*N^{*T}T_u^{-T}(\beta)D^{-1}(ho) \tag{34}$$

For the analytically tractable special case of a scalar matrix R, that is

$$\underset{(N+1)(N+1)}{R} = \sigma_v I \tag{35}$$

Eq. (34) becomes

$$\sum_{3x3} \tilde{a}\tilde{a} = D^{-1}(ho)T_u^{-1}(\beta)(N^TN)^{-1}T_u^{-T}(\beta)D^{-1}(ho) \tag{36}$$

since

$$(N^*N^{*T}) = \underset{3x3}{(N^TN)^*} = (N^TN)^{-1} \tag{37}$$

or by Eq. (20) in Eq. (36)

$$\sum_{3x3} \tilde{a}\tilde{a} = \lambda T_{hou}^{-1} \begin{bmatrix} 3N^2+3N+2 & -(12N+6) & 10 \\ -(12N+6) & \frac{4(2N+1)(8N-3)}{N(N-1)} & \frac{-60}{N-1} \\ 10 & \frac{-60}{N-1} & \frac{60}{N(N-1)} \end{bmatrix} T_{hou}^{-T} \tag{38}$$

where λ and T_{hou}^{-1} are given by Eq. (29) and (30).

Case 2. Mid-point Polynomials

The mid-point case for the monomial base of polynomials is given by Eq. (128) of sec (5) for k = 0

$$\langle t(m) = \langle m\ T_u(\beta)D(ho) \tag{39}$$

and the package of fitting functions at the m indices by Eq. (129) as

$$\underset{(N+1)x3}{T} = \begin{bmatrix} \langle t(-M) \\ \langle t(-M+1) \\ \cdot \\ \cdot \\ \cdot \\ \langle t(-1) \\ \langle t(0) \\ \langle t(1) \\ \cdot \\ \cdot \\ \cdot \\ \langle t(M) \end{bmatrix} = \underset{(N+1)x3}{M}\ T_u(\beta)D(ho) \tag{40}$$

where N = 2M

The matrix M^T is given by Eq. (66) of sec (5) as

$$\underset{3x(N+1)}{M^T} = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 & 1 & 1 & 1 & \ldots & 1 \\ -M & -(M-1) & \ldots & -2 & -1 & 0 & 1 & 2 & \ldots & M \\ M^2 & (M-1)^2 & \ldots & 4 & 1 & 0 & 1 & 4 & \ldots & M^2 \end{bmatrix} \quad (41)$$

The matrix $(T^T T)^{-1}$ for this case is by Eq. (18)

$$(T^T T)^{-1} = T_{hou}^{-1} \, (M^T M)^{-1} \, T_{hou}^{-1} \quad (42)$$

The matrix $(M^T M)^{-1}$ is given by Eq. (120) sec (5) as

$$(M^T M)^{-1} = \begin{bmatrix} \frac{3(3N^2+6N-4)}{4(N^2-1)(N+3)} & 0 & \frac{-15}{(N^2-1)(N+3)} \\ 0 & \frac{12}{N(N+1)(N+2)} & 0 \\ \frac{-15}{(N^2-1)(N+3)} & 0 & \frac{180}{N(N+2)(N+3)(N^2-1)} \end{bmatrix} \quad (43)$$

The estimate of the parameters by Eq. (3), Eq. (40) and Eq. (42) is

$$\hat{a} \left\langle 3 \right\rangle = T_{hou}^{-1} \, (M^T M)^{-1} \, M^T \, z \, \left(2M+1 \right\rangle \quad (44)$$

where

$$\left\langle 2M+1 \right) z = (z(-M), \ldots z(-1), z(0), z(1) \ldots z(M)) \quad (45)$$

The variance of the estimate of the parameters is given by analogy with Eq. (38) for the scalar matrix noise case as

$$\sum_{\tilde{a}\tilde{a}} = \lambda T_{hou}^{-1} \begin{bmatrix} \frac{3(3N^2+6N-4)}{4(N^2-1)(N+3)} & 0 & \frac{-15}{(N^2-1)(N+3)} \\ 0 & \frac{12}{N(N+1)(N+2)} & 0 \\ \frac{-15}{(N^2-1)(N+3)} & 0 & \frac{180}{N(N+2)(N+3)(N^2-1)} \end{bmatrix} T_{hou}^{-1} \quad (46)$$

Case 3. Front to Back Polynomials

The indexing from the most recent "real time" data point to the back as given by Eq. (135) sec (5) is

$$\underset{(N+1)x3}{F} = T = \underset{(N+1)x3}{\Gamma} \; T_u(\beta) D(ho) \quad (47)$$

and

$$F^T = D(ho)T_u^T(\beta)\Gamma^T \tag{48}$$

The inverse of the discrete-metric matrix is

$$(F^TF)^{-1} = T_{hou}^{-1}\ (\Gamma^T\Gamma)^{-1}\ T_{hou}^{-1} \tag{49}$$

The "number-theoretic discrete metric matrix" inverse by Eq. (279) sec (5) is

$$(\Gamma^T\Gamma)^{-1} = \begin{bmatrix} 3N^2+3N+2 & 12N+6 & 10 \\ 12N+6 & \frac{4(2N+1)(8N-3)}{N(N-1)} & \frac{60}{N-1} \\ 10 & \frac{60}{N-1} & \frac{60}{N(N-1)} \end{bmatrix} g_o \tag{50}$$

where

$$g_o = \frac{3}{(N+1)(N+2)(N+3)} \tag{51}$$

Note that Eq. (50) is the same as Eq. (20) except for the alternating sign changes in the diagonals from right to left. Hence, the estimate of $\hat{a}\rangle$ and the variance will be the same except for this sign changes and will not be presented here.

Case 4. Discrete Orthogonal Polynomials Back to Front

This case is the result of a Gram-Schmidt process on the case 1 equations, that is by Eq. (21)

$$\hat{a}\ \langle 3\rangle_j = D^{-1}(ho)T_u^{-1}(\beta)(N^TN)^{-1}\ N^T\ z\ \ \langle N+1\rangle_j \tag{52}$$

By Eq. (207) sec (5)

$$\underset{(N+1)\times 3}{G(N)} = \underset{(N+1)\times 3}{N}\ \underset{3\times 3}{B_g(N)} \tag{53}$$

where by Eq. (200) sec (5)

$$B_g(N) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -\frac{2}{N} & -\frac{6}{N-1} \\ 0 & 0 & \frac{6}{N(N-1)} \end{bmatrix} \tag{54}$$

The inverse discreet metric $(N^TN)^{-1}$ is given by Eq. (223) as

$$(N^TN)^{-1} = B_g M_{gg}^{-1} B_g^T \tag{55}$$

where by Eq. (210) sec (5)

$$M_{gg}^{-1} = \left[G^T(N)\ G(N) \right]^{-1} \tag{56}$$

or elementwise

$$M_{gg}^{-1} = \begin{bmatrix} \frac{1}{N+1} & 0 & 0 \\ 0 & \frac{3N}{(N+1)(N+2)} & 0 \\ 0 & 0 & \frac{5N(N-1)}{(N+1)(N+2)(N+3)} \end{bmatrix} \tag{57}$$

By Eq. (53)

$$N = G(N)B_g^{-1} \tag{58}$$

and

$$N^T = B_g^{-T}G^T(N) \tag{59}$$

The psuedo-inverse of N is

$$\underset{3x(N+1)}{N^*} = (N^TN)^{-1}\ N^T \tag{60}$$

and using Eq. (59) and Eq. (55) in Eq. (60)

$$(N^TN)^{-1}\ N^T = B_g M_{gg}^{-1} B_g^T\ (B_g^{-1}G^T(N)) \tag{61}$$

or

$$(N^TN)^{-1}\ N^T = B_g\ (G^T(N)G(N))^{-1}\ G^T(N) \tag{62}$$

or by Eq. (53)

$$(N^TN)^{-1}\ N^T = B_g M_{gg}^{-1} B_g^T N^T \tag{63}$$

Using Eq. (63) in Eq. (52)

$$\hat{a}\ \langle 3 \rangle_j = D^{-1}(ho)T_u^{-1}(\beta)B_g M_{gg}^{-1} B_g^T N^T\ z\ \langle N+1 \rangle_j \tag{64}$$

Multiply Eq. (64) on the left by $D(ho)\ T_u(\beta)B_g^{-1}$ to obtain

$$B_g^{-1}T_u(\beta)D(ho)\hat{a}\rangle = \hat{a}^g\rangle = M_{gg}^{-1}B_g^TN^Tz\rangle \tag{65}$$

or

$$\hat{a}^{g}\rangle = M_{gg}^{-1} G^{T}(N) z \langle k \rangle \quad (66)$$

By Eq. (205) and Eq. (207) sec (5)

$$\underset{(N+1)x3}{G(N)} = NB_g(N) \quad (67)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ 1 & n & n^2 \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ 1 & N & N^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -\frac{2}{N} & -\frac{6}{N-1} \\ 0 & 0 & \frac{6}{N(N-1)} \end{bmatrix} \quad (68)$$

and by Eq. (208) sec (5)

$$\langle^{n} 3) \; g(N) = (1, n, n^2) \begin{bmatrix} 1 & 1 & 1 \\ 0 & -\frac{2}{N} & -\frac{6}{N-1} \\ 0 & 0 & \frac{6}{N(N-1)} \end{bmatrix} \quad (69)$$

$$\langle^{n} 3) \; g = \left(1, 1 - \frac{2n}{N}, 1 - \frac{6n}{N} + \frac{6n(n-1)}{N(N-1)}\right)$$

hence the G matrix by Eq. (69) in Eq. (68) is for n = 0, 1 ... N)

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 - \frac{2}{N} & 1 - \frac{6}{N} \\ 1 & 1 - \frac{4}{N} & 1 - \frac{12}{N} + \frac{12(2-1)}{N(N-1)} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & 1 & 1 \end{bmatrix} \quad (70)$$

By Eq. (70) and Eq. (57) in (66)

$$\hat{a}^g\rangle = \begin{bmatrix} \frac{1}{N+1} & 0 & 0 \\ 0 & \frac{3N}{(N+1)(N+2)} & 0 \\ 0 & 0 & \frac{5N(N-1)}{(N+1)(N+2)(N+3)} \end{bmatrix} G^T z\rangle \tag{71}$$

where

$$G^T z\rangle = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1-\frac{2}{N} & 1-\frac{4}{N} & \cdots & 1 \\ 0 & 1-\frac{6}{N} & 1-\frac{12}{N}+\frac{12}{N(N-1)} & \cdots & 1 \end{bmatrix} \begin{bmatrix} z(0) \\ z(1) \\ \cdot \\ \cdot \\ \cdot \\ z(N) \end{bmatrix} \tag{72}$$

For polynomials of degree higher than the second degree, the elements of G^T are given by Eq. (209) sec (5).

The variance is given by Eq. (55) in Eq. (36) as

$$\sum_{\tilde{a}\tilde{a}} = D^{-1} T_u^{-1} B_g M_{gg}^{-1} B_g^T T_u^{-1} D^{-1} \tag{73}$$

or

$$T_u(\beta) D(ho) \sum_{\tilde{a}\tilde{a}} D(ho) T_u^T(\beta) = \sum_{\tilde{a}\tilde{a}}^{g} \tag{74}$$

where $\sum_{\tilde{a}\tilde{a}}^{g}$ is a congruent transformation on the variance; hence is a base change, and

$$\sum_{\tilde{a}\tilde{a}}^{g} = B_g M_{gg}^{-1} B_g^T \tag{75}$$

or in open form

$$\sum_{\tilde{a}\tilde{a}}^{g} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -\frac{2}{N} & -\frac{6}{N-1} \\ 0 & 0 & \frac{6}{N(N-1)} \end{bmatrix} \begin{bmatrix} \frac{1}{N+1} & 0 & 0 \\ 0 & \frac{3N}{(N+1)(N+2)} & 0 \\ 0 & 0 & \frac{5N(N-1)}{(N+1)(N+2)(N+3)} \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 1 & -\frac{2}{N} & 0 \\ 1 & -\frac{6}{N-1} & \frac{6}{N(N-1)} \end{bmatrix} \tag{76}$$

Case 5. Midpoint Discrete Orthogonal Polynomials (Ortho-normal)

This case is the orthogonal conditions for Case 2 and by Eq. (249) sec (5)

$$\underset{(N+1)\times 3}{G(M)} = MB_g(M) \tag{77}$$

where by Eq. (250) sec ( )

$$B_g(M) = \begin{bmatrix} 1 & 0 & \frac{-N(N+2)}{12} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{78}$$

or by Eq. (77)

$$M = G(M)B_g^{-1}(M) \tag{79}$$

Using Eq. (79) in the transpose of the fitting function F matrix of Eq. (40)

$$F^T = T^T = DT_u^T(\beta)B_g^{-T}(M)G^T(M) \tag{80}$$

By Eq. (42) the inverse relation becomes

$$(T^TT)^{-1} = T_{hou}^{-1}\ (M^TM)^{-1}\ T_{hou}^{-1} \tag{81}$$

and by Eq. (79) and its transpose

$$M^TM = B_g^{-T}(M)G^T(M)G(M)B_g^{-1}(M) \tag{82}$$

and the inverse is

$$(M^TM)^{-1} = B_g(M)\left(G^T(M)G(M)\right)^{-1}\ B_g^T(M) \tag{83}$$

By Eq. (77)

$$M^T = \underset{3\times 3}{B_g^{-T}(M)}\quad \underset{3\times(N+1)}{G^T(M)} \tag{84}$$

By Eq. (44)

$$\hat{a}\rangle = T_{hou}^{-1}(M^TM)^{-1}M^Tz\rangle \tag{85}$$

The psuedo inverse of M by Eq. (83) and (84) is

$$(M^TM)^{-1}M^T = B_g(M)[G^T(M)G(M)]^{-1}\ G^T(M) \tag{86}$$

Using Eq. (86) in Eq. (85)

$$B_g^{-1}(M)T_{hou}^{-1}\ \hat{a}\rangle = \hat{a}^g\rangle = \left[G^T(M)G(M)\right]^{-1} G^T(M)z\rangle \tag{87}$$

The matrix $G^T(M)$ by Eq. (77) is

$$G^T(M) = B_g^T M^T \tag{88}$$

and by Eq. (78) and Eq. (41) in Eq. (88)

$$G^T(M) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-N(N+2)}{12} & 0 & 1 \end{bmatrix} \begin{bmatrix} \langle 1 & 1 & \langle 1 \\ -\langle cL_c & 0 & \langle c \\ -\langle c^2 L_c & 0 & \langle c^2 \end{bmatrix} \tag{89}$$

where Lc is the linear convolution matrix

$$\underset{M\times M}{L_c} = \begin{pmatrix} 0 & \dots & 1 \\ & \dots & 1 \\ 1 & \dots & 0 \end{pmatrix} \tag{90}$$

Eq. (89) becomes

$$\underset{3x(N+1)}{G^T(M)} = \begin{bmatrix} \langle M)\ 1 & 1 & \langle M)\ 1 \\ -\langle M)cL_c & 0 & \langle M)\ c \\ \frac{-N(N+2)\ \langle M)\ 1 - \langle M)c^2 L_c}{12}, & 0 & \frac{-N(N+2)\ \langle M) 1}{12} + \langle M)c^2 \end{bmatrix} \tag{91}$$

The parameter estimate in the new base now becomes by Eq. (87) and Eq. (255) sec (5) for the inverse metric matrix

$$\hat{a}^g\rangle = \begin{bmatrix} \frac{1}{N+1} & 0 & 0 \\ 0 & \frac{12}{N(N+1)(N+2)} & 0 \\ 0 & 0 & \frac{180}{N(N+1)(N+2)(N^2+2N-3)} \end{bmatrix} G^T(M)z\rangle \tag{92}$$

The midpoint variance matrix of Eq. (46) is

$$\Sigma_{\tilde{a}\tilde{a}} = \lambda\ T_{hou}^{-1}\ (M^T M)^{-1}\ T_{hou}^{-T}. \tag{93}$$

and by Eq. (83) in Eq. (93)

$$\Sigma_{\tilde{a}\tilde{a}} = \lambda\ T_{hou}^{-1}\ B_g(M)\ \ G^T(M)G(M)^{-1}\ B_g^T(M)\ T_{hou}^{-T} \tag{94}$$

Do the base change on Eq. (94) and obtain

$$\sum^{g}_{\tilde{a}\tilde{a}} = \lambda^{-1} T_{hou} \sum_{\tilde{a}\tilde{a}} T^{T}_{hou} \tag{95}$$

where

$$\sum_{\tilde{a}\tilde{a}} = B_g(M) \begin{bmatrix} \frac{1}{N+1} & 0 & 0 \\ 0 & \frac{12}{N(N+1)(N+2)} & 0 \\ 0 & 0 & \frac{180}{N(N+1)(N+2)(N^2+2N-3)} \end{bmatrix} B^{T}_{g}(M) \tag{96}$$

The $\hat{a}\rangle$ of Eq. (92) is for the orthogonal midpoint polynomials. If we use the ortho-normal relations given by Eq. (258) sec (5) as

$$\underset{(N+1)x3}{S} = \underset{(N+1)x3}{M} \; B_s(N) \tag{97}$$

where by Eq. (260) sec (5)

$$B_s(N) = \begin{bmatrix} \frac{1}{\sqrt{N+1}} & 0 & \frac{-N(N+2)\sqrt{180}}{12\ N(N+1)(N+2)(N^2+2N-3)^{\ 1/2}} \\ 0 & \frac{\sqrt{12}}{\sqrt{N(N+1)(N+2)}} & 0 \\ 0 & 0 & \frac{\sqrt{180}}{12\left[N(N+1)(N+2)(N^2+2N-3)\right]^{1/2}} \end{bmatrix} \tag{98}$$

By Eq. (97)

$$SB^{-1}_{s} = M \tag{99}$$

and the metric is

$$M^T M = B^{-1}_{s}\ S^T S B^{-1}_{s} = B^{-T}_{s}\ B^{-1}_{s} \tag{100}$$

and the inverse of Eq. (100) is

$$(M^T M)^{-1} = B_s B^{T}_{s} \tag{101}$$

and since these are ortho-normal polynomials the metric is the identity, that is

$$S^T S = \underset{3x3}{I} \tag{102}$$

hence

$$(M^TM)^{-1} = B_sB_s^T \tag{103}$$

By Eq. (44)

$$\hat{a}\ (3\rangle = T_{hou}^{-1}\ (M^TM)^{-1}\ M^Tz\rangle \tag{104}$$

The psuedo-inverse of M is by Eq. (97) and Eq. (103)

$$M^* = (M^TM)^{-1}\ M^T = B_sB_s^TB_sB_s^{-T}S^T = B_sS^T \tag{105}$$

Using Eq. (105) in Eq. (104)

$$\hat{a}\ (3\rangle = T_{hou}^{-1}\ B_sS^Tz\rangle \tag{106}$$

or

$$B_s^{-1}\ T_{hou}\ \hat{a}\rangle = \ \hat{a}^s\rangle = S^Tz\rangle \tag{107}$$

the parameter in the new base $\hat{a}^s\rangle$ is given by Eq. (107) or

$$M = SB_s^{-1}$$

where by Eq. (97), Eq. (98) and Eq. (41)

$$S^T = B_s^TM^T \tag{108}$$

$$\begin{pmatrix} \frac{1}{\sqrt{N+1}} & 0 & 0 \\ 0 & b_{22} & 0 \\ b_{13} & 0 & b_{33} \end{pmatrix} \begin{pmatrix} \langle 1 & 1 & \langle 1 \\ -\langle cL_c & 0 & \langle c \\ -\langle c^2L_c & 0 & \langle c^2 \end{pmatrix} \tag{109}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{N+1}}\ \langle M)\ 1 & \frac{1}{\sqrt{N+1}} & \langle M)\ \frac{1}{\sqrt{N+1}} \\ -b_{22}\langle M)\,cL_c & 0 & b_{22}\langle M)\,c \\ b_{13}\langle M)\ 1 - b_{33}\langle c^2L_c & b_{13} & b_{13}\langle M)1 + b_{33}\langle c^2 \end{bmatrix}$$

where B_s by Eq. (98) is

$$B_s = \begin{pmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix} \tag{110}$$

Equator (109) is to be used with Eq. (106) to multiply $z\rangle$ to obtain $\hat{a}\rangle$.

The variance matrix by Eq. (101) in Eq. (93) yields

$$\Sigma_{\tilde{a}\tilde{a}} = \lambda T_{hou}^{-1} B_s B_s^T T_{hou}^{-T} . \tag{111}$$

<u>Case 6.</u> <u>Front to Back or (Backward) Orthogonal Polynomials</u>

This case will not be presented here since it is similar to the forward case except for the alternating sign changes mentioned before. The congruent transformation on the $(\Gamma^T\Gamma)^{-1}$ matrix is given by Eq. (278) sec (5) to obtain the diagonal metric matrix.

<u>Case 7.</u> <u>Backward Exponential Weighted Polynomials</u>

This case will consider the fitting of the data by exponentially weighted polynomials with the indices running from the front of the data (real-time or front-time) point to the back. The measurement data will be assumed to go back in time to negative infinity, with zero values prior to the actual time the data is available. By Eq. (169) sec (5) the time variable will be indexed as

$$t(\ell, \gamma) = ho(\beta + \ell) - ho\gamma \tag{112}$$

with the span-generating index ℓ taken to be zero in this case (for the recursive case it will not be zero). By Eq. (171) sec (5) the fitting functions are

$$\langle f_e(t) = \langle t_e = (1,\ t,\ t^2 \ \ldots\ )\ e^{\,at/2} \tag{113}$$

or

$$z_j(t) = \langle f_e(t)a\rangle + v_j(t) = \langle f_e\hat{a}\rangle_j + \tilde{z}_j(t) \tag{114}$$

The index γ (for $\ell = 0$) by Eq. (176) sec (5) is

$$\langle f_e = \ ^{\alpha o/2}\ \Theta^{\gamma/2} \langle \gamma\ T_u(\beta)D(ho) \tag{115}$$

where by Eq. (177) sec ( )

$$\langle \gamma = (1,\ -\gamma,\ \gamma^2,\ -\gamma^3,\ \gamma^4\ \ldots\ ) \tag{116}$$

and packagewise by Eq. (178) sec (5)

$$T_e = \underset{(N+1)xd}{F} = e^{\,\alpha o/2}\ W^{1/2}\ \ \Gamma T_u(\beta)D(ho) \tag{117}$$

and the discrete metric by Eq. (179) sec (5) is for the 3x3 case

$$\underset{3\times3}{T_e^T T_e} = D(ho)\ T_u^T(\beta)\ \Gamma^T W\Gamma T_u(\beta) D(ho)\ e^{\alpha o} \tag{118}$$

The inverse is

$$(T_e^T T_e)^{-1} = T_{hou}^{-1}\ \ (\Gamma^T W\Gamma)^{-1}\ T_{hou}^{-T}\ e^{-\alpha o} \tag{119}$$

The weighted metric inverse is given by Eq. (343) sec (5) as

$$(\Gamma^T W\Gamma)^{-1} = \begin{bmatrix} (1-\theta)(1+\theta+\theta^2) & \frac{3}{2}(1+\theta)(1-\theta)^2 & \frac{(1-\theta)^3}{2} \\ \frac{3}{2}(1+\theta)(1-\theta)^2 & \frac{(1-\theta)^3(1+10\theta+9\theta^2}{4\theta^2} & \frac{(1-\theta)^4(1+3\theta)}{4\theta^2} \\ \frac{(1-\theta)^3}{2} & \frac{(1-\theta)^4(1+3\theta)}{4\theta^2} & \frac{(1-\theta)^5}{4\theta^2} \end{bmatrix} \tag{120}$$

where θ is given by Eq. (305) sec (5)

$$\theta = e^{-aho} \tag{121}$$

and a is related to the time constant in the exponential of Eq. (113).

The estimate of the parameters is given by Eq. (31) as

$$\hat{a}\ (3\rangle = \underset{3\times3}{(T_e^T T_e)^{-1}}\ \ \underset{(3\times N+1)}{T_e^T}\ \ \underset{(N+1)\times1}{z\ (N+1\rangle} = T_e^* z\rangle \tag{122}$$

Notice that

$$\underset{3\times\infty}{T_e^T}\ z\ (\infty\rangle = \left[\underset{3\times(N+1)}{T_e^T,}\ \ \underset{3\times\surd}{T_e}\right] \begin{pmatrix} z\ (N+1\rangle \\ z\ (\surd\rangle \end{pmatrix} \tag{123}$$

where

$$\mu = \infty - N+1$$

and

$$z\ (\infty\rangle = \begin{pmatrix} z\ (N+1\rangle \\ 0\ (\surd\rangle \end{pmatrix} \tag{124}$$

The psuedo-inverse of T_e is

$$T_e^* = (T_e^T T_e)^{-1}\ T_e^T \tag{125}$$

and by Eq. (119) and Eq. (117)

$$\underset{3xN+1}{T_e^*} = T_{hou}^{-1} \ (\Gamma^T W \Gamma)^{-1} \ \Gamma^T W^{1/2} \ e^{-\alpha o/2} \tag{126}$$

Using Eq. (126) in Eq. (122)

$$T_{hou} \hat{a}\rangle = \hat{a}^e\rangle = (\Gamma^T W \Gamma)^{-1} \ \Gamma^T W^{1/2} \ z\rangle \ e^{-\alpha o/2} \tag{127}$$

where by Eq. (174) sec (5)

$$e^{\alpha o/2} = e^{\frac{aho\beta}{2}} = e^{\frac{ato}{2}} \tag{128}$$

and if $t_o = 0$

$$e^{\frac{\alpha o}{2}} = 1 \tag{129}$$

The vector $\Gamma^T W^{1/2} z\rangle$ is by Eq. (180) sec (5)

$$\underset{3x(N+1)}{\Gamma^T} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ -N & -(N-1) & \cdots & -2 & -1 & 0 \\ N^2 & (N-1)^2 & \cdots & 2^2 & 1 & 0 \end{bmatrix} \tag{130}$$

and the $W^{1/2}$ matrix by Eq. (151) sec (5) is

$$W^{1/2} = \begin{bmatrix} 1 & & & & & & & 0 \\ & \theta^{1/2} & & & & & & \\ & & \theta^{2/2} & & & & & \\ & & & \theta^{3/2} & & & & \\ 0 & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & & \theta^{N/2} \end{bmatrix} \tag{131}$$

Note also that Eq. (125) by Eq. (181) sec (5) is

$$\underset{3x(N+1)}{\Gamma^T} = \begin{bmatrix} \langle (N+1) n^o \\ -\langle (N+1) n^1 L_c \\ \langle (N+1) n^2 L_c \end{bmatrix} \tag{132}$$

where

$$\langle (N+1) \ n = (0, 1, 2, 3 \ldots N) \tag{133}$$

and L_c is the linear convolution matrix.

Using Eq. (128) and Eq. (126) in the vector

$$\Gamma^T W^{1/2} z\rangle = \begin{bmatrix} \langle N+1)\ n_o\ W^{1/2}\ z\ (N+1\rangle \\ -\langle N+1)\ n_1\ Lc\ W^{1/2}\ z\ (N+1\rangle \\ \langle N+1)\ n^2\ Lc\ W^{1/2}\ z\ (N+1\rangle \end{bmatrix} \tag{134}$$

Thus $\hat{a}^e\rangle$ is obtained by the products of Eq. (120) and Eq. (128) and Eq. (123).

The variance is given by Eq. (7) for scalar-matrix noise as

$$\sum_{\tilde{a}\tilde{a}} = T_e^* \, T_e^{*T} \, \sigma_v \tag{135}$$

Using Eq. (126) and its transpose in Eq. (129)

$$\sum_{\tilde{a}\tilde{a}} = T_{hou}^{-1} \, (\Gamma^T W \Gamma)^{-1} \, T_{hou}^{-1} \quad e^{-\alpha o} \, \sigma_v \tag{136}$$

Section 12 UNCONSTRAINED UNWEIGHTED STATE ESTIMATION

This section relates the state estimation relations to the parameter estimation relations of the previous sections.

By Eq. (13) sec (10)

$$z(N+1\rangle = \underset{(N+1)xd}{F}\, a(d\rangle + v(N+1\rangle = F\,\hat{a}(d\rangle + \bar{z}\rangle \tag{1}$$

By Eq. (13) sec (4)

$$\begin{bmatrix} x(t) \\ \dot{x} \\ \cdot \\ \cdot \\ x^{(d+1)} \end{bmatrix} = \begin{bmatrix} \langle f(t) \\ \langle f(t)V_f \\ \cdot \\ \cdot \\ \langle f(t)V_f^{\,d-1} \end{bmatrix} a(d\rangle \tag{2}$$

For the class of fitting functions with

$$\langle f(o) = (1,\ 0,\ 0\ \ldots\ ) = {}_{1}\langle e \tag{3}$$

We have by Eq. (2)

$$X(o\rangle_s = \underset{dxd}{F_v(o)}\, a(d\rangle \tag{4}$$

Solving for the parameters

$$a(d\rangle = F_v^{-1}(o)\ X(o)\ \ (d\rangle_s \tag{5}$$

<u>Polynomials Back to Front</u>

The discrete state transition is given by Eq. (160) sec (4)

where

$$t^1 = t = t_b$$

with

$$t = t_b + Nho$$

or

$$t^1 = Nho$$

hence Eq. (160) sec (4) becomes for the 3x3 case of polynomial functions

$$\Phi_x(N,o) = \Phi_x^{\,N} = \begin{bmatrix} 1 & Nho & \frac{N^2ho^2}{2} \\ 0 & 1 & Nho \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

and the inverse by Eq. (164) sec (4) is

$$\Phi_x^{-N} = \begin{bmatrix} 1 & -Nho & \frac{N^2 ho^2}{2} \\ 0 & 0 & -Nho \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

The discrete state transition matrix of Eq. (6) by Eq. (158) sec (4) yields

$$x(N) \; (3\rangle_s = \Phi_x^N X(o) \; (3\rangle_s \tag{8}$$

or

$$x(o) \; (3\rangle_s = \Phi_x^{-N} X(N) \; (3\rangle_s \tag{9}$$

Using Eq. (9) in Eq. (5)

$$a(d\rangle = F_v^{-1}(o) \; \Phi_x^{-N} \; x(N) \; (3\rangle_s \tag{10}$$

The measurements of Eq. (1) are

$$\begin{bmatrix} z(o) \\ z(1) \\ \cdot \\ \cdot \\ z(N) \end{bmatrix} = \begin{bmatrix} x(o) \\ x(1) \\ \cdot \\ \cdot \\ x(N) \end{bmatrix} + \begin{bmatrix} v(o) \\ v(1) \\ \cdot \\ \cdot \\ v(N) \end{bmatrix} \tag{11}$$

hence by Eq. (10)

$$x \; (N+1\rangle = \begin{bmatrix} x(o) \\ x(1) \\ \cdot \\ \cdot \\ x(N) \end{bmatrix} = \underset{(N+1)x3}{F} \; a\rangle = \underset{(N+1)x3}{F} \; \underset{3x3}{F_v^{-1}(o)} \; \Phi_x^{-N} \; x(N) \; (3\rangle_s \tag{12}$$

which relates the states of the function at time corresponsing to N to the function values at the (N+1) time points. Inverting Eq. (12)

$$x(N) \; (3\rangle_s = \Phi_x^N \; F_v(o) \; \underset{3x(N+1)}{F^*} \; x \; (N+1\rangle \tag{13}$$

The above relations will be investigated for the monomial polynomial base and the exponentially weighted polynomial base.

Consider first the 3x3 matrix of Eq. (10) by Eq. (7) and Eq. (16) sec (4)

$$F_v^{-1}(o)\ \Phi_x^{-N} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -Nho & N^2h^o/2 \\ 0 & 1 & -Nho \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

or

$$F_v^{-1}(o)\ \Phi_x^{-N} = \begin{bmatrix} 1 & -Nho & N^2h_o^2/2 \\ 0 & 1 & -Nho \\ 0 & 0 & 1/2 \end{bmatrix} \tag{15}$$

Also the inverse of Eq. (15) is

$$\Phi_x^N\ F_v(o) = \begin{bmatrix} 1 & Nho & N^2h_o^2/2 \\ 0 & 1 & Nho \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

By Eq. (30) sec ( ) for a forward cycling filter

$$T^* = D^{-1}(ho)\ T_u^{-1}(\beta)\ N^* \tag{17}$$

For the special case of $t_o=0$, the matrix product of Eq. (13) becomes

$$\Phi_x^N\ F_v(o)F^* = \Phi_x^N\ T_v(o)\ D^{-1}(ho)(N^TN)^{-1}\ N^T \tag{18}$$

The intermediate matrix product term of Eq. (18) is

$$D(ho)\ T_v^{-1}(o)\ \Phi_x^{-N} = \begin{bmatrix} 1 & -Nho & N^2ho^2 \\ 0 & ho & -Nho^2 \\ 0 & 0 & \frac{ho^2}{2} \end{bmatrix} \tag{19}$$

The matrix product term of Eq. (12) is for the forward cycling filter by Eq. (14) sec (11)

$$F\ F_v^{-1}(o)\ \Phi^{-N} = ND(ho)\ F_v^{-1}(o)\ \Phi^{-N} \tag{20}$$

By Eq. (16) sec (11) and Eq. (19)

$$
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 1 \\
1 & 2 & 2^2 \\
1 & 3 & 3^2 \\
\cdot & & \\
\cdot & & \\
\cdot & & \\
1 & (N-1) & (N-1)^2 \\
1 & N & N^2
\end{bmatrix}
\begin{bmatrix}
1 & -Nho & \frac{N^2ho^2}{2} \\
0 & ho & -Nho^2 \\
0 & 0 & \frac{h^2o}{2}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
1 & -Nho & N^2ho^2/2 \\
1 & -(N-1)ho & (N-1)^2ho^2/2 \\
1 & -(N-2)ho & (N-2)^2ho^2/2 \\
\cdot & & \\
\cdot & & \\
\cdot & & \\
1 & -ho & ho^2/2 \\
1 & 0 & 0
\end{bmatrix} \tag{21}
$$

Note that Eq. (21) used in Eq. (12) relates the states at Nho time points ahead to the function values. The normal Taylor series expansion is to relate the function values to the initial states, that is for example

$$
\begin{aligned}
x(1) &= x(o) + \dot{x}(o)ho + \ddot{x}(o)\frac{ho^2}{2:} \\
x(2) &= x(o) + \dot{x}(o)zho + \ddot{x}(o)\frac{(zho^2)}{2:} \\
&\cdot \\
&\cdot \\
&\cdot \\
x(N) &= x(o) + \dot{x}(o)Nho + \ddot{x}(o)\frac{(Nho^2)}{2:}
\end{aligned} \tag{22}
$$

or packaging

$$
\begin{bmatrix}
x(o) \\
x(1) \\
x(2) \\
\cdot \\
\cdot \\
\cdot \\
x(N)
\end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 \\
1 & ho & h^2o/2 \\
1 & zho & (zho)^2/2: \\
 & & \\
 & & \\
1 & Nho & (Nho)^2/2:
\end{bmatrix}
\begin{pmatrix}
x(o) \\
\dot{x}(o) \\
\ddot{x}(o)
\end{pmatrix} \tag{23}
$$

The above matrix can be compared with the matrix of Eq. (21).

Observe that the state transition matrix for the polynomials yields

$$x(1) = {}^{1}\langle e\ \Phi\ x(o)\rangle_s = {}^{1}\langle e\ x(1)\rangle_s$$

$$x(2) = {}^{1}\langle e\ x(2)\rangle_s = {}^{1}\langle e\ \Phi^2\ x(o)\rangle_s$$

$$\vdots$$

$$x(N) = {}^{1}\langle e\ x(N)\rangle_s = {}^{1}\langle e\ \Phi^N\ x(o)\rangle_s$$

or

$$\begin{bmatrix} x(o) \\ x(1) \\ x(2) \\ \cdot \\ \cdot \\ \cdot \\ x(N) \end{bmatrix} = \begin{bmatrix} {}^{1}\langle e\ \Phi^o \\ {}^{1}\langle e\ \Phi \\ {}^{1}\langle e\ \Phi^2 \\ \\ \\ \\ {}^{1}\langle e\ \Phi^N \end{bmatrix} x(o)\rangle_s \tag{24}$$

By Eq. (11) and Eq. (1)

$$x\ \langle N+1\rangle = \underset{(N+1)x3}{F}\ a\ \langle d\rangle \tag{25}$$

and by Eq. (5) in Eq. (25)

$$x\ \langle N+1\rangle = F\ F_v^{-1}(o)\ x(o)\ \langle 3\rangle_s \tag{26}$$

By Eq. (24) and Eq. (26)

$$\underset{(N+1)x3}{F\ F_v^{-1}(o)} = \begin{bmatrix} {}^{1}\langle e\ \Phi^o \\ {}^{1}\langle e\ \Phi \\ \cdot \\ \cdot \\ \cdot \\ {}^{1}\langle e\ \Phi^N \end{bmatrix} \tag{27}$$

we have also

$$x(N)\ \langle 3\rangle_s = \Phi^N x(o)\ \langle 3\rangle_s \tag{28}$$

or

$$x(o)\ \langle 3\rangle_s = \Phi^{-N} x(N)\ \langle 3\rangle_s \tag{29}$$

and using Eq. (29) in Eq. (24)

$$x\ (N+1\rangle = \begin{bmatrix} 1\langle e\ \Phi^{-N} \\ 1\langle e\ \Phi^{-N+1} \\ 1\langle e\ \Phi^{-N+2} \\ \cdot \\ \cdot \\ \cdot \\ 1\langle e\ \Phi^{-1} \\ 1\langle e\ \Phi^{0} \end{bmatrix} \quad x(N)\ (3\rangle_s \tag{30}$$

which is the same as Eq. (12); hence, by Eq. (12) and Eq. (30)

$$\underset{(N+1)x3}{F} \quad F_v^{-1}(o)\ \Phi_x^{-N} = \begin{bmatrix} 1\langle e\ \Phi^{-N} \\ 1\langle e\ \Phi^{-N+1} \\ 1\langle e\ \Phi^{-N+2} \\ \cdot \\ \cdot \\ \cdot \\ 1\langle e\ \Phi^{0} \end{bmatrix} \tag{31}$$

Using Eq. (12) in Eq. (11)

$$z\ (N+1\rangle = FF_v^{-1}(o)\ \Phi^{-N}\ x(N)\ (3\rangle_s\ +\ v\ (N+1\rangle \tag{32}$$

$$= FF_v^{-1}(o)\ \Phi^{-N}\ \hat{x}(N,N)\ (2\rangle_s\ +\ \tilde{z}(N\rangle \tag{33}$$

where

$$\hat{x}(N,N)\ (3\rangle_s\ = \Phi^N F_v(o) F^*\ z\ (N+1\rangle \tag{34}$$

where $\hat{x}(N,N)\ (3\rangle_s$ is the unweighted estimate of the states at t=Nho based on using all of the first N+1 measurements, or data up to time Nho.

The matrix product terms of Eq. (34) are given by Eq. (18) and by Eq. (11)

$$\Phi^N F_v(o) F^* = \Phi^N T_v(o) D^{-1}(ho)(N^T N)^{-1}\ N^T. \tag{35}$$

By Eq. (10) we have

$$\tilde{x}(N,N)\ (d\rangle_s\ = \Phi^N F_v(o)\ \tilde{a}\ (d\rangle \tag{36}$$

and

$$\tilde{x}(N,N) = \langle d)\ \tilde{a}\ F_v^T(o)(\Phi^A)^T \tag{37}$$

and the variance matrix under a base change transforms via a congruent transformation, that is

$$E(\tilde{x}(N,N) >< \tilde{x}(N,N)) = \Sigma_{\tilde{x}\tilde{x}} \tag{38}$$

$$= \Phi^N F_v(o) \Sigma_{\tilde{a}\tilde{a}} F_v^T(o)(\Phi^N)^T \tag{39}$$

By Eq. ( ) sec ( ) in Eq. (39)

$$\Sigma_{\tilde{x}\tilde{x}} = \Phi^N F_v(o) D^{-1}(ho)(N^T N)^{-1} D^{-1}(ho) F_v^T(o)(\Phi^N)^T \sigma_v \tag{40}$$

Section 13 RECURSIVE PSUEDO INVERSE AND PROJECTORS

Given a matrix T of rank $d \leq k$, the psuedo inverse of T is given as

$$\underset{d\times k}{T^{*}} = \underset{d\times d}{(T^{T}T)^{-1}}\;\underset{d\times k}{T^{T}} \tag{1}$$

thus the computation of the psuedo inverse can be obtained as the inverse of a symmetric (Grammian) matrix $T^{T}T$. If we now have a new matrix (k + 1)xd in size, with $\underset{k\times d}{T}$ as a submatrix

$$\underset{(k+1)\times d}{T} = \begin{pmatrix} \underset{k\times d}{T} \\ \langle t_{k+1} \end{pmatrix} \tag{2}$$

the psuedo inverse is given as

$$\underset{d\times(k+1)}{T^{*}} = \underset{d\times d}{(T^{T}T)^{-1}}\;\underset{d\times(k+1)}{T^{T}} \tag{3}$$

where

$$\underset{d\times(k+1)}{T^{T}} = \left[\underset{d\times k}{T^{T}},\; t\rangle_{k+1}\right] \tag{4}$$

and the symetric Grammian is

$$\underset{d(k+1)d}{T^{T}T} = \underset{d(k)d}{T^{T}T} + t\rangle_{k+1}\,{}_{k+1}\langle t \tag{5}$$

the previous Grammian plus a dyad.

The inverse of the new Grammian as a function of the inverse of the old Grammian is given by Householder as

$$\underset{d(k+1)d}{(T^{T}T)^{-1}} = \underset{d(k)d}{(T^{T}T)^{-1}}\left[I - \frac{t\rangle_{k+1}\,{}_{k+1}\langle t\;\underset{d(k)d}{(T^{T}T)^{-1}}}{1 + {}_{k+1}\langle t\,\underset{d(k)d}{(T^{T}T)^{-1}}\,t\rangle_{k+1}}\right] \tag{6}$$

If we partition the new psuedo inverse of Eq. (3) as

$$\underset{d\times(k+1)}{T^{*}} = \left[\underset{d\times k}{T^{*}(k+1)},\; t^{*}\rangle_{k+1}\right] \tag{7}$$

$$= \left[\underset{d(k+1)d}{(T^{T}T)^{-1}}\;\underset{d\times k}{T^{T}};\; \underset{d(k+1)d}{(T^{T}T)^{-1}}\;t\rangle_{k+1}\right] \tag{8}$$

Using Eq. (6) in Eq. (8)

$$\underset{dx(k+1)}{T^*} = \left[\left(\underset{dxk}{T^*} - \frac{\underset{d(k)d}{(T^TT)^{-1}} t \,|d\rangle_{k+1} \langle d|\, t\, T^*}{1 + \langle t\, \underset{d(k)d}{(T^TT)^{-1}}\, t \rangle_{k+1}} \right) ; \; \frac{\underset{d(k)d}{(T^TT)^{-1}} t \,|d\rangle_{k+1}}{1 + \langle t\, \underset{d(k)d}{(T^TT)^{-1}}\, t \rangle_{k+1}} \right] \tag{9}$$

Note that

$$\underset{dxk}{T^*}(k+1) = \underset{dxk}{T^*} - \frac{(T^TT)^{-1} \,|t\rangle_{k+1}\langle t|\, \underset{dxk}{T^*}}{1 + \langle t\, (T^TT)^{-1}\, t \rangle_{k+1}} \tag{10}$$

and

$$t^* |d\rangle_{k+1} = \frac{(T^TT)^{-1} t \,|d\rangle^{k+1}}{1 + \langle t\, (T^TT)^{-1}\, t \rangle_{k+1}} \tag{11}$$

If we consider $\underset{kxd}{T}$ as d column vectors in k space, then the k x k matrix projector onto the subspace spanned by the d vectors is

$$\underset{kxk}{TT^*} = \underset{kxk}{P_{TT^*}} \tag{12}$$

The Orthogonal compliment projector is

$$\underset{kxk}{\tilde{P}_{TT^*}} = I - \underset{kxk}{P_{TT^*}} \tag{13}$$

The expanded matrix is

$$\underset{(k+1)xd}{T} = \begin{pmatrix} \underset{kxd}{T} \\ \langle d|\, t_{k+1} \end{pmatrix} \tag{14}$$

and the expanded projector by Eq. (7) is

$$TT^*_{(k+1)(k+1)} = \begin{pmatrix} T_{kxd} \\ {}^{k+1}\langle d| t \end{pmatrix} \left[T^*_{dxk}(k+1),\ t^* |d\rangle_{k+1} \right]$$

$$= \begin{bmatrix} TT^*_{kxk}(k+1) & T_{kxd}\, t^* |d\rangle_{k+1} \\ {}_{k+1}\langle d| t\ T^*_{dxk}(k+1) & {}_{k+1}\langle t \quad t^*\rangle_{k+1} \end{bmatrix} \tag{15}$$

The k x k submatrix of Eq. (15) is

$$T_{kxk}\, T^*(k+1) = T_{kxd}\left[T^*_{dxk} - \frac{(T^TT)^{-1}_{d(k)d}\, t\rangle \langle tT^*_{dxk}}{1 + \langle t(T^TT)^{-1}t\rangle}\right]$$

or

$$TT^*_{k(d)k}(k+1) = TT^*_{kxk} - \frac{T^{*T}_{kxd}\, t\rangle_{k+1}\ {}^{k+1}\langle t\ T^*}{1 + {}_{k+1}\langle t\ (T^TT)^{-1}t\rangle_{k+1}} \tag{16}$$

The other submatrices are by Eq. (11)

$$Tt^*\rangle_{k+1} = \frac{T(T^TT)^{-1}t\rangle_{k+1}}{1 + {}_{k+1}\langle t(T^TT)^{-1}t\rangle_{k+1}} \tag{17}$$

The inner-product of Eq. (15) is

$${}_{k+1}\langle tt^*\rangle_{k+1} = \frac{{}^{k+1}\langle t(T^TT)^{-1}t\rangle^{k+1}}{1 + {}_{k+1}\langle t(T^TT)^{-1}t\rangle_{k+1}} \tag{18}$$

and the remaining term of Eq. (15) is

$${}_{k+1}\langle tT^*(k+1) = \frac{{}^{k+1}\langle tT^*}{1 + \langle t(T^TT)^{-1}t\rangle} \tag{19}$$

Using (16), (17), (18) and (19) in (15)

$$TT^*_{(k+1)(k+1)} = \begin{bmatrix} TT^*_{kxk} - \dfrac{T^{*T}t\rangle \langle tT^*}{\Delta} & \dfrac{T^{*T}t\rangle}{\Delta} \\ \dfrac{\langle tT^*}{\Delta} & \dfrac{\Delta - 1}{\Delta} \end{bmatrix} \tag{20}$$

$$\Delta = 1 + \langle t (T^T T)^{-1} t \rangle \tag{21}$$

Note that the commute of Eq. (20) is

$$\underset{(dxk+1)}{T^*} \underset{(k+1)xd}{T} = \underset{dxd}{I} = \left[\underset{dxk}{T^*(k+1)}, \; t^* \langle d \rangle_{k+1} \right] \begin{bmatrix} \underset{kxd}{T} \\ \langle t \end{bmatrix} \tag{22}$$

or

$$I = \underset{dxk}{T^*(k+1)} \; \underset{kxd}{T} + t^* \rangle_{k+1}^{k+1} \langle t \tag{23}$$

(B) Recursive Psuedo Inverses of Full Rank Expanding Rectangular Matrices Times Full Rank Square Weighting Matrix and Recursions on Their Projectors.

The matrix relations in this section are useful in weighted least squares. Consider the product of a k x k full rank matrix M times the full rank matrix T, or

$$\underset{kxk}{M} \; \underset{kxd}{T} = \underset{kxd}{T_M} \tag{24}$$

and

$$T_M{}^T = T^T M^T \tag{25}$$

The generalized inverse is

$$\underset{dxh}{T_M{}^*} = \underset{dxd}{(T_M{}^T \, T_M)^{-1}} \; T_M{}^T \tag{26}$$

$$\underset{dxk}{T_M{}^*} = (T^T \, M^T \, MT)^{-1} \, T^T \, M^T \tag{27}$$

The k x k matrix orthogonal projector onto the space spanned by the columns of Eq. (24) is

$$\underset{kxk}{T_M T_M{}^*} = MT \; (T^T M^T MT)^{-1} \; T^T M^T \tag{28}$$

The expanded T matrix times the corresponding (k + 1) x (k + 1) M matrix is

$$\underset{(k+1)xd}{T_M} = \underset{(k+1)(k+1)}{M} \quad \underset{(k+1)xd}{T} \tag{29}$$

with psuedo inverse

$$\underset{dx(k+1)}{T_M{}^*} = (\underset{dx(k+1)}{T^T} \quad \underset{(k+1)}{M^T} \quad \underset{(k+1)xd}{MT})^{-1} \quad \underset{dx(k+1)}{T^T} \quad M^T \tag{30}$$

The expanded T matrix and its transpose is given in partitioned form in Eq. (2) and Eq. (4), hence it is convenient to partition M as

$$\underset{(k+1)(k+1)}{M} = \begin{bmatrix} \underset{k\times k}{M(k+1)} & m\,|k\rangle_{k+1} \\ \langle\mu_{k+1} & m_{(k+1)k+1} \end{bmatrix} \tag{31}$$

and the expanded Grammian is

$$\underset{(k+1)(k+1)}{M^T M} = \begin{bmatrix} \underset{k\times k}{M^T(k+1)} & \mu\rangle_{k+1} \\ \langle m_{k+1} & m_{(k+1)_{k+1}} \end{bmatrix} \begin{bmatrix} M(k+1) & m\,k\rangle \\ \langle\mu & m_{k+1} \end{bmatrix} \tag{32}$$

$$= \begin{bmatrix} M^T(k+1)M(k+1) + \mu\rangle\langle\mu & M^T(k+1)m\rangle + \mu\rangle m_{(k+1)} \\ \langle m M(k+1) + m\langle\mu & \langle mm\rangle + m^2 \end{bmatrix}_{k+1,\ k+1}$$

or

$$\underset{(k+1)(k+1)}{G} = \begin{bmatrix} \underset{k\times k}{G(k+1)} & g\,|k\rangle_{k+1} \\ \langle k|\,g_{k+1} & g_{k+1,\ k+1} \end{bmatrix} \tag{33}$$

Using (33) in square matrix inverse term of Eq. (30)

$$(\underset{d\times k+1}{T^T}\underset{(k+1)}{G}\ \underset{(k+1)\times d}{T})^{-1}$$

$$= \left(\begin{bmatrix} \underset{d\times k}{T^T}, & t\rangle \end{bmatrix} \begin{bmatrix} \underset{k\times k}{G(k+1)} & g\rangle \\ \langle g & g \end{bmatrix} \begin{pmatrix} \underset{k\times d}{T} \\ \langle t \end{pmatrix} \right)^{-1}$$

$$= \begin{bmatrix} \underset{d(k)d}{T^T G(k+1)\, T} + \underset{k+1}{t\rangle\langle g}\ \underset{k+d}{T} + T^T g\rangle\langle t + t\rangle\langle t\ g \end{bmatrix}^{-1} \tag{34}$$

The sums of the three dyads of Eq. (34) can be written as

$$t\rangle\langle gT + T^T g\rangle\langle t + t\rangle\langle t\ g$$

$$= \begin{bmatrix} t\,|d\rangle, & \underset{d\times k\ \times\ k\times 1}{T^T g\rangle}, & t\,|d\rangle \end{bmatrix} \begin{bmatrix} \langle gT \\ \langle t \\ \langle t\ g \end{bmatrix} = \underset{d(3)d}{BC^T} \tag{35}$$

where

$$\underset{dx3}{B} = [t\rangle,\ T^T g\rangle,\ t\rangle] \qquad (36)$$

$$\underset{dx3}{C} = \left[T^T g\rangle,\ t\rangle,\ t\rangle\, g_{k+1,\ k+1}\right] \qquad (37)$$

Equation (34) can be written as

$$\left(\underset{d(k+1)}{T^T}\ \ \underset{(k+1)(k+1)}{G}\ \ \underset{(k+1)xd}{T}\right)^{-1}$$

$$= \left[\underset{dxk}{T^T}\ \underset{kxk}{G(k+1)}\underset{kxd}{T} + \underset{dx3}{B}\ \underset{3xd}{C^T}\right]^{-1}$$

$$= \underset{dxd}{(T^T G(k+1)T)^{-1}}\ x$$

$$\left\{\underset{dxd}{I} - \underset{dx3}{B}\ \left[\underset{3x3}{I} + \underset{3xd}{C^T}\ \underset{dxd}{(T^T G(k+1)T)^{-1}}\ \underset{dx3}{B}\right]^{-1} C^T (T^T G(k+1)T)^{-1}\right\} \qquad (38)$$

Equation (38) comes from Householders inversion of modified matrices (ref.39) which is

$$(A + UV^T)^{-1} = A^{-1} - A^{-1}U\ (I + V^T A^{-1} U)^{-1} V^T A^{-1} \qquad (39)$$

If we impose the constraint (inspired by knowledge of known properties when applied to variance matrices) that the k x k submatrix of the expanded Grammian be equal to the previous Grammian, by Eq. (32)

$$\underset{kxk}{G(k+1)} = \underset{kxk}{G} = \underset{kxk}{M^T(k+1)}\ \underset{kxk}{M(k+1)} + \mu\, k\rangle\langle k\, \mu \qquad (40)$$

then we can obtain the expanded psuedo inverse as a function of the previous one. Let

$$\underset{dxk+1}{T_M^{\ *}} = \left[\underset{dx(k)}{T_M^{\ *}(k+1)},\ t_m^{\ *}\, d\rangle\right]_{k+1} \qquad (41)$$

$$= \underset{d(k+1)d}{(T^T G T)^{-1}}\ \ \underset{(k+1)\ x\ (k+1)}{T^T M^T}$$

where

$$T^T M^T = (T^T,\ t\rangle) \begin{bmatrix} M^T(k+1) & \mu\rangle \\ \langle m & m \end{bmatrix} \qquad (42)$$

$$= \left(\underset{dxk}{T^T M^T(k+1)} + \underset{dxk}{t(d\rangle \langle m}, \ \underset{dx1}{T^T \mu\rangle} + t\rangle m\right)$$

The first submatrix of Eq. (41) by Eq. (42) and Eq. (34) is

$$\underset{dxk}{T_m^*(k+1)} = \underset{dxd}{[T^TGT + BC^T]^{-1}} [T^TM^T + t\rangle\langle m] \tag{43}$$

$$T_m^*(k+1) = (T^TGT)^{-1}\left\{I - B[I + C^T(T^TGT)^{-1}B]^{-1}C^T(T^TGT)^{-1}\right\}(T^TM^T + t\rangle\langle m)$$

$$= \underset{dxh}{T_m^*} + (T^TGT)^{-1} B[I + C^T(T^TGT)^{-1}B]^{-1}C^TT_m^* \tag{44}$$

$$+ \ (T^TGT)^{-1}\left\{I - B[I + C^T(T^TGT)^{-1}B]^{-1}C^T(T^TGT)^{-1}\right\} t\rangle\langle m \tag{45}$$

The (k + 1)th column vector of Eq. (41) is

$$t_m^* \langle d\rangle_{k+1} = \underset{d(k+1)d}{(T^TGT)^{-1}} [\underset{dxk}{T^T\mu} \langle k\rangle + t\langle d\rangle m]$$

$$= \underset{d(k)d}{(T^TGT)^{-1}}\left\{I - B[I + C^T(T^TGT)^{-1}B]^{-1}C^T(T^TGT)^{-1}\right\} [T^T\mu\rangle + t\rangle m] \tag{46}$$

Equation (45) and (46) are quite messy. They can be simplified by assuming G is diagonal which implies that M is an orthogonal matrix. They can be further simplified by assuming that M is diagonal which is equivalent in the variance applications to the uncorrelated assumptions.

The assumptions henceforth will be assumed.

$$g\langle k\rangle = m\langle k\rangle = \mu\langle k\rangle = 0\rangle , \tag{47}$$

and the matrix of Eq. (38) becomes

$$\underset{d(k+1)d}{(T^TGT)^{-1}} = [\underset{d(k)d}{T^TGT} + t\rangle\langle t \ g_{k+1,\ k+1}]^{-1}$$

and by the Householder inversion of Eq. (6)

$$\underset{d(k+1)d}{(T^TGT)^{-1}} = (T^TGT)^{-1} [I - \frac{t\rangle\langle t(T^TGT)^{-1}g]}{1 + g\langle t(T^TGT)^{-1}t\rangle} \tag{48}$$

The transpose of Eq. (42) becomes

$$\underset{dxk+1}{T^TM^T} = [\underset{dxk}{T^TM^T}, \ t\langle d\rangle \ m_{k+1.\ 1+1}] \tag{49}$$

The expanded space psuedo inverse of Eq. (41) by Eq. (48) and (49) is

$$\underset{dxk}{T_M^*(k+1)} = \underset{dxk}{T_M^*} - \frac{(T^TGT)^{-1}t\rangle\langle t \ T_M^* \ g}{1 + g\langle t(T^TGT)^{-1}t\rangle} \tag{50}$$

and the (k + 1)st column of Eq. (41) is

$$t_m^* \langle d \rangle_{k+1} = \frac{(T^T GT)^{-1} t \langle d \rangle_{k+1} \; g}{1 + g \langle t (T^T GT)^{-1} t \rangle} \tag{51}$$

where

$$g = m^2 \tag{52}$$

The expanded space (k + 1) x (k + 1) matrix projector for the diagonal constraints above and using the transpose of Eq. (42)

$$\underset{(k+1)xd}{T_M} = \begin{bmatrix} \underset{kxd}{MT} \\ m \overset{k+1}{\langle} t \end{bmatrix} \tag{53}$$

is

$$\underset{(k+1)xd}{T_M} \; \underset{dxk+1}{T_M^*} = \begin{bmatrix} \underset{kxd}{MT} \\ m \overset{k+1}{\langle} t \end{bmatrix} \begin{bmatrix} T_M^*(k+1), & t_M^* \rangle_{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} \underset{kxd \; dxk}{MTT_M^*(k+1)} & MTt_m^* \rangle_{k+1} \\ m \langle t \; T_M^*(k+1) & m \langle t \; t_M^* \rangle_{k+1} \end{bmatrix} \tag{54}$$

The first submatrix of Eq. (54) is by (50)

$$MTT_M^*(k+1) = MTT_M^* - \frac{MT(T^T GT)^{-1} t \quad t \; T_M^* g}{1 + g \langle t (T^T GT)^{-1} t \rangle}$$

$$= \underset{k(d)h}{T_M T_M^*} - \frac{T_M^{*T} t \rangle \langle t \; T_M^* g}{1 + g \langle t (T^T GT)^{-1} t \rangle} \tag{55}$$

The first row second column vector of Eq. (54) is

$$MTt_M^* \rangle = \frac{MT[(T^T GT)^{-1} t \rangle \; \sqrt{g}}{1 + g \langle t (T^T GT)^{-1} t \rangle}$$

$$= \frac{\underset{kxd}{T^{*T} t \rangle} \sqrt{g}}{1 + g \langle t (T^T GT)^{-1} t \rangle} \tag{56}$$

and due to the symmetry of the projector of (54)

$$m \left\langle t\; T_M^*(k+1) = \frac{\sqrt{g}}{\Delta g} \left\langle t \right. \underset{dxk}{\;\;T^*} \right. \tag{57}$$

The second row, second column element of Eq. (54) is

$$\begin{aligned} m \left\langle tt_M^* \right\rangle &= g \frac{\left\langle t(T^TGT)^{-1}t \right\rangle}{1 + g \left\langle t(T^TGT)^{-1}t \right\rangle} \\ &= \frac{\Delta g - 1}{\Delta g} \end{aligned} \tag{58}$$

where

$$\Delta g = 1 + g \left\langle t(T^TGT)^{-1}t \right\rangle \tag{59}$$

Thus the expanded space projector in terms of the previous projector is

$$\underset{(k+1)(k+1)}{T_M T_M^*} = \begin{bmatrix} \underset{kxk}{T_M T_M^*} - \dfrac{T_M^{*T} t \rangle \langle t\; T_M^* g}{\Delta g} & \dfrac{T_M^{*T} t \rangle \sqrt{g}}{\Delta g} \\ \underset{k+1}{\left\langle t\; \dfrac{T_M^*}{\Delta g} \right.} & \dfrac{\Delta g - 1}{\Delta g} \end{bmatrix} \tag{60}$$

II <u>Recursive Psuedo Inverse of Fixed Size Rectangular Matrices Generated by Shifting all Rows Up, the First Row Out and Adding a New Last Row, and Recursion on Corresponding Projectors.</u>

This section considers a two index sequence of vectors in d space

$$\left\langle \!\!\!\! \bigcirc\!\!\!\!\! d \right.\; t\;(i+n) \tag{61}$$

where i is an open ended running index and n runs from 0 to N, then i goes to i+1, etc. The open ended string of vectors are

$$\begin{array}{l} \langle t(i+o) \\ \langle t(i+1) \\ \quad\vdots \\ \langle t(i+N) \\ \langle t(i+N+1) \\ \langle t(i+N+2) \\ \quad\vdots \\ \langle t(i+2N) \\ \quad\vdots \end{array}$$

(N + 1 Vectors bracketed from $t(i+o)$ to $t(i+N)$; N + 1 Vectors bracketed from $t(i+1)$ to $t(i+N+1)$)

We want to psuedo invert the (N + 1) x d matrices (full rank assumptions for simplicity).

$$\underset{(N+1)xd}{T(i)} = \begin{bmatrix} \langle t(i+o) \\ \cdot \\ \cdot \\ \cdot \\ \langle t(i+N) \end{bmatrix} \tag{62}$$

and the succeeding fixed size rectangular matrix of the same size and rank d, defined as

$$\underset{(N+1)xd}{T(i+1)} = \begin{bmatrix} \langle t(i+1) \\ \cdot \\ \cdot \\ \cdot \\ \langle t(i+N) \\ \langle t(i+N+1) \end{bmatrix} \tag{63}$$

Partition T(i) as

$$T(i) = \begin{bmatrix} \langle t(i+o) \\ \underset{Nxd}{T_n} \end{bmatrix} \tag{64}$$

and

$$T(i+1) = \begin{pmatrix} \underset{Nxd}{T_n} \\ \langle t(i+N+1) \end{pmatrix} \tag{65}$$

where the vectors in the intersection of the two sequences is designated T_n.

The psuedo inverse of Eq. (62) is

$$\underset{dxN+1}{T^*(i)} = \underset{d(N+1)d}{[T^T(i)\ T(i)]^{-1}} \underset{dxN+1}{T^T} \tag{66}$$

The rank d assumption implies that

$$N + 1 \geqq d \tag{67}$$

The inverse of the symmetric matrix is

$$\begin{aligned} \left[T^T(i)\ T(i)\right]^{-1} &= [t\rangle_o {}^o\langle t + T_n^T T_n]^{-1} \\ &= (T_n^T T_n)^{-1}\ I - \frac{t\rangle_o {}^o\langle t(T_n^T T_n)^{-1}}{1 + {}_o\langle t(Tn^T T_n)^{-1} t\rangle_o} \end{aligned} \tag{68}$$

where

$$_o\langle t = \langle t(i+o) \tag{69}$$

The psuedo inverse of (66) is

$$\underset{dxn}{T^*(i)} = [T^T(i)T(i)]^{-1}[t\rangle_o,\ T_n^T] \tag{70}$$

$$= (t^*(i)\rangle_o,\ \underset{dxn}{T^*(i)})$$

The first column vector of Eq. (70) by Eq. (68) is

$$T^*(i)\rangle_o = \frac{(T_n^T T_n)^{-1} t\rangle_o}{1 + \langle t_o\,(T_n^T T_n)^{-1} t\rangle_o} \tag{71}$$

The second matrix of Eq. (70) is

$$\underset{dxN}{T^*(i)} = \underset{dxN}{T_n^*} - \frac{(T_n^T T_n)^{-1} t\rangle_o\ {}^o\langle t\ T_n^*}{1 + {}_o\langle t\ (T_n^T T_n)^{-1} t\rangle_o} \tag{72}$$

where

$$\underset{dxn}{T_n^*} = (T_n^T T_n)^{-1} T_n^T \tag{73}$$

The generalized inverse by Eq. (71) and Eq. (72) in Eq. (70) yields

$$\underset{dxN+1}{T^*(i)} = \left[\frac{(T_n^T T_n)^{-1} t\rangle_o}{\Delta_o},\ T_n^* - \frac{(T_n^T T_n)^{-1} t\rangle_o\,{}^o\langle t\ T_n^*}{\Delta_o}\right] \tag{74}$$

where

$$\Delta_o = 1 + {}_o\langle t\,(T_n^T T_n)^{-1} t\rangle_o \tag{75}$$

The projector onto the space of T(i) is

$$\underset{(N+1)(N+1)}{T(i)\ T^*(i)} = \begin{bmatrix} {}^o\langle d)\ t \\ T_n \end{bmatrix} \left[\frac{(T_n^T T_n)^{-1} t\rangle_o}{\Delta},\ T_n^* - \frac{(T_n^T T_n)^{-1} t\rangle_o\,{}^o\langle t\ T_n^*}{\Delta}\right]$$

$$= \begin{bmatrix} \dfrac{\Delta_o - 1}{\Delta o} & \dfrac{{}^o\langle t\ T_n^*}{\Delta o} \\ \dfrac{T_n^{*T} t\rangle_o}{\Delta o} & T_n T_n^* - \dfrac{T_n^{*T} t\rangle^o\ {}^o\langle t\ T_n^*}{\Delta o} \end{bmatrix} \tag{76}$$

The psuedo inverse of T(i + 1) is

$$\underset{dx(N+1)}{T^*(i+1)} = \underset{dxd}{[T^T(i+1)\ T(i+1)]^{-1}}\ \underset{dxN+1}{T^T(i+1)} \tag{77}$$

The Grammian is

$$T^T(i+1)\ T(i+1) = T_n^T T_n + t\rangle_{N+1}\,{}^{N+1}\langle t \tag{78}$$

where

$${}^{N+1}\langle t = \langle t(i+N+1) \tag{79}$$

By the Householder inversion relation

$$[T^T(i+1)\ T(i+1)]^{-1} = (T_n^T T_n)^{-1}\ \left[I - \frac{t\rangle_{N+1}\,{}^{N+1}\langle t\ (T_n^T T_n)^{-1}}{\Delta N+1}\right] \tag{80}$$

where

$$\Delta_{N+1} = 1 + {}_{N+1}\langle t(T_n^T T_n)^{-1} t\rangle_{N+1} \tag{81}$$

and

$$T^T(i+1) = [T_n^T;\ t\rangle_{N+1}] \tag{82}$$

$$\underset{d\times(N+1)}{T^*(i+1)} = [\underset{d\times N}{T^*(i+1)};\ t^*\rangle_{N+1}] \tag{83}$$

Using Eq. (80) and Eq. (82) in Eq. (77) the first matrix of Eq. (83) is

$$\underset{d\times N}{T^*(i+1)} = T_n^* - \frac{(T_n^T T_n)^{-1} t\rangle_{N+1}\,{}^{N+1}\langle t\ T_n^*}{\Delta N+1} \tag{84}$$

the (N + 1)st vector of Eq. (83) is

$$t^*\rangle_{N+1} = \frac{(T_n^T T_n)^{-1} t\rangle_{N+1}}{\Delta N+1} \tag{85}$$

The generalized inverse by (84) and (85) in (83) yields

$$\underset{d\times(N+1)}{T^*(i+1)} = \left[T_n^* - \frac{(T_n^T T_n)^{-1} t\rangle_{N+1}\,{}^{N+1}\langle t\ T_n^*}{\Delta N+1},\ \frac{(T_n^T T_n)^{-1} t\rangle_{N+1}}{\Delta N+1}\right] \tag{86}$$

and the projector onto the space of T(i + 1) is

$$\underset{(N+1)}{T(i+1)}\ \underset{(N+1)}{T^*(i+1)} = \begin{pmatrix} T_n \\ \langle t_{N+1} \end{pmatrix} \left[T^*(i+1),\ t^*\rangle_{N+1}\right]$$

$$= \begin{bmatrix} \underset{N\times N}{T_nT_n^*} - \dfrac{T_n^{*T}t\rangle_{N+1}\ {}^{N+1}\langle t\ T_n^*}{\Delta N+1} & \dfrac{T_n^{*T}t\rangle_{N+1}}{\Delta N+1} \\ {}_{N+1}\langle t\ \dfrac{T_n^*}{\Delta N+1} & \dfrac{\Delta N+1\ -\ 1}{\Delta N+1} \end{bmatrix} \qquad (87)$$

Equation (86) and (87) obtained the psuedo inverse of the updated matrix and the projector as a function of the corresponding matrices in the intersection. An Alternate derivation could be obtained using the shift up and out matrix of Eq. (5) Sec. (6) or

$$\underset{(N+1)(N+1)xd}{S_{uo}\ T(i)} = \begin{bmatrix} \langle t(i+1) \\ \cdot \\ \cdot \\ \langle t(i+N) \\ \langle o \end{bmatrix} = \begin{bmatrix} T_n \\ \\ \langle o \end{bmatrix} \qquad (88)$$

and

$$\underset{(N+1)xd}{T(i+1)} = \underset{(N+1)xd}{S_{uo}T(i)} + e_{N+1}\rangle_{N+1}\ {}^{N+1}\langle d\ t \qquad (89)$$

An extension of Householder's inversion lemma to the psuedo inverse of rectangular matrices plus a dyad would yield a direct result, however this route will not be pursued at this point. The psuedo inverse of Eq. (89) is

$$\underset{dx(N+1)}{T^*(i+1)} = [\underset{dxd}{T^T(i)\ S_{uo}^T\ S_{uo}T(i)} + t\rangle_{N+1}\ {}^{N+1}\langle t]^{-1}\underset{d(N+1)}{T^T(i)\ S_{uo}^T} \qquad (90)$$

where

$$S_{uo}^T\ S_{uo} = \underset{(N+1)(N+1)}{\begin{pmatrix} \underset{N\times N}{I} & o\rangle \\ \langle o & o \end{pmatrix}} \qquad (91)$$

A less powerful approach can be used; let

$$\underset{(N+1)xd}{T(i+1)} = \underset{(N+1)xd}{T(i)} - e\,{}_{N+1}\rangle_1\ {}^{o}\langle d\ t + e_{N+1}\rangle_{N+1}\ {}^{N+1}\langle d\ t \qquad (92)$$

or

$$\underset{(N+1)xd}{T(i+1)} = T(i) + [-\mathscr{e}_1, \mathscr{e}_{N+1}] \begin{bmatrix} \overset{o}{\langle} t_{N+1} \\ \langle t \end{bmatrix} \tag{93}$$

$$\underset{(N+1)xd}{T(i+1)} = \underset{(N+1)xd}{T(i)} + \underset{(N+1)x2xd}{E\ F} \tag{94}$$

where

$$\langle \mathscr{e}_1 = (1,\ 0,\ 0\ .\ .\ .)$$

$$\underset{dx(N+1)}{T^*(i+1)} = \underset{dx(N+1)}{T^*(i)} - T^*(i)\ E(I + FT^*E)^*\ FT^*(i)$$

Equation (95) is the rectangular analog of Eq. (39).

$$\underset{dx(N+1)}{T^*(i+1)} = \underset{dx(N+1)}{T^*(i)} - T^*(i)\ E\ \underset{2x2}{(I + FT^*(i)E)^*}\ \underset{2xd}{F}\ T^*(i) \tag{95}$$

$$\underset{(N+1)(N+1)}{T(i+1)\ T^*(i+1)} = (T(i) + EF)\ T^*(i)\ \{I - E[I + FT^*E]^*FT^*\}$$

$$= T(i)\ T^*(i)\ \{I - E[I + FT^*(i)E]^*FT^*(i)\} \tag{96}$$

$$+ EFT^*(i)\ \{I - E[I + FT^*E]^*FT^*(i)\}$$

Discrete Transition Matrix of Moving Fixed Span for Polynomial Functions

When the row vectors of the T matrix are polynomials as considered in this paper, then the updated fixed size matrix becomes very simple in structure. The three cases of interest are:

$t(i, n) \quad n = 0, 1, 2, . . . N$
$t(k, m) \quad m = -M, . . . -1, 0, 1, . . . M$
$t(1, \gamma) \quad \gamma = 0, 1, 1, . . . N$

and the polynomial basis by Eq. (38), Eq. (41) and Eq. (43) Section (5) are

$$\langle t(i, n) = \langle \tau(i, n)\ T_u(\beta)\ D(ho) \tag{97}$$
$$\langle t(k, m) = \langle \tau(k, m)\ T_u(\beta)\ D(ho) \tag{98}$$
$$\langle t(1, \gamma) = \langle \tau(1, \gamma)\ T_u(\beta)\ D(ho) \tag{99}$$

The non-dimensional quantities are

$$\langle \tau(i, n) = \langle n\ T_u(i) \tag{100}$$
$$\langle \tau(k, m) = \langle m\ T_u(k) \tag{101}$$
$$\langle \tau(1, \gamma) = \langle \gamma\ T_u(1) \tag{102}$$

where

$$\left.\begin{aligned} &\langle n = (0,\ n,\ n^2,\ n^3,\ .\ .\ .\ n^{d-1}) \\ &\langle m = (1,\ \pm m,\ m^2,\ \pm m^3,\ .\ .\ .\ (\pm m)^{d-1}) \\ &\langle \gamma = (1,\ -\gamma,\ \gamma^2,\ -\gamma^3,\ .\ .\ .\ (-\gamma)^{d-1}) \end{aligned}\right\} \tag{103}$$

If we pack the N + 1 row vector of Eq. (97) into a column as the index n varies

$$\underset{(N+1)xd}{T(i)} = \begin{pmatrix} \langle{}^{0}_{n} \\ \langle{}^{1}_{n} \\ \cdot \\ \cdot \\ \cdot \\ \langle{}^{N}_{n} \end{pmatrix} T_u(i)\ T_u(\beta)\ D(ho) \tag{104}$$

or by Eq. (123) Sec. (5).

$$\underset{(N+1)xd}{T(i)} = \underset{(N+1)xd}{N}\ T_u(i)\ T_u(\beta)\ D(ho) \tag{105}$$

where

$$\underset{(N+1)xd}{N} = \underset{(N+1)xd}{\begin{bmatrix} 1 & 0 & . . . 0 \\ 1 & 1 & . . . 1 \\ 1 & 2 & 2^2 . . . 2^{d-1} \\ 1 & 3 & 3^2 . . . 3^{d-1} \\ 1 & N & N^2 . . . N^{d-1} \end{bmatrix}} = \begin{bmatrix} \langle{}^{0}_{n} \\ \langle{}^{1}_{n} \\ \cdot \\ \cdot \\ \langle{}^{N}_{n} \end{bmatrix} \tag{106}$$

The psuedo inverse of Eq. (105) with full rank factor is

$$\underset{dx(N+1)}{T^*(i)} = D^{-1}(ho)\ T_u^{-1}(\beta)\ T_u^{-1}(i)\ \underset{dxN+1}{N^*} \tag{107}$$

The square matrix inverses are given by Eq. (164) Section (4).

The rectangular matrix psuedo inverse of Eq. (107) is

$$\underset{dx(N+1)}{N^*} = \underset{dxd}{(N^TN)^{-1}}\underset{dx(N+1)}{N^T} \tag{108}$$

The Grammian N^TN is given by Eq. (60) Section 5 in terms of the sums of the power of the natural number. The inverse for a 3x3 matrix is given by Eq. (224) Section (5) as

$$(N^TN)^{-1} = g_o \begin{bmatrix} 3N^2 + 3N + 2 & -(12N + 6) & 10 \\ -(12N + 6) & \frac{4(2N + 1)(8N-3)}{N(N+1)} & \frac{-60}{N-1} \\ 10 & \frac{-60}{N-1} & \frac{60}{N(N+1)} \end{bmatrix} \tag{109}$$

$$g_o = \frac{3}{(N+1)(N+2)(N+3)} \tag{110}$$

and

$$\underset{3xN+1}{N^T} = \begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 2 & 3 & \ldots & N \\ 0 & 1 & 2^2 & 3^2 & \ldots & N^2 \end{bmatrix} = \left[n\rangle_o, n\rangle_1, \ldots n\rangle_N \right] \tag{111}$$

$$\underset{3x(N+1)}{N^*} = [n^* ③\rangle_o \quad n^* ③\rangle_1 \ldots n^* ③\rangle_N] \tag{112}$$

$$= g_o \begin{bmatrix} 3N^2+3N+2 & -(12N+6) & 10 \\ -(12N+6) & \frac{4(2N+1)(8N-3)}{N(N+1)} & \frac{-60}{N-1} \\ 10 & \frac{-60}{N-1} & \frac{60}{N(N+1)} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 2 & 3 & \ldots & N \\ 0 & 1 & 2^2 & 3^2 & \ldots & N^2 \end{bmatrix} \tag{113}$$

$$= g_o \left[(N^TN)^{-1} n\rangle_o, (N^TN)^{-1} n\rangle_1, (N^TN)^{-1} n\rangle_2 \ldots (N^TN)^{-2} n\rangle_N \right] \tag{114}$$

$$= g_o \underset{3x3}{(N^TN)^{-1})} \; [n ③\rangle_o, n ③\rangle_1, \ldots n ③\rangle_N] \tag{115}$$

The time point at indices i and n is

$$t(i, n) = [\beta_o + (i+n)]ho \tag{116}$$

where

$$\beta_o = to/ho \tag{117}$$

and the powers

$$\langle t(i, n) = \langle n \; T_u(i) \; T_u(\beta) \; D(ho) \tag{118}$$

The time point at i + 1 and n is

$$t(i+1, n) = [\beta_o + (i+1+n)]ho \tag{119}$$

and the powers by Eq. (154) appendix (A)

$$\langle t(i+1, n) = \langle n \; T_u(i) \; T_u(1) \; T_u(\beta) \; D(ho) \tag{120}$$

using the commutative property

$$T_u(1) \; T_u(\beta) = T_u(\beta) \; T_u(1) \tag{121}$$

we have

$$\begin{aligned} \langle t(i+1, n) &= \langle n \; T_u(i) \; T_u(\beta) \; D(ho) \; D^{-1}(ho) \; T_u(1) \; D(ho) \\ \langle t(i+1, n) &= \langle t(i, n) \; \Phi_t(i+1, i) \end{aligned} \tag{122}$$

where

$$\underset{dxd}{\Phi_t}(i+1,\ i) = D^{-1}(ho)\ T_u(1)\ D(h0) \tag{123}$$

Packaging the row vector of Eq. (118) in a column

$$\underset{(N+1)xd}{T(i)} = \underset{(N+1)xd}{N\ T_u(i)}\ T_u(\beta)\ D(ho) \tag{124}$$

and packaging Eq. (122)

$$\underset{(N+1)xd}{T(i+1)} = \underset{(N+1)xd}{T(i)}\quad \underset{dxd}{\Phi_t}(i+1,\ i) \tag{125}$$

and

$$\underset{dx(N+1)}{T^*(i+1)} = \underset{dxd}{\Phi_t^{-1}}(i+1,\ i)\ \underset{dx(N+1)}{T^*(i)} \tag{126}$$

Thus the transition matrix Φ_t has inverse.

$$\Phi^{-1} = D^{-1}(ho)\ T_u^{-1}(1)\ D(ho) \tag{127}$$

The new projector is

$$\underset{(N+1)xd}{T(i+1)}\ \underset{dx(N+1)}{T^*(i+1)} = \underset{(N+1)(N+1)}{T(i)\ T^*(i)} = NN^* \tag{128}$$

It is interesting to verify these results with some simple examples. Consider a second degree polynomial with N = 3, or

$$d = 3$$
$$N+1 = 4$$

and for simplicity let β = ho = 1 than

$$\underset{4x3}{T(i)} = \underset{4x3}{N\ T_u(i)} = \begin{pmatrix} {}^{0}\lambda_n \\ {}^{1}\lambda_n \\ {}^{2}\lambda_n \\ {}^{3}\lambda_n \end{pmatrix} T_u(i) \tag{129}$$

or

$$T_{\underset{4x3}{u}}(i) = \begin{bmatrix} [(i+1)^{o}, & (i+1), & (i+1)^{2}] \\ [(i+1)^{o}, & (i+1), & (i+1)^{2}] \\ [(i+2)^{o}, & (i+2), & (i+2)^{2}] \\ [(i+3)^{o}, & (i+3), & (i+3)^{2}] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2^{2} \\ 1 & 3 & 3^{2} \end{bmatrix} \begin{bmatrix} 1 & i & i^{2} \\ 0 & 1 & 2i \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & i & i^{2} \\ 1 & i+1 & i^{2}+2i+1 \\ 1 & i+2 & i^{2}+4i+4 \\ 1 & i+3 & i^{2}+6i+4 \end{bmatrix} \qquad (130)$$

at time point i + 1

$$T(i+1) = \begin{bmatrix} \langle(i+1+0) \\ \langle(i+1+1) \\ \langle(i+1+2) \\ \langle(i+1+3) \end{bmatrix} \quad T_u(i) = \begin{bmatrix} \langle(i+0) \\ \langle(i+1) \\ \langle(i+2) \\ \langle(i+3) \end{bmatrix} \quad T_u(i+1) = N\, T_n(i) T_u(1) \qquad (131)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2^{2} \\ 1 & 3 & 3^{2} \end{bmatrix} \begin{bmatrix} 1 & i & i^{2} \\ 0 & 1 & 2i \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2^{2} \\ 1 & 3 & 3^{2} \end{bmatrix} \begin{bmatrix} 1 & 1+i & 1+2i+i^{2} \\ 0 & 1 & 2(i+1) \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1+i & 1+2i+i^{2} \\ 1 & i+2 & i^{2}+4i+4 \\ 1 & i+3 & i^{2}+6i+9 \\ 1 & i+4 & i^{2}+8i+16 \end{bmatrix}$$

$$
= \begin{bmatrix} (i+1)^o & (i+1) & (i+1)^2 \\ (i+2)^o & (i+2) & (i+2)^2 \\ (i+3)^o & (i+3) & (i+3)^2 \\ (i+4)^o & (i+4) & (i+4)^2 \end{bmatrix} \tag{132}
$$

Midpoint of Span

This section considers the T matrix when the indices are referenced to the time point k at the center of the span and the index m running $\pm$ about k. By Eq. (35, 41) Section (5)

$$
\langle t(k,m) = \langle m\ T_u(k)\ T_u(\beta)\ D(ho) \tag{133}
$$

and advancing k by one point

$$
\langle t(k+1,\ m) = \langle m\ T_u(k)\ T_u(1)\ T_u(\beta)\ D(ho) \tag{134}
$$

By Figure (1) Section (1) we see the time points are related

$$
\begin{pmatrix} t(i+0) \\ t(i+1) \\ \cdot \\ \cdot \\ (t(i+\frac{N}{2}) \\ \\ \cdot \\ \cdot \\ \cdot \\ t(i+N) \end{pmatrix} = \begin{pmatrix} t(k-m) \\ \cdot \\ \cdot \\ \cdot \\ t(k+0) \\ \\ \cdot \\ \cdot \\ \cdot \\ t(k+m) \end{pmatrix} \tag{135}
$$

and

$$
\begin{pmatrix} \langle t(i+0) \\ \langle t(i+1) \\ \cdot \\ \cdot \\ \cdot \\ \langle t(i+\frac{N}{2}) \\ \\ \cdot \\ \cdot \\ \cdot \\ \langle t(i+N) \end{pmatrix} = \begin{pmatrix} \langle t(k-M) \\ \langle t(k-M+1) \\ \cdot \\ \cdot \\ \cdot \\ \langle t(k+0) \\ \\ \cdot \\ \cdot \\ \cdot \\ \langle t(k+M) \end{pmatrix} \tag{136}
$$

or

$$\underset{(N+1)xd}{T(i)} = \underset{(N+1)xd}{T(k)}$$

The time point is

$$t(k,m) = [\beta_o + (k \pm m)]ho \tag{137}$$

and at k+1

$$t(k+1), m) = [\beta_o + (k+1 \pm m)]ho \tag{138}$$

where by Eq. (66) Section (5)

$$\underset{(N+1)xd}{M} = \begin{bmatrix} 1 & -M & M^2 & -m^3 \; . \; . \; . & (-M)^{d-1} \\ 1 & -(M-1) & (M-1)^2 & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 1 & -1 & 1 & -1 \; . \; . \; . \; . & . \; . \; . \; . \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 1 & M & M^2 & & M^{d-1} \end{bmatrix} \tag{139}$$

By Equations (35) and (43) Section (5)

$$\langle t(k+1, m) = \langle m \; T_u(k) \; T_u(1) \; T_u(\beta) \; D(ho) \tag{140}$$

and packagewise

$$\underset{(N+1)xd}{T(k+1)} = \underset{(N+1)xd}{M} \; T_u(k) \; T_u(1) \; T_u(\beta) \; D(ho) \tag{141}$$

and also for Eq. (133) the matrix is

$$\underset{(N+1)xd}{T(k)} = M \; T_u(k) \; T_u(\beta) \; D(ho) \tag{142}$$

As before commuting in Eq. (140)

$$T_u(k) \; T_u(1) = T_u(1) \; T_u(k) \tag{143}$$

we obtain

$$T(k+1) = T(k) \; \phi_t(k+1, k) \tag{144}$$

where the transition matrix is

$$\Phi_t(k+1,\ k) = D^{-1}(ho)\ T_u(1)\ D(ho) = \Phi_t(i+1,\ i) \tag{145}$$

and the psuedo inverse is

$$T^*(k+1) = \Phi_t^{-1} T^*(k) \tag{146}$$

By Eq. (142)

$$T^*(k) = D^{-1}(ho)\ T_u^{-1}(\beta)\ T_u^{-1}(k)\ \underset{dx(N+1)}{M^*} \tag{147}$$

And the psuedo inverse of the rectangular matrix M is

$$\underset{d(N+1)}{M^*} = \underset{dxd}{(M^TM)^{-1}} M^T \tag{148}$$

The Grammian (M^TM) is given by Eq. (119) and the inverse by Eq. (120) Section (5).

The projectors are

$$\underset{dxd}{T^*(k+1)\ T(k+1)} = \underset{dxd}{T^*(k)\ T(k)} = \underset{dxd}{I} \tag{149}$$

and

$$\underset{(N+1)}{T(k+1)}\ \underset{(N+1)}{T^*(k+1)} = \underset{(N+1)}{T(k)}\ \underset{(N+1)}{T^*(k)} = \underset{(N+1)(N+1)}{MM^*} \tag{150}$$

Front of Span

This section considers the T matrix when the indices are referenced to the time point ℓ at the front or leading point of the span and the index γ runs from 0 to N. By Eq. (99)

$$\langle t(\ell,\gamma) = \langle\gamma\ T_u(\ell)\ T_u(\beta)\ D(ho) \tag{151}$$

and advancing ℓ by one time point

$$\langle t(\ell\ 1,\ \gamma) = \langle\gamma\ T_u(\ell)\ T_u(\beta)\ T_u(1)\ D(ho) \tag{152}$$

where by Eq. (44) Sec. (5)

$$\langle\gamma = (1,\ -\gamma,\ \gamma^2,\ -\gamma^3,\ .\ .\ .) \tag{153}$$

A vector with alternating signs.

The matrix of N+1 vectors of Eq. (151) is

$$\underset{(N+1)xd}{T(\ell)} = \underset{(N+1)(d)}{\Gamma}\, T_u(\ell)\ T_u(\beta)\ D(ho) \tag{154}$$

and advancing the front point ℓ by 1

$$\underset{(N+1)xd}{T(\ell+1)} = \Gamma\, T_u(\ell)\ T_u(\beta)\ T_u(1)\ D(ho) \tag{155}$$

or in terms of the transition matrix

$$T(\ell+1) = T(\ell)\ \Phi_t(\ell+1,\ \ell) \tag{156}$$

where as for the two previous cases the transition matrix is

$$\Phi_t(\ell+1,\ \ell) = D^{-1}(ho)\ T_u(1)\ D(ho) \tag{157}$$

and for example for a 4 x 4 matrix one can obtain similar results to equation (131).

SECTION 14

Recursive-Unconstrained-Unweighted Parameter Estimation

This section considers the infinite memory case as opposed to the fixed memory or recursive fixed span cases considered later. From Eq. (13) sec (10)

$$z\rangle_j = \underset{k\times d}{F} \; a\rangle + v\rangle_j = F\hat{a}\rangle_j + \tilde{z}\rangle_j \tag{1}$$

with estimate of the parameters given by Eq. (33) sec (10)

$$\hat{a}\rangle_j = F^* z\rangle_j = (F^T F)^{-1} F^T z\rangle_j \tag{2}$$

Assume now that another measurement is available z(k+1) such that the vector of measurements becomes

$$\begin{pmatrix} z\ (k\rangle \\ z\ (k+1) \end{pmatrix} = z\ (k+1\rangle \tag{3}$$

where

$$\begin{aligned} z\ (k+1) &= \langle f\ (k+1)\ a\rangle + v\ (k+1) \\ &= \langle f\ (k+1)\ \hat{a}\ (k+1)\rangle + \tilde{z}\ (k+1) \end{aligned} \tag{4}$$

we now seek a new parameter vector $\hat{a}(k+1)\rangle$ determined by all of the data points of Eq. (3) and such that

$$\begin{bmatrix} z(1) \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_k \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} \langle f(1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \langle f(k+1) \end{bmatrix} \hat{a}\ (k+1)\rangle + \tilde{z}\ (k+1)\rangle \tag{5}$$

Since this is the unweighted case the sequence j will be omitted as a subscript quite often. Eq. (5) is

$$z\rangle = \underset{(k+1)\times d}{F} \; \hat{a}(k+1)\rangle + \tilde{z}(k+1)\rangle \tag{6}$$

Multiply Eq. (6) on the left by

$$\underset{d\times(k+1)}{F^*} = \underset{d(k+1)d}{(F^T F)^{-1}} \; \underset{d\times(k+1)}{F^T} \tag{7}$$

The psuedo-inverse is given by sec ~~(8)~~ 13 Eq. (9) as

$$\underset{dx(k+1)}{F^*} = \left[F^* - (F^TF)^{-1} \frac{f\rangle\langle f \; F^*}{1+\langle f \; (F^TF)^{-1} \; f\rangle}, \right. \tag{8}$$

$$\left. \frac{(F^TF)^{-1} \; f\rangle}{1+\langle f \; (F^TF)^{-1} \; f\rangle} \right]$$

By Eq. (6) and Eq. (7)

$$\hat{a}\;(k+1)\rangle = \underset{dx(k+1)}{F^*} \; z\rangle \tag{9}$$

which in partitioned form becomes by Eq. (3) and Eq. (8)

$$\hat{a}(k+1)\rangle = F^* \; z\rangle - \frac{(F^TF)^{-1} \; f\rangle \langle f \; F^* \; z \; (k\rangle}{1+\langle f \; (F^TF)^{-1} \; f\rangle} \tag{10}$$

$$+ \frac{(F^TF)^{-1} \; f\rangle \; z(k+1)}{1+\langle f \; (F^TF)^{-1} \; f\rangle}$$

By Eq. (2)

$$\langle f \; F^* \; z \; (k\rangle = \langle f \; \hat{a} \; (d\rangle , \tag{11}$$

but the predicted value of the measurement at k+1 based on the parameter at k is

$$\hat{z}\;(k+1,\, k) = \langle f \; (k+1) \; \hat{a} \; (d\rangle \tag{12}$$

Using Eq. (12) in Eq. (10) with Eq. (2)

$$\hat{a}\;(k+1)\rangle = \hat{a}\;(k)\rangle + w\;(k+1)\rangle \; \tilde{z}\;(k+1,\, k) \tag{13}$$

with

$$\tilde{z}\;(k+1,\, k) = z\;(k+1) - \hat{z}\;(k+1,\, k) \tag{14}$$

and the weight vector is

$$w(k+1)\rangle = \frac{(F^TF)^{-1} \; f(k+1)\rangle}{1+\langle f(k+1) \; (F^TF)^{-1} \; f \; (k+1\rangle} \tag{15}$$

when F is the matrix for the discrete polynomial fitting functions from Eq. (124) sec (13)

$$\underset{kxd}{F} = \underset{kxd}{N} \; \underset{dxd}{T_{u\beta ho}} \tag{16}$$

and

$$\underset{dxk}{F^*} = T_{u\beta ho}^{-1} N^* \tag{17}$$

with

$$N^* = (N^T N)^{-1} N^T \tag{18}$$

where for the second degree polynomial

$$\underset{3x(k+1)}{N^T} = \begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 2 & 3 & \ldots & k \\ 0 & 1 & 2^2 & 3^2 & \ldots & k^2 \end{bmatrix} \tag{19}$$

and for the next time point

$$\underset{3x(k+2)}{N^T} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 1 & 2 & \ldots & k & k+1 \\ 0 & 1 & 2^2 & \ldots & k^2 & (k+1)^2 \end{bmatrix} \tag{20}$$

The Grammian of Eq. (16)

$$\underset{d(k+1)d}{F^T F} = T_{u\beta ho}^T (N^T N) T_{u\beta ho} \tag{21}$$

and

$$(F^T F)^{-1} = T_{u\beta}^{-1} (N^T N)^{-1} T_{u\beta}^{-T} \tag{22}$$

also

$$f(k+1) \; (d\rangle = T_{u\beta ho} \; (k+1) \; (d\rangle \tag{23}$$

and by Eq. (23) and Eq. (22)

$$(F^T F)^{-1} f(k+1)\rangle = T_{u\beta}^{-1} (N^T N)^{-1} N^T \; (k+1)\rangle \tag{24}$$

The inner-product term in the denomonator of Eq. (15) can be written as

$$\langle f(k+1) \; (F^T F)^{-1} f(k+1)\rangle = \langle (k+1) \; (N^T N)^{-1} \; (k+1)\rangle \tag{25}$$

By Eq. (395) sec (1)

$$(N^T N)^{-1} = B_g M_{gg}^{-1} B_g^{\;T} \tag{26}$$

and in terms of the notation change

$$(N^T N)^{-1} = B_g(k) \; M_{gg}^{-1} \; B_g^{\;T}(k) \tag{27}$$

where

$$B_g(k) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -\frac{2}{k} & -\frac{6}{k-1} \\ 0 & 0 & \frac{6}{k(k-1)} \end{bmatrix} \tag{28}$$

and

$$M_{gg}^{-1} = \begin{bmatrix} \frac{1}{k+1} & & \\ & \frac{3k}{(k+1)(k+2)} & \\ & & \frac{5k(k-1)}{(k+3)(k+2)(k+1)} \end{bmatrix} \tag{29}$$

The inner product of Eq. (25) can be computed as

$$\left\langle (k+1)\ (N^TN)^{-1}\ (k+1) \right\rangle = \left\langle (k+1)\ B_g M_{gg}^{-1}\ B_g^{\ T}\ (k+1) \right\rangle \tag{30}$$

we note also that

$$\left[1,\ k+1,\ (k+1)^2\right] = \left\langle k\ T_u(1)\right. \tag{31}$$

with

$$T_u(1) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \tag{32}$$

hence

$$\left\langle (k+1)\ (N^TN)^{-1}\ (k+1) \right\rangle = \left\langle k\ T_u)1)\ (N^TN)^{-1}\ T_u^{\ T}(1)\ k \right\rangle \tag{33}$$

By tedious attention to detail one can show

$$\left\langle (k+1)\ (N^TN)^{-1}\ (k+1) \right\rangle = \tag{34}$$

$$go \left[2N^2 + \frac{(N+1)}{N(N-1)}\ (61\,N^3 - 103N^2 - 150N - 48)\right.$$

The weighting vector of Eq. (15) can easily be put together for the orthogonal polynomial cases over the infinite span for exponential weighting. The other polynomials are orthogonal over a finite discrete point set and can be used for fixed span filters.

Section 15 RECURSIVE UNCONSTRAINED UNWEIGHTED STATE ESTIMATION

By Eq. (32) and Eq. (33) of section (12)

$$z\ (N+1\rangle = F_{zx}\ x(N)\ (3\rangle_s + v\ (N+1\rangle \tag{1}$$

$$= F_{zx}\ \hat{x}(N, N)\ (3\rangle_s + \tilde{z}(N, N)\ (N+1\rangle \tag{2}$$

where by Eq. (34)

$$\hat{x}(N, N)\ (3\rangle_s = F^*_{zx}\ z\ (N+1\rangle \tag{3}$$

and

$$F_{zx} = FF_v^{-1}(0)\ \Phi^{-N} \tag{4}$$

and

$$F^*_{zx} = \Phi^N\ F_v(0)\ F^* \tag{5}$$

Consider an additional measurement

$$\begin{bmatrix} z(0) \\ z(1) \\ \cdot \\ \cdot \\ \cdot \\ z(N) \\ z(N+1) \end{bmatrix} = \begin{bmatrix} z\ (N+1\rangle \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z(N+1) \end{bmatrix} \tag{6}$$

with

$$z(N+1) = x(N+1) + v(N+1) \tag{7}$$

and

$$x(N+1) = {}^{1}\langle e\ \ x(N+1)\ (3\rangle_s \tag{8}$$

also

$$x(N+1)\ (3\rangle_s = \Phi\ x(N)\ (3\rangle_s \tag{9}$$

or

$$x(N)3\rangle_s = \Phi^{-1}\ x(N+1)\ (3\rangle_s \tag{10}$$

Using Eq. (1) and Eq. (12) sec (12)

$$x(N+1\rangle = FF_v^{-1}(0)\ \Phi^{-N}\ \Phi^{-1}\ x(N+1)\ (3\rangle_s \qquad (11)$$

$$= F_{zx}\Phi^{-1}\ x(N+1)\ (3\rangle_s \qquad (12)$$

$$x\ (N+1\rangle = FF_v^{-1}(0)\ \Phi^{-(N+1)}\ x(N+1)\ (3\rangle_s \qquad (13)$$

By Eq. (12) in Eq. (6)

$$\begin{bmatrix} z\ (N+1\rangle \\ z\ (N+1) \end{bmatrix} = \begin{bmatrix} FF_v^{-1}(0)\ \Phi^{-(N+1)}\ x(N+1)\ (3\rangle_s \\ \langle 3)_1\ e\ x(N+1)\ (3\rangle_s \end{bmatrix} + v\ (N+2 \qquad (14)$$

$$= \begin{bmatrix} FF_v^{-1}(o)\ \Phi^{-(N+1)} \\ {}_1\langle 3)\ e \end{bmatrix} \quad x(N+1)\ (3\rangle_s + v\ (N+2 \qquad (15)$$

or

$$z\ (N+2\rangle = \begin{pmatrix} F_{zx}\ \Phi^{-1} \\ {}_1\langle 3)\ e \end{pmatrix} x(N+1)\ (3\rangle_s + v\ (N+2\rangle \qquad (16)$$

The estimate equation is

$$z\ (N+2\rangle = \begin{pmatrix} F_{zx}\ \Phi^{-1} \\ {}_1\langle 3)\ e \end{pmatrix} \hat{x}\ (N+1,\ N+1)\ (3\rangle_s + \tilde{z}\ (N+2\rangle \qquad (17)$$

$$z(N+2\rangle = F_{zxE}\ \hat{x}\ (N+1,\ N+1)\ (3\rangle_s + \tilde{z}\ (N+2\rangle \qquad (18)$$

where

$$F_{zxE} = \begin{pmatrix} F_{zx}\ \Phi^{-1} \\ {}_1\langle e \end{pmatrix} \qquad (19)$$

The psuedo-inverse of Eq. (19) is given by Eq. (9) sec (13) as

$$F^*_{zxE} = \left[\Phi F^*_{zx} - \frac{(\Phi^{-T}F^T_{zx}F_{zx}\Phi^{-1})^{-1}\ e\rangle_1\ {}_1\langle e\ \Phi F^*_{zx}}{1 + {}_1\langle e\ \Phi(F^T_{zx}F_{zx})^{-1}\Phi^T\ e\rangle_1}, \right. \qquad (20)$$

$$\left. \frac{\Phi(F^T_{zx}F_{zx})^{-1}\ \Phi^T\ e\ (3\rangle_1}{1 + \langle 3)_1\ e\ \Phi(F^T_{zx}F_{zx})^{-1}\ \Phi^T\ e\ (3\rangle_1} \right]$$

The unweighted estimate of the states based upon N+1 measurements is by Eq. (20) in Eq. (18)

$$\hat{x}\,(N+1,\ N+1)\ (3\rangle_s = F^*_{zxE}\ z(N+2\rangle \tag{21}$$

or

$$\hat{x}\,(N+1,\ N+1)\ (3\rangle_s = \Phi F^*_{zx} - \frac{\left(\Phi(F^T_{zx}F_{zx})^{-1}\Phi^T\ e(3\rangle_1\ {}_1\langle e\ \Phi F^*_{zx}\right) z(N+1}{1+\ {}_1\langle e\,\Phi(F^T_{zx}F_{zx})^{-1}\ \Phi^T\ e\rangle_1} + \frac{\Phi(F^T_{zx}F_{zx})^{-1}\ \Phi^T\ \ e(3\rangle_1\ \ z(N+1)}{1+\ {}_1\langle e\ \Phi(F^T_{zx}F_{zx})^{-1}\ \Phi^T\ e\rangle_1} \tag{22}$$

The first term on the right hand side of Eq. (22) is by Eq. (3)

$$\Phi F^*_{zx}\ z\ (N+1\rangle = \hat{x}(N,\ N)\ (3\rangle_s \tag{23}$$

The scalar element in the second term of Eq. (22) is

$${}^1\langle 3)\ e\ \Phi F^*_{zx}\ z(N+1) = {}^1\langle e\ \Phi\hat{x}\ (N,\ N)\ 3(\rangle_s \tag{24}$$

$$= {}^1\langle 3)\ e\ \hat{x}\ (N+1,\ N)\ (3\rangle_s \tag{25}$$

$$= \hat{x}\ (N+1,\ N) \tag{26}$$

where

$$\Phi\hat{x}\ (N,\ N)\ (3\rangle_s = \hat{x}\ (N+1,\ N)\ (3\rangle_s \tag{27}$$

is the predicted value of the state estimates at time N+1, based on N measurements. The position level term is the first coordinate of the state vector, that is

$$\hat{x}\ (N+1,\ N) = {}_1\langle e\ \ \hat{x}\ (N+1,\ N)\ (3\rangle_s = (1,\ 0,\ 0)\begin{pmatrix}\hat{x}\ (N+1,\ N)\\ \hat{\dot{x}}\ (N+1,\ N)\\ \hat{\ddot{x}}\ (N+1,\ N)\end{pmatrix} \tag{28}$$

Since the matrix relating the predicted measurement to the predicted state is

$$\hat{z}\ (N+1,\ N) = {}^1\langle 3)\ e\ \hat{x}\ (N+1,\ N)\ (3\rangle_s = \hat{x}\ (N+1,\ N) \tag{29}$$

we have using Eq. (29) in Eq. (23) and Eq. (27) in Eq. (23)

$$\hat{x}\ (N+1,\ N+1)\ (3\rangle_s = \hat{x}\ (N+1,\ N)\ (3\rangle_s + w\ (N+1)\ (3\rangle\ \tilde{z}\ (N+1,\ N) \tag{30}$$

where

$$\tilde{z}\ (N+1,\ N) = z\ (N+1) - \hat{z}\ (N+1,\ N) \tag{31}$$

and the weighting vector is

$$w(N+1)\ (3\rangle = \frac{\Phi(F_{zx}^{T}F_{zx})^{-1}\Phi^{T}\ e\ (3\rangle_{1}}{1+\ {}^{1}\langle 3)\ e\ \Phi(F_{zx}^{T}F_{zx})^{-1}\ {}^{T}\ e\rangle_{1}} \tag{32}$$

One can proceed to get an update on the w vector for the next measurement as well as an update on $(F_{zx}^{T}F_{ZX})^{-1}$ etc. One must also obtain the variance of the predicted states as well as of the corrected estimate of the states.

These relations are derived under the discrete Kalman filter.

Section 16 UNWEIGHTED RECURSIVE ESTIMATION OVER FIXED SPAN.

Consider a polynomial signal

$$x(t) = \langle d| t\,a |d\rangle + \langle r| t\,a |r\rangle \tag{1}$$

where

$$\langle d| t = (1,\ t,\ t^2\ \ldots\ t^{d-1}) \tag{2}$$

and

$$\langle \mu| t = (t^d,\ t^{d+1}\ \ldots\ t^{d+\alpha}) \tag{3}$$

where α is finite or infinite. One can approximate x(t) as

$$x(t) = \hat{x}(t) + \tilde{x}(t) \tag{4}$$

where

$$\hat{x}(t) = \langle d| t\,\hat{a} |d\rangle \tag{5}$$

The discreet vector (for N+1) points representation of $\hat{x}(t)$ is given by Eq. (124) Section (5) as

$$\begin{bmatrix} \hat{x}(i+o) \\ \hat{x}(i+1) \\ \vdots \\ \hat{x}(i+N) \end{bmatrix} = \begin{pmatrix} \langle t(i+0) \\ \vdots \\ \langle t(i+N) \end{pmatrix} \hat{a}(i,w_p)\,|d\rangle = \underset{w_p \times d}{T(i)}\ \hat{a}(i,w_p\rangle \tag{6}$$

In this section recursive relations over a fixed span of size $N+1 = w_p$ will be developed. For a given time point i, the index n will run from 0 to N, then i will advance to i+1 and n will repeat.

One can express this relation matrix-wise as (the hat symbols over the x are ommitted thus Eq. (14)

$$\underset{(N+1)\times 1}{\begin{bmatrix} x(i+0) \\ x(i+1) \\ x(i+2) \\ \vdots \\ x(i+N) \end{bmatrix}} = x(i)(wp\rangle \tag{7}$$

and for the next set of data

$$x(i+1)\rangle = \begin{pmatrix} x(i+1+0) \\ x(i+1+1) \\ \cdot \\ \cdot \\ \cdot \\ x(i+1+N) \end{pmatrix} = \begin{pmatrix} x(i+1) \\ x(i+2) \\ \cdot \\ \cdot \\ \cdot \\ x(i+N+1) \end{pmatrix} = S_{uo}x(i)\rangle + \mathbf{e}\langle N+1\rangle\, x(i+N+1) \quad (8)$$

where the shift-up and out matrix

$$S_{uo} = \begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 & \\ 0 & 0 & 1 & . & . & . & . & 0 \\ & & & & & & & \\ 0 & 0 & . & . & . & . & . & 1 \\ 0 & 0 & . & . & . & . & . & 0 \end{pmatrix} \quad (9)$$

and

$$\langle N+1\rangle\, \mathbf{e} = (0,\ 0\ .\ .\ .\ 1) \quad (10)$$

Also for a span indexed with respect to a midpoint we have

$$\begin{bmatrix} x(k-M) \\ x(k-M+1) \\ \cdot \\ \cdot \\ \cdot \\ x(k-1) \\ x(k+0) \\ x(k+1) \\ \cdot \\ \cdot \\ \cdot \\ x(k+M) \end{bmatrix} = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & & \\ & & & 0 \\ & & 0 & 1 \end{bmatrix} \begin{bmatrix} x(i+0) \\ x(i+1) \\ \cdot \\ \cdot \\ \cdot \\ x(i+M) \\ \cdot \\ \cdot \\ \cdot \\ x(i+N) \end{bmatrix} \quad (11)$$

and advancing k by one point

$$\begin{bmatrix} x(k+1-M) \\ \cdot \\ \cdot \\ \cdot \\ x(k+1+0) \\ \cdot \\ \cdot \\ \cdot \\ x(k+1+M) \end{bmatrix} = S_{uo} \begin{bmatrix} x(k-M) \\ \cdot \\ \cdot \\ \cdot \\ x(k+0) \\ x(k+1) \\ \cdot \\ \cdot \\ x(k+M) \end{bmatrix} + \mathbf{e}\langle wp\rangle\, x(k+1+M) \quad (12)$$

and likewise the connection between a midpoint span and a front point span is

$$\begin{bmatrix} x(k-M) \\ x(k-M+1) \\ \cdot \\ \cdot \\ \cdot \\ x(k+0) \\ \cdot \\ \cdot \\ \cdot \\ x(k+M) \end{bmatrix} = \begin{bmatrix} x(\gamma-N) \\ x(\gamma-N+1) \\ \cdot \\ \cdot \\ \cdot \\ x(\gamma-M) \\ \cdot \\ \cdot \\ \cdot \\ x(\gamma-0) \end{bmatrix} \tag{13}$$

and advancing the span

$$\begin{bmatrix} x(\gamma+1-N) \\ \cdot \\ \cdot \\ \cdot \\ x(\gamma-0) \\ x(\gamma+1) \end{bmatrix} = S_{uo} x(\gamma)\rangle + \mathbf{e}\,(wp\rangle\, x(\gamma+1) \tag{14}$$

The following time relations will be used

$$\begin{aligned} \langle t(i,n) &= \langle \tau(i,n)\ T_u(\beta)\ D(ho) \\ \langle t(k,m) &= \langle \tau(k,m)\ T_u(\beta)\ D(ho) \\ \langle t(1,\gamma) &= \langle \tau(1,\gamma)\ T_u(\beta)\ D(ho) \end{aligned} \tag{15}$$

and

$$\begin{aligned} \langle \tau(i,n) &= \langle n\ T_u(i) \\ \langle \tau(k,m) &= \langle m\ T_u(h) \\ \langle \tau(1,\gamma) &= \langle \gamma\ T_u(1) \end{aligned} \tag{16}$$

and

$$\begin{aligned} \langle n &= (1,\ n,\ n^2,\ n^3\ .\ .\ .\ n^{d-1}) \\ \langle m &= (1,\ \pm m,\ m^2,\ \pm m^3\ .\ .\ .) \\ \langle \gamma &= (1,\ -\gamma,\ \gamma^2,\ -\gamma^3\ .\ .\ .) \end{aligned} \tag{17}$$

If we package the N+1 measurements as a function of i, we have

$$\begin{pmatrix} z(i+0) \\ z(i+1) \\ \cdot \\ \cdot \\ \cdot \\ z(i+N) \end{pmatrix} = z(i)\,(wp\rangle = \underset{(N+1)xd}{T(i)}\ \hat{a}(i;\ wp)\rangle + \tilde{z}(i,\ wp)\rangle \tag{18}$$

or as a function of k

$$\begin{pmatrix} z(k-M) \\ \cdot \\ \cdot \\ \cdot \\ z(k+M) \end{pmatrix} = \underset{(N+1)xd}{T(k)}\ \hat{a}(k;\ wp)\rangle + \tilde{z}(k,\ wp)\rangle \tag{19}$$

and as a function of the span front index.

$$z(\ell)\ wp\rangle = \underset{(N+1)xd}{T(\ell)}\ \hat{a}(\ell;\ wp)\rangle + \tilde{z}(\ell;\ wp)\rangle \tag{20}$$

if we advance the span one point Eq. (18) becomes

$$\begin{pmatrix} z(i+1) \\ z(i+1) \\ \cdot \\ \cdot \\ \cdot \\ z(i+N+1) \end{pmatrix} = T(i+1)\ \hat{a}(i+1;\ wp)\rangle + \tilde{z}(i+1)\rangle \tag{21}$$

and similarly for Eq. (19) and Eq. (20),

$$z(i+1)\ wp\rangle = S_{uo} z(i)\rangle + e\ wp\rangle_{N+1}\ z(i+N+1) \tag{22}$$

and by Eq. (89) Section (13)

$$\underset{(N+1)xd}{T(i+1)} = \underset{(N+1)x(N+1)}{S_{uo}}\ T(i) + e\ wp\rangle_{N+1}\langle t(i+N+1) = \begin{pmatrix} t(i+1) \\ t(i+2) \\ \cdot \\ \cdot \\ \cdot \\ t(i+N+1) \end{pmatrix} \tag{23}$$

The unweighted solutions to Equations (18, 19, 20) requires computing the psuedo inverses

$$\underset{dx(N+1)}{T^{*}(i)} = \underset{d(N+1)d}{(T^{T}(i)\ T(i))^{-1}}T^{T}(i) \tag{24}$$

and similar for $T^{*}(k)$, and $T^{*}(\ell)$.

The update solution of Eq. (21) requires

$$T^{*}(i+1) = [T^{T}(i+1)\ T(i+1)]^{-1}T^{T}(i+1) \tag{25}$$

These relations have been derived in Section (13).

Partitioning the measurement vector

$$z(i)\ wp\rangle = \begin{pmatrix} z(i+0) \\ z\ N\rangle \end{pmatrix} = \begin{pmatrix} z_0 \\ z\ N\rangle \end{pmatrix} \tag{26}$$

then

$$\begin{pmatrix} z_o \\ z\,(N)\rangle \end{pmatrix} = \begin{pmatrix} \langle t_o \\ T_n \end{pmatrix} \hat{a}(i)\,(d)\rangle + \tilde{z}(i)\rangle \qquad (27)$$

and the psuedo inverse of the partitioned matrix by Eq. (71 - 72) Section (13) yields

$$\hat{a}(i)\,(d)\rangle = \frac{(T_n^T T_n)^{-1} t\rangle_o}{\Delta_o}\, z_o + \left[T_n^* - \frac{(T_n^T T_n)^{-1} t\rangle_o \langle t}{\Delta_o}\, T_n^*\right] z\,(N)\rangle \qquad (28)$$

advancing the time point to i+1

$$\begin{pmatrix} z\,(N)\rangle \\ z(i+N+1) \end{pmatrix} = \begin{pmatrix} z\,(N)\rangle \\ z_{N+1} \end{pmatrix} = \begin{pmatrix} T_n \\ \langle t_{N+1} \end{pmatrix} \hat{a}(i+1)\rangle + \tilde{z}(i+1)\rangle \qquad (29)$$

and using the psuedo inverse relations of Eq. (84 - 85) Section (13)

$$\hat{a}(i+1)\,(d)\rangle = \left[T_n^* - \frac{(T_n^T T_n)^{-1} t\rangle_{N+1} \langle t\, T_n^*}{\Delta_{N+1}}\right] z\,(N)\rangle + \frac{(T_n^T T_n)^{-1} t\rangle_{N+1}}{\Delta_{N+1}}\, z_{N+1} \qquad (30)$$

Subtracting Eq. (28) from Eq. (30)

$$\hat{a}(i+1)\rangle = \hat{a}(i)\rangle + (T_n^T T_n)^{-1} \left[\frac{t\rangle^{N+1}}{\Delta N+1}\left(z_{N+1} - \langle t^{N+1} T_n^* z\,(N)\rangle\right) - \frac{t\rangle^{o}}{\Delta_o}\left(z_o - \langle t T_n^* z\,(N)\rangle\right)\right] \qquad (31)$$

Eq. (31) can be rewritten as

$$\hat{a}(i+1)\,(d)\rangle = \hat{a}(i)\,(d)\rangle + (T_n^T T_n)^{-1} \left\{\frac{t\rangle_{N+1}}{\Delta N+1} z_{N+1} - t\rangle^{o} z_o + \left(\frac{t\rangle_o \langle t^{o}}{\Delta o} - \frac{t\rangle_{N+1} \langle t^{N+1}}{\Delta_{N+1}}\right) T_n^* z\,(N)\rangle\right\} \qquad (32)$$

Another expression for updating $\hat{a}(i+1)\,(d)\rangle$ can be obtained from Eq. (22)

$$S_{uo} z(i)\rangle + e\rangle_{N+1} z_{N+1} = T(i+1)\,\hat{a}(i+1) + \tilde{z}(i+1)\rangle \qquad (33)$$

and by Eq. (27)

$$S_{uo} T(i)\hat{a}(i)\rangle + S_{uo}\tilde{z}(i)\rangle + e\rangle_{N+1} z_{N+1} = T(i+1)\,\hat{a}(i+1)\rangle + \tilde{z}(i+1)\rangle \qquad (34)$$

Operating on (34) by $T^*(i+1)$

$$\hat{a}(i+1)\rangle = T^*(i+1) S_{uo} T(i)\hat{a}(i)\,(d)\rangle + T^*(i+1)\, e\rangle_{N+1} z_{N+1} \qquad (35)$$

since

$$T^*(i)\tilde{z}(i)\rangle = T^*(i+1)\tilde{z}(i+1)\rangle = 0\rangle \tag{36}$$

and by Eq. ( )

$$T^*(i+1) = \Phi_t^{-1}T^*(i) \tag{37}$$

hence

$$T^*(i+1)\tilde{z}(i)\rangle = \Phi_t^{-1}T^*(i)\tilde{z}(i)\rangle = 0\rangle \tag{38}$$

By Eq. (28)

$$\hat{a}(i)\rangle = T^*(i)z(i)\rangle \tag{39}$$

and

$$\hat{z}(i+N+1,\ N+1) = \langle t^{N+1}\hat{a}(i)\rangle = \langle t\ T^*(i)z(i)\rangle \tag{40}$$

where the hat symbol is the predicted measurement.

By Eq. (86) Section (13)

$$\frac{T^*(i+1)}{dx(N+1)}\, e\rangle_{N+1} = \frac{(T_n^T T_n)^{-1} t}{\Delta_{N+1}}\ {N+1} \tag{41}$$

and adding and subtracting to Eq. (35) Eq. (39) times Eq. (40)

$$\begin{aligned}\hat{a}(i+1)\rangle &= T^*(i+1)S_{uo}T(i)\hat{a}(i)\rangle + \frac{(T_n^T T_n)^{-1}t\rangle_{N+1}}{\Delta_{N+1}}\tilde{z}(i+N+1;\ N+1) \\ &+ \frac{(T_n^T T_n)^{-1}t\rangle_{N+1}}{\Delta_{N+1}}\hat{z}(i+N+1;\ N+1)\end{aligned} \tag{42}$$

Using Eq. (39) in the last term of Eq. (41)

$$\begin{aligned}\hat{a}(i+1)\rangle &= [T^*(i+1)S_{uo}T(i) + \frac{(T_n^T T_n)^{-1}t\rangle_{N+1}}{\Delta_{N+1}}\langle t]^{N+1}\ \hat{a}(i)\rangle \\ &+ \frac{(T_n^T T_n)^{-1}t\rangle_{N+1}}{\Delta_{N+1}}\ \tilde{z}(i+N+1;\ N+1)\end{aligned} \tag{43}$$

now

$$S_{uo}\ T(i) = S_{uo}\begin{pmatrix}\langle t \\ T_n\end{pmatrix} = \begin{pmatrix}T_n \\ \langle o\end{pmatrix} \tag{44}$$

and by Eq. (86) Section (13)

$$\underset{d\times(N+1)}{T^*(i+1)} = \left[T_n^* - \frac{(T_n^T T_n)^{-1} t\rangle_{N+1}\, {}^{N+1}\langle t\, T_n^*,\; (T_n^T T_n)^{-1} t\, \textcircled{d}\rangle_{N+1}}{\Delta_{N+1}}\right] \tag{45}$$

$$\underset{d\times(N+1)(N+1)\times(N+1)(N+1)\times d}{T^*(i+1)\, S_{uo}\, T(i)} = \underset{d\times n}{\left[T_n^* - \frac{(T_n^T T_n)^{-1} t\rangle_{N+1}\langle t\, T_n^*}{\Delta N+1}\right] T_n}. \tag{46}$$

and

$$\underset{d\times d}{T_n^*}\;\underset{d\times d}{T_n} = I \tag{47}$$

and (46) and (45) in the multiplicative matrix term of $\hat{a}(i)\rangle$ in Eq. (42) yields

$$T^*(i+1)\, S_{uo}\, T(i) + \frac{(T_n^T T_n)^{-1} t\rangle_{N+1}\, {}^{N+1}\langle t}{\Delta_{N+1}} = I \tag{48}$$

and Eq. (47) in (42)

$$\hat{a}(i+1)\rangle = \hat{a}(i)\rangle + \frac{(T_n^T T_n)^{-1} t\rangle_{N+1}}{\Delta_{N+1}}\tilde{z}(i+N+1;\, N+1) \tag{49}$$

$$\hat{a}(i+1)\rangle = \hat{a}(i)\rangle + w(i+1)\, \textcircled{d}\rangle\, \tilde{z}(i+N+1;\, N+1)$$

where

$$w(i+1)\, \textcircled{d}\rangle = \frac{(T_n^T T_n)^{-1} t\, \textcircled{d}\rangle_{N+1}}{1 + {}_{N+1}\langle\textcircled{d}\, t (T_n^T T_n)^{-1} t\, \textcircled{d}\rangle_{N+1}} \tag{50}$$

Section 17 CONSTRAINED LEAST-SQUARES RECURSIVE ESTIMATION FOR ONE DEGREE OF FREEDOM

This section considers an increasing batch of data being fitted to a d-1 degree polynomial, or a d dimensional vector of parameters where the first d-1 parameters are constrained to be the initial estimates, thus one has a scalar recursive relation. For k measurements partitioned as shown

$$\begin{pmatrix} z_1 \\ \cdot \\ z_k \end{pmatrix} = \underset{(kxd-1)}{[T,} \; t(k\rangle_d] \begin{pmatrix} \hat{a}(k)(d-1\rangle \\ \hat{a}(k)_d \end{pmatrix} + \tilde{z}(k\rangle \tag{1}$$

or

$$z(k)\rangle = \underset{kx(d-1)}{T} \hat{a}(k)(d-1\rangle + t(k\rangle \hat{a}(k)_d + \tilde{z}(k\rangle \tag{2}$$

It is assumed that d-1 of the parameters

$$\hat{a}(k)(d-1\rangle = \hat{a}(k-1)(d-1\rangle = \hat{a}(d-1\rangle \tag{3}$$

is known from previous stage, then

$$z(k\rangle - \underset{kx(d-1)}{T} \hat{a}(d-1\rangle = t(k\rangle_d \hat{a}_d(k) + z(k\rangle \tag{4}$$

Multiplying by the psuedo-inverse

$$\frac{{}_d\langle k)t}{{}_d\langle k)t \; t(k\rangle_d} [z(k\rangle - T \hat{a}(d-1\rangle] = \hat{a}_d(k), \tag{5}$$

where

$$\frac{{}_d\langle k)t \; \tilde{z}(k)\rangle}{{}_d\langle k)t \; t(k\rangle_d} = 0 \tag{6}$$

If we now have k+1 observations

$$\begin{matrix} z(k\rangle \\ z_{k+1} \end{matrix} = \begin{bmatrix} \underset{(k+1)x(d-1)}{T} & t(k+1\rangle_d \end{bmatrix} \begin{pmatrix} \hat{a}(d-1\rangle \\ a(k+1)_d \end{pmatrix} + \tilde{z}(k+1)\rangle \tag{7}$$

$$z(k+1\rangle - \underset{(k+1)xd-1}{T} \hat{a}(d-1\rangle = t(k+1\rangle_d \hat{a}(k+1)_d + \tilde{z}(k+1\rangle \tag{8}$$

or

$$\hat{a}(k+1)_d = \frac{{}_d\langle k+1)t}{{}_d\langle k+1)tt(k+1\rangle_d} [z(k+1\rangle - \underset{(k+1)x(d-1)}{T} \hat{a}(d-1\rangle \tag{9}$$

The partitioned matrix is

$$\underset{(k+1)\times d}{T} = \begin{bmatrix} \underset{k\times d-1}{T} & t(k\rangle_d \\ \langle d-1)\underset{k+1}{t} & t_{k+1,d} \end{bmatrix} = \begin{bmatrix} \underset{k\times d}{T} \\ \langle d)\underset{k+1}{t} \end{bmatrix} = \begin{bmatrix} [\underset{(k+1)(d-1)}{T}, \; t(k+1\rangle_d] \end{bmatrix} \tag{10}$$

or

$$\underset{(k+1)\times(d-1)}{T} = \begin{pmatrix} \underset{k\times d-1}{T} \\ \langle d-1)\underset{k+1}{t} \end{pmatrix} \tag{11}$$

and

$$t(k+1\rangle_d = \begin{pmatrix} t(k\rangle_d \\ t_{k+1,d} \end{pmatrix} \tag{12}$$

and

$$\begin{aligned} g_{(k+1)} = {}_d\langle k+1)tt(k+1\rangle_d &= {}_d\langle k)tt(k\rangle_d + t^2_{k+1,d} \\ &= g_k + t^2_{k+1,d} \end{aligned} \tag{13}$$

Using Eq. (11) and (12) in Eq. (9)

$$\hat{a}(k+1)_d = \frac{1}{g(k+1)} \left({}_d\langle k)t, \; t_{k+1,d} \right) \begin{pmatrix} z(k\rangle \\ z_{k+1} \end{pmatrix} - \begin{pmatrix} \underset{k\times d-1}{T} \hat{a}(d-1\rangle \\ \langle d-1)\underset{k+1}{t}\hat{a}(d-1\rangle \end{pmatrix} \tag{14}$$

$$= \frac{1}{g(k+1)} \left\{ {}_d\langle k)t[z(k\rangle - \underset{k\times d-1}{T}\hat{a}(d-1\rangle] + t_{k+1,d}(z_{k+1} - {}_{k+1}\langle d-1)t \; \hat{a}(d-1\rangle) \right\} \tag{15}$$

The predicted value of the measurement is

$$\hat{z}(k+1;\, k) = {}_{k+1}\langle d)t \; \hat{a}(k)(d\rangle = \left({}_{k+1}\langle d-1)t, \; t_{k+1,d} \right) \begin{pmatrix} \hat{a}(d-1\rangle \\ \hat{a}_d(k) \end{pmatrix}$$

$$\hat{z}(k+1,k) = {}_{k+1}\langle d-1)t \; \hat{a}(d-1\rangle + t_{k+1,d}\hat{a}_d(k) \tag{16}$$

Adding and subtracting Eq. (16) to last term of Eq. (15)

$$\hat{a}(k+1)_d = \frac{1}{g(k+1)} \left\{ {}_d\langle k)t[z(k\rangle - \underset{k\times(d-1)}{T} \hat{a}(d-1\rangle] + t_{k+1,d}(z_{k+1} - \hat{z}(k+1,k)) \right. \tag{17}$$

$$\left. + \; t_{k+1,d}\hat{a}_d(k) \right.$$

By Eq. (5) in (17)

$$\hat{a}(k+1)_d = \hat{a}(k)_d \frac{({}^d\langle k) tt (k\rangle_d + t^2_{k+1,d})}{g_{k+1}} + \tilde{z}(k+1;\ k)\ \frac{t_{k+1,d}}{g_{(k+1)}} \tag{18}$$

or

$$\hat{a}(k+1)_d = \hat{a}(k)_d + \frac{t_{k+1,d}\ \tilde{z}(k+1,\ k)}{t(k+1)\rangle\langle t(k+1)} \tag{19}$$

which is a scalar recursive equation.

Constrained Least Squares Recursive Estimation for Multiple Degrees of Freedom.

This section partitions the unknown d dimensional $a(d\rangle$ vector into two sub-vectors of dimensions d_1 and d_2; thus for k measurements

$$z(k\rangle = [\underset{kxd}{T}\ ,\ \underset{kxd_2}{T}] \begin{pmatrix} \hat{a}(k)(d-1\rangle \\ a(k)(d_2\rangle \end{pmatrix} + \tilde{z}(k\rangle \tag{20}$$

where the constraint is imposed

$$\hat{a}(k)(d_1\rangle = \hat{a}(k-1)(d_1\rangle = \hat{a}(d_1\rangle \tag{21}$$

and

$$d_1 + d_2 = d \tag{22}$$

or

$$z(k\rangle - \underset{kxd_1}{T}\ \hat{a}(d_1\rangle = \underset{kxd_2}{T}\ \hat{a}(k)(d_2\rangle + \tilde{z}(k\rangle \tag{23}$$

Multiplying Eq. (23) by the psuedo inverse of $\underset{kxd_2}{T}$

$$\underset{d_2xk}{T^*}\ [z(k\rangle - \underset{kxd_1}{T}\ \hat{a}(d_1\rangle\ ] = \hat{a}(k)(d_2\rangle \tag{24}$$

The predicted scalar measurement at the next stage is

$$\hat{z}(k+1;\ k) = \langle d) \underset{k+1}{t}\ \hat{a}(k)(d\rangle = [\langle d_1) \underset{k+1}{t},\ \langle d_2) \underset{k+1}{t}] \begin{pmatrix} \hat{a}(d_1\rangle \\ \hat{a}(k)(d_2 \end{pmatrix}$$

$$\hat{z}(k+1;\ k) = \langle d_1) \underset{k+1}{t}\ \hat{a}(d_1\rangle + \langle d_2) \underset{k+1}{t}\ \hat{a}(k)(d_2\rangle \tag{25}$$

The partitioning for k+1 measurements is

$$\begin{pmatrix} z(k\rangle \\ z_{k+1} \end{pmatrix} = \begin{pmatrix} T_{kxd_1} & T_{kxd_2} \\ \langle d_1)t_{k+1}, & \langle d_2)t_{k+2} \end{pmatrix} \begin{pmatrix} \hat{a}(d_1\rangle \\ a(k+1)(d_2\rangle \end{pmatrix} + \tilde{z}(k+1\rangle \tag{39}$$

also

$$\begin{pmatrix} z(k\rangle \\ z_{k+1} \end{pmatrix} = \begin{bmatrix} T_{(K+1(xd_1}, & T_{(k+1)xd_2} \end{bmatrix} \begin{pmatrix} \hat{a}(d_1\rangle \\ \hat{a}(k+1)(d_2\rangle \end{pmatrix} + \tilde{z}(k+1) \tag{40}$$

or

$$z(k+1\rangle - T_{(k+1)xd_1} \hat{a}(d_1\rangle = T_{(k+1)xd_2} \hat{a}(k+1(d_2\rangle + \tilde{z}(k+1\rangle \tag{41}$$

Multiplying Eq. (41) by the psuedo inverse as shown

$$T^*_{d_2x(k+1)} [z(k+1\rangle - T_{(k+1)xd_1} \hat{a}(d_1\rangle] = \hat{a}(k+1)(d2\rangle \tag{42}$$

By Eq. (39)

$$T_{(k+1)xd_2} = \begin{pmatrix} T_{kxd_2} \\ \langle d_2 t_{k+1} \end{pmatrix} \tag{43}$$

and

$$T^*_{(d_2x(k+1)} = (T^T_{d_2(k+1)} T_{(k+1)xd_2})^{-1} T^T_{d_2x(k+1)} \tag{44}$$

The Grammian is

$$[T^T_{(d_2xk)}, t(d2\rangle_{k+1}] \begin{bmatrix} T_{kxd_2} \\ d_2 t_{k+1} \end{bmatrix} = T^T T_{d_2(k)d_2} + t(d2\rangle\langle d2)t_{k+1} \tag{45}$$

By the Householder inversion lemma, the Grammian inverse is

$$(T^T T)^{-1}_{d_2(k+1)d_2} = (T^T T)^{-1}_{d_2(k)d_2} [I - \frac{t(d2\rangle\langle d2)t \ (T^T T)^{-1}_{d_2(k)d_2}}{\Delta d2}] \tag{46}$$

$$\Delta_{d2} = 1 + \langle d2t_{k+1} \ (T^T T)^{-1}_{d_2(k)d_2} t(d2\rangle_{k+1} \tag{47}$$

Carrying out the analysis one obtains

$$\underset{d_2(k+1)}{T^*}(k+1) = \left[\underset{d_2xk}{T^*}(k) - \frac{\underset{d_2(k)d_2}{(T^TT)^{-1}} t(d2\rangle \underset{k+1}{\langle d2)t} \underset{d_2xk}{T^*}(k)}{\Delta_{d2}} , \frac{\underset{d_2(k)d_2}{(T^TT)^{-1}} t(d2\rangle}{\Delta_{d2}} \right]$$

$$= [\underset{d_2xk}{T^*}(k+1), t^*(d2\rangle_{k+1}] \qquad (48)$$

The first term on the left side of Eq. (42) is

$$\underset{d_2(k+1)}{T^*} z(k+1\rangle = \underset{d_2xk}{T^*}(k)z(k\rangle - \frac{\underset{d_2(k)d_2}{(T^TT)^{-1}} t(d2\rangle \underset{k+1}{\langle d2)t} \underset{d_2xk}{T^*}(k)\, z(k\rangle}{\Delta_{d2}} \qquad (49)$$

$$+ \underset{d_2(k)d_2}{(T^TT)^{-1}} t(d2\rangle_{k+1} \frac{1}{\Delta_{d2}} z_{k+1}$$

Consider the second matrix product term of Eq. (42)

$$\underset{d_2x(k+1)(k+1xd_1}{T^* \quad T} = [\underset{d_2xk}{T^*}(k+1), t^*(d2\rangle_{k+1}] \begin{bmatrix} \underset{kxd_1}{T} \\ \underset{k+1}{d1)t} \end{bmatrix}$$

$$= \underset{(d_2xk)(kxd_1)}{T^*(k+1)T} + t^*(d2\rangle \underset{k+1}{\langle d1)t} \qquad (50)$$

and by Eq. (48) in Eq. (50)

$$\underset{d_2x(k+1)(k+1)xd_1}{T^*T} = [\underset{d_2(k)dxd_1}{T^*(k)T} - \underset{d_2(k)d_2}{(T^TT)^{-1}} t(d2\rangle \underset{k+1}{\langle d2)t} \underset{d_2xk}{T^*(k)} \underset{kxd_1}{T} \frac{1}{\Delta_{d2}}, \frac{\underset{d_2(k)d_2}{(T^TT)^{-1}} t(d2\rangle \langle d1t}{\Delta_{d2}}] \qquad (51)$$

The second matrix vector term on the left of Eq. (42) which is Eq. (51) times the vector $a(d1\rangle$ is

$$\underset{d_2(k+1)(k+1)xd_1}{T^*T} \hat{a}(d1\rangle = \underset{d_2xkkxd_1}{T^*T} \hat{a}(d1\rangle - \underset{d_2(k)d_2}{(T^TT)^{-1}} t(d2\rangle \frac{\underset{k+1}{\langle d2)t}}{\Delta_{d2}} \times \qquad (52)$$

$$\underset{(d_2xk)}{T^*(k)} \underset{kxd_1}{T} \hat{a}(d1\rangle + \frac{\underset{d_2(k)d_2}{(T^TT)^{-1}} t(d2\rangle}{\Delta_{d2}} \langle d1)t\, \hat{a}(d1\rangle$$

By Eq. (25)

$$\underset{k+1}{\langle d1)t} \hat{a}(d1\rangle = \hat{z}(k+1;\ k) - \underset{k+1}{\langle d2)t} \hat{a}(k)(d2\rangle \qquad (53)$$

Using Eq. (53) in the last term of Eq. (52)

$$T^{*}_{d_2(k+1)}\; T_{(k+1)xd_1}\; \hat{a}(d1\rangle = T^{*}_{d_2xk}\; T_{kxd_1}\; \hat{a}(d1\rangle - (T^TT)^{-1}_{d_2(k)d_2}\; t(d2\rangle_{k+1} \langle d1)t\; T^{*}_{d_2xk}\; T_{kxd_1}\; \hat{a}(d1\rangle \frac{1}{\Delta_{d2}} \tag{54}$$

$$\frac{1}{\Delta_{d2}} + (T^TT)^{-1}_{d_2(k)d_2}\; t(d2\rangle_{k+1} (\hat{z}(k+1;\ k) - \langle d2)t_{k+1}\; \hat{a}(k)(d2\rangle)$$

By Eq. (42) using Eq. (49) and Eq. (54)

$$\hat{a}(k+1)(d2\rangle = [T^{*}_{d_2xk}(k) - (T^TT)^{-1}_{d_2(k)d_2}\; t(d2\rangle_{k+1} \langle d2)t\; \frac{T^{*}_{d_2xk}(k)}{\Delta d2}]z(k\rangle$$

$$+ (T^TT)^{-1}_{d_2(k)d_2}\; \frac{t(d2\rangle_{k+1}}{\Delta_{d2}}\; z_{k+1} \tag{55}$$

$$- [T^{*}_{d_2xk} - (T^TT)^{-1}_{d_2(k)d_2}\; \frac{t(d2\rangle_{k+1} \langle d2)t}{\Delta_{d2}}\; T^{*}_{d_2k}]\; T_{kxd_1}\; \hat{a}(d1\rangle$$

$$- \frac{1}{\Delta_{d2}} (T^TT)^{-1}_{d_2(k)d_2}\; t(d2\rangle_{k+1} (\hat{z}(k+1;\ k) - \langle d2)t_{k+1}\; \hat{a}(d2\rangle)$$

Rearranging Eq. (55)

$$\hat{a}(k+1)(d2\rangle = T^{*}_{d_2xk}(k)\; [z(k\rangle - T_{kxd_1}\; \hat{a}(d1\rangle]$$

$$- (T^TT)^{-1}_{d_2(k)d_2}\; t(d2\rangle_{k+1} \langle d1)t\; T^{*}_{d_2xk}(k)z(k\rangle \frac{1}{\Delta_{d2}}$$

$$+ (T^TT)^{-1}_{d_2(k)d_2}\; t(d2\rangle_{k+1} \langle d2)t\; T^{*}_{d_2xk}\; T_{kxd_1}\; \hat{a}(d1\rangle \frac{1}{\Delta_{d2}} \tag{56}$$

$$+ (T^TT)^{-1}_{d_2(k)d_2}\; t(d2\rangle_{k+1} \langle d2)t\; \hat{a}(k)(d2\rangle\; \Delta_{d2}$$

$$+ (T^TT)^{-1}_{d_2(k)d_2}\; \frac{t(d2\rangle}{\Delta_{d2}}\; \tilde{z}(k+1,k)$$

The first term by Eq. (56) is $\hat{a}(k)(d2\rangle$ hence

$$\hat{a}(k+1)(d2\rangle = \hat{a}(k)(d2\rangle + \frac{1}{\Delta_{d2}} (T^TT)^{-1}_{d_2(k)d_2}\; t(d2\rangle_{k+1}\; \tilde{z}(k+1,k) \tag{57}$$

The three middle terms on the right of Eq. (56) equal zero since

$$-\underset{d_2 x d_2}{(T^T T)^{-1}} A(d2\rangle \langle d2)t \underset{d_2 xk}{T^*(k)} z(k\rangle \frac{1}{\Delta_{d2}}$$

$$+ \underset{d_2 x d_2}{(T^T T)^{-1}} t(d2\rangle_{k+1} \langle d2)t \underset{d_2 xk}{T^*} \underset{kxd_1}{T} \hat{a}(d1\rangle \frac{1}{\Delta_{d2}} \tag{58}$$

$$+ \underset{d_2 x d_2}{(T^T T)^{-1}} t(d2\rangle_{k+1} \langle d2)t \frac{\hat{a}(k)(d2}{\Delta_{d2}}\rangle =$$

$$\frac{1}{\Delta_{d2}} \underset{d_2 x d_2}{(T^T T)^{-1}} t(d2\rangle \langle d2)t \left\{ [-\underset{d_2 xk}{T^*} z(k\rangle + \underset{d_2 xk}{T^*} \underset{kxd_1}{T} \hat{a}(d1\rangle] \right.$$

$$\left. + \hat{a}(k)(d2\rangle \right\} = 0\rangle$$

since by Eq. (24)

$$\hat{a}(k)(d2\rangle = \underset{d_2 xk}{T^*} [z(k\rangle - \underset{kxd_1}{T} \hat{a}(d1\rangle] \tag{59}$$

Thus we see by Eq. (57) we need only estimate d_2 parameters instead of $d_1 + d_2$, and the recursive relation is

$$\hat{a}(k+1)(d2\rangle = \hat{a}(k)(d2\rangle + W(k+1)(d2\rangle \tilde{z}(k+1), k)$$

where the weight vector is

$$W(k+1)(d2\rangle = \underset{d_2(k)d_2}{(T^T T)^{-1}} \frac{t(d2\rangle}{\Delta_{d2}} \tag{60}$$

and by Eq. (47)

$$\Delta_{d2} = 1 + {}_{N+1}\langle d2)t \underset{d_2(k)d_2}{(T^T T)^{-1}} t(d2\rangle_{N+1} \tag{61}$$

SECTION 18
OPTIMAL WEIGHTING MATRIX FOR PARAMETER ESTIMATION

This section obtains the optimal weighting matrix for minimizing the trace of the variance of the estimate of the parameter vector given by Eq (89) sec (10) as

$$\underset{d\times d}{\Sigma_{\tilde{a}\tilde{a}}} = \underset{d\times k}{W}\ \underset{k\times k}{R}\ \underset{k\times d}{W^T} \tag{1}$$

subject to the linear constraint of Eq (18) sec (10) (R and F given)

$$\underset{d\times d}{L_o} = \underset{d\times k}{W}\ \underset{k\times d}{F} = I \tag{2}$$

we saw in previous sections the unweighted solution occurs for

$$W = F^* \tag{3}$$

Minimize trace of Eq (1) subject linear constraint of Eq (2).

The minimum variance estimate is derived in most test books using LaGrange multiples. A few of the recent psuedo-inverse books obtain the solution via algebraic utilization of the psuedo-inverse. These derivations are believed to be original and appealing to geometrical intuition. The derivation will be step-wise, starting with the case d=1. By Eq (1)

$$\sigma_{\tilde{a}\tilde{a}} = \langle \underset{k\times k}{W\ R\ W} \rangle \tag{4}$$

and by Eq (2)

$$\ell_o = \langle w\ f \rangle \tag{5}$$

where for the polynomial case

$$\langle f = \langle 1 = (1,1,1,\ldots). \tag{6}$$

The variance matrix R is assummed full rank (one can solve the non-full rank case but it will not be presented here). Since R is full rank and symmetric (positive definite) it has factors

$$R = B\ B^T \tag{7}$$

or Eq (7) in Eq (4)

$$\sigma_{\tilde{a}} = \langle w\ B\ B^T w \rangle \tag{8}$$

$$\sigma_{\tilde{a}} = \langle y\, y\rangle \tag{9}$$

where

$$\langle y = \langle w\, B \tag{10}$$

and

$$\langle y B^{-1} = \langle w \tag{11}$$

Using Eq (11) in Eq (5)

$$\ell_o = \langle y\, B^{-1}\, f\rangle = \langle y\, c\rangle \tag{12}$$

where

$$B^{-1}\, f\rangle = c\rangle \tag{13}$$

we see the problem to minimize the quadratic form in the unknown vector $\langle w$ of Eq (8) subject to the one degree of freedom linear constraint of Eq (12) can now be stated as finding the vector $\langle y$ of minimum magnitude (norm) which is constrained to lie on the hyperplane of Eq (12).

Multiply Eq (12) by the psuedo-inverse of $c\rangle$ or

$$\ell_o \langle c^* = \langle y\, c\rangle\langle c^* \tag{14}$$

Decompose the vector $\langle y$ into a component along $c\rangle$ and one perpendicular, that is

$$\langle y = \langle \hat{y} + \langle \tilde{y} \tag{15}$$

with

$$\langle \hat{y} = \langle y\, P_{cc^*} = \langle y(c\rangle\langle c^*) \tag{16}$$

and

$$\langle \tilde{y} = \langle y\, \tilde{P}_{cc^*} = \langle y(I - c\rangle\langle c^*) \tag{17}$$

If we obtain the normal form of the equation of the plane of Eq (12)

$$\frac{\ell_o}{\langle cc\rangle^{1/2}} = \langle y\, \frac{c\rangle}{\langle cc\rangle^{1/2}} = \langle y\, n\rangle \tag{18}$$

where the unit normal is

$$n\rangle = \frac{c\rangle}{\langle cc\rangle^{1/2}} \tag{19}$$

we know that the distance out from the origin along the normal to the plane is by Eq (18)

$$d = \frac{\ell_o}{\langle cc\rangle^{1/2}} \tag{20}$$

The picture is now as shown in Fig (1)

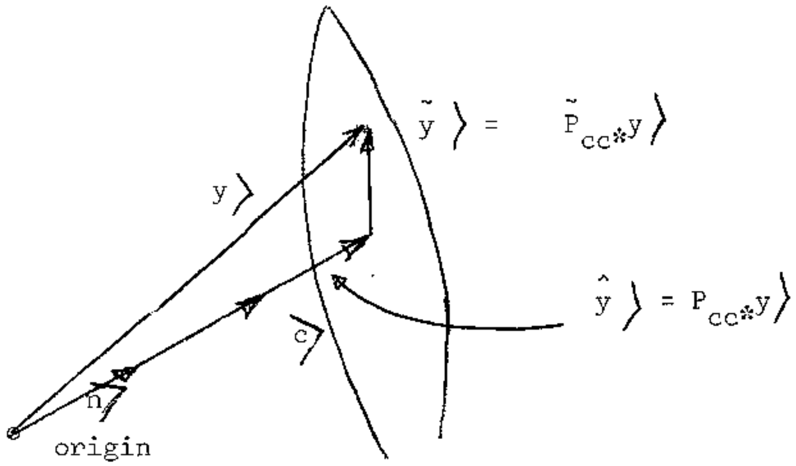

FIG (1) ORTHOGONAL PROJECTION

Clearly (via the orthogonal projection theorem) by obvious geometrical intuition the nearest approach to the origin for the set of vectors constrained to terminate on the plane is the vector along the normal, or $\langle \hat{y}$.

By Eq (13)

$$\langle c^* = \frac{\langle c}{\langle cc \rangle} = \frac{\langle f\ B^{-T}}{\langle f\ B^{-T}\ B^{-1}\ f \rangle} \tag{21}$$

or

$$\langle c^* = \frac{\langle f\ B^{-T}}{\langle f\ (B\ B^T)^{-1}\ f \rangle} \tag{22}$$

Using Eq (22) in Eq (~~14~~) \\10

$$\langle \hat{y} = \ell_o \frac{\langle f\ B^{-T}}{\langle f(BB^T)^{-1}\ f \rangle} \tag{23}$$

By Eq (23) in Eq (11)

$$\langle \hat{w} = \langle \hat{y} B^{-1} = \ell_o \frac{\langle f\ (BB^T)^{-1}}{\langle f\ (BB^T)^{-1} f \rangle} \tag{24}$$

or By Eq (7)

$$\langle \hat{w} = \ell_o \frac{\langle f\ Q^{-1}\ R^{-1}}{\langle f\ Q^{-1}\ f \rangle}\ R^{-1} \tag{25}$$

when $\ell_o = 1$ and $\langle f = \langle 1$

$$\langle \hat{w} = \frac{\langle 1\ Q^{-1}}{\langle 1\ Q^{-1}\ 1 \rangle} \quad (26)$$

which is the well known solution for the minimum variance sequence of weights to estimate a constant.

Clearly when the quadratic surface is a hyperplane Q = I and ?

$$\sigma_{\tilde{a}} = \langle w\ w \rangle \quad (27)$$

which is the equation of a family of spheres. The minimum sphere touches the fixed hyperplane as shown in Fig (2)

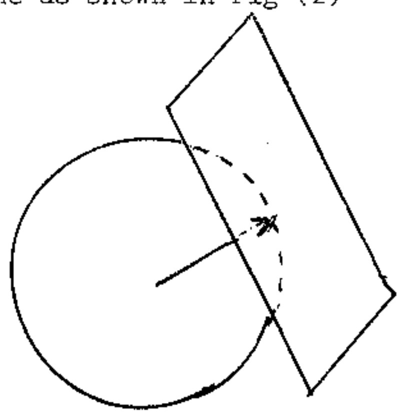

FIG 2 MINIMUM SPHERE TANGENT TO PLANE

By Eq (27) the minimum sphere has

$$\langle \hat{w}\ \hat{w} \rangle = r_o^2 \quad (28)$$

By Eq (5) the normal form for the Equation of the plane is

$$\frac{\ell_o}{\langle ff \rangle^{1/2}} = \langle w \frac{f \rangle}{\langle ff \rangle^{1/2}} \quad (29)$$

which is the distance to the plane (that is the sphere touching the hyperplane)

$$r_o = \frac{\ell_o}{\langle ff \rangle^{1/2}} \quad (30)$$

for $R = BB' = I$

By Eq (24) in Eq (28)

$$\langle \hat{w}\ \hat{w} \rangle = r_o^2 = \frac{\langle f}{\langle ff \rangle} \frac{f \rangle \ell_o^2}{\langle ff \rangle} = \frac{\ell_o^2}{\langle ff \rangle} \quad (31)$$

hence Eq (30) and (31) agree.

Consider next the case where the parameter vector has dimension d, that is a d-1 degree polynomial.

By Eq (1)

$$\text{tr}\ \Phi_{\tilde{a}\tilde{a}} = \text{tr}\ wRw^T \qquad (32)$$

Partition w into d row vectors in K-space

$$\text{tr}\ \Phi_{\tilde{a}\tilde{a}} = \text{tr} \begin{bmatrix} {}^1\langle w \\ {}^2\langle w \\ \vdots \\ {}^d\langle w \end{bmatrix} R[w\rangle_1 \ldots w\rangle_d] \qquad (33)$$

or

$$\text{tr}\ \Phi_{\tilde{a}\tilde{a}} = {}^1\langle wRw\rangle_1 + \ldots + {}_d\langle wRw\rangle_d \qquad (34)$$

The problem is now to minimize Eq (34) subject to the linear constraints of Eq (2). Using the full rank factors of Eq (7) in Eq (32)

$$\text{tr}\ \Phi_a = wAA^Tw^T = YY^T \qquad (35)$$

$$= {}_1\langle yy\rangle_1 + \ldots + {}_d\langle yy\rangle_d \qquad (36)$$

where

$$\underset{dxk}{Y} = wA \qquad (37)$$

or

$$w = YA^{-1} \qquad (38)$$

Using Eq (38) in Eq (2)

$$L_o = YA^{-1}F = \underset{dxk\ kxd}{YC} \qquad (39)$$

with

$$\underset{kxd}{C} = A^{-1}F \qquad (40)$$

Multiply Eq (39) by C^*

$$L_o C^* = YCC^* = \hat{Y} \qquad (41)$$

where

$$Y = \hat{Y} + \tilde{Y} \qquad (42)$$

and

$$\tilde{Y} = (I - CC^{*})Y \qquad (43)$$

The psuedo inverse of Eq (40) is

$$\underset{d \times k}{C^{*}} = (C^{T}C)^{-1} C^{T} = (F^{T}A^{-T}A^{-1}F)^{-1} F^{T}A^{-T} \qquad (44)$$

or

$$C^{*} = (F^{T} R^{-1} F)^{-1} F^{T} A^{-T} \qquad (45)$$

Using Eq (45) in Eq (41)

$$\hat{Y} = L_{o}C^{*} = L_{o}(F^{T} R^{-1} F)^{-1} F^{T} A^{-T} \qquad (46)$$

and Eq (46) in Eq (38)

$$w = L_{o}(F^{T} R^{-1} F)^{-1} F^{T} R^{-1} \qquad (47)$$

while $A^{-T}A^{-1} = (A^{T}A)^{-1} = R^{-1}$

SOLUTION OF MATRIX CASE USING KRONECKER MATRIX PRODUCT.

The solution to the matrix case can be done using the Kronecker matrix product. The technique requires a simple extension to the vector case where d=1, that is by Eq (4) for $\underset{o \times k}{W}$ is $\langle w$ $(1 \times k)$

$$\sigma_{\tilde{a}} = \langle wRw \rangle \qquad (48)$$

and assume a vector of linear constraints instead of a matrix as in Eq (2), that is

$$\langle r)\ell_{o} = \langle w\underset{k \times r}{F_{o}} \qquad (49)$$

where the F_{o} of Eq (49) is not to be confused with the matrix F of Eq (2) which is the matrix of fitting functions.

Using Eq (11) in Eq (49)

$$\langle \ell_{o} = \langle yB^{-1}F_{o} = \langle y \underset{k \times r}{C} \qquad (50)$$

Multiply Eq (50) as before by C^{*}

$$\langle \ell_{o} C^{*} = \langle y B^{-1} F_{o}C^{*} = \langle y C C^{*}, \qquad (51)$$

where as before

$$\langle y = \langle \hat{y} + \langle \tilde{y} \qquad (52)$$

with

$$\langle \hat{y} = \langle yC C^{*} \qquad (53)$$

Note that the projector CC^* is kxk and if k=r (that is if there are k linear constraints) then there are no remaining degrees of freedom left and (full rank C)

$$\langle y = \langle \hat{y} = \langle \ell_o \tag{54}$$

a constant vector. Equation (54) implies that the $\langle w$ vector is also a constant, hence the quadratic of equation (48) can not be minimized since it is also a constant. The problems of interest is for $r < k$, then

$$\underset{nxk}{C^*} = \underset{nxr}{(C^T C)^{-1}} \, C^T \tag{55}$$

$$c^* = (F_o^T \, B^{-T} \, B^{-1} \, F_o)^{-1} \, F_o^T \, B^{-T} \tag{56}$$

$$C^* = (F_o^T \, (BB^T)^{-1} F_o)^{-1} \, F_o^T \, B^{-T} \tag{57}$$

or using Eq (57) in Eq (51)

$$\langle \hat{y} = \langle \ell_o \, [F_o^T \, R^{-1} \, F_o]^{-1} \, F_o^T \, B^{-T} \tag{58}$$

and Eq (58) in Eq (11)

$$\langle w = \langle \ell_o [F_o^T \, R^{-1} \, F_o]^{-1} \, F_o^T \, R^{-1} \tag{59}$$

The Kronecker product enters the picture via Eq (38)

$$\text{vec } w = \text{vec}(YA^{-1}) = (A^{-T} \otimes I) \text{ vec } Y \tag{60}$$

where Eq (49) sec (G) is used.

Also by Eq (37) and Eq (49) sec (G)

$$\text{vec } Y = \text{vec}(wA) = (A^T \otimes I) \text{ vec } w \tag{61}$$

we use the trace properties

$$\text{tr } Y \, Y^T = \text{tr } Y^T Y = (\text{vec } Y)^T \text{ vec } Y \tag{62}$$

hence the inner product expression of Eq (62)

$$(\text{vec } Y^T) \text{ vec } Y = \langle p) \, y \, y \, (p \rangle \tag{63}$$

where

$$p = k + d$$

is the same form as the problem of Eq (48) and Eq (50) where we use the vec operator on Eq (39), that is

$$\text{vec } L_o = \text{vec}(YC) = (C^T \otimes I)\text{vec } Y \tag{64}$$

The matrix C by Eq (40) is k x d in size and the vec L_o is d^2 x 1 in size that is

$$\underset{d^2 x1}{\text{vec } L_o} = [\underset{dxk}{C^T} \otimes \underset{dxd}{I}] \underset{kdx1}{\text{vec } Y} = \Gamma \text{ vec } Y \tag{65}$$

or

$$C^T \otimes I = \underset{d^2xkd}{\Gamma} \tag{66}$$

we now need the psuedo inverse of Γ or

$$\underset{kdxd^2}{\Gamma^*} = [C^T \otimes I]^* = \underset{kdxd^2}{\Gamma^T} (\Gamma\, \Gamma^T)^{-1} \tag{67}$$

By Eq (83) sec (G)

$$\Gamma^T = (C^T \otimes I)^T = C \otimes I \tag{68}$$

Using Eq (68) and Eq (66)

$$\Gamma\Gamma^T = (C^T \otimes I)(C \otimes I) \tag{69}$$

By Eq (54) sec (G) $\quad C^TC \otimes I$

$$\Gamma\Gamma^T = C^TC \otimes I \tag{70}$$

The inverse of the Kronecker product of Eq (70) is given by Eq (55) sec (G) as

$$[\Gamma\Gamma^T]^{-1} = [C^TC \otimes I]^{-1} = (C^TC)^{-1} \otimes I \tag{71}$$

Using Eq (71) AND EQ 68 in Eq (67) and Eq (54) sec (G)

$$\Gamma^* = (C \otimes I)[(C^TC)^{-1} \otimes I] = C(C^TC)^{-1} \otimes I \tag{72}$$

By Eq (44)

$$C^* = (C^TC)^{-1}\, C^T \tag{73}$$

or transposing

$$C^{*T} = C(C^TC)^{-1} \tag{74}$$

hence using Eq (74) in Eq (72)

$$\Gamma^* = C^{*T} \otimes I \tag{75}$$

Eq (75) in Eq (65)

$$\text{vec}\,\hat{Y} = \overset{*}{\Gamma}\,\text{vec}\,L_o = \Gamma^{*}\Gamma\,\text{vec}\,Y \tag{76}$$

$$= [C^{*T} \otimes I]\,\text{vec}\,L_o \tag{77}$$

and the de-vec or unpack operation given by Eq (49) sec (G)

$$\text{vec}\,\hat{Y} = \text{vec}\,L_o C^{*} \tag{78}$$

or

$$\hat{Y} = L_o C^{*} \tag{79}$$

which agrees with Eq (41), And hence the solution for the weighting matrix W of EQ 47 follows.

By Eq (13) sec (10)

$$z\rangle = \underset{kxd}{F}\, a\rangle + v\rangle = F\,\hat{a}\rangle + \tilde{z}\rangle \tag{80}$$

and premultiplying Eq (80) by $\underset{dxk}{w}$

$$\underset{dxk}{w}\, z\rangle = \hat{a}\rangle \tag{81}$$

One can multiply Eq (80) by a full rank matrix M kxk in size to obtain

$$\underset{kxk}{M}\, z\rangle = M\,F\,\hat{a}\rangle + M\,\hat{z}\rangle \tag{82}$$

or

$$M\,z\rangle = F_m\,\hat{a}\rangle + M\,\tilde{z}\rangle \tag{83}$$

with

$$\underset{kxd}{F_m} = M\,F \tag{84}$$

Multiply Eq (83) by the psuedo inverse of Eq (84)

$$F_m^{*}\,M\,z\rangle = F_m^{*}\,F_m\,\hat{a}\rangle = \hat{a}\rangle \tag{85}$$

The psuedo-inverse of Eq (84) is

$$F_m^{*} = (F_m^T\,F_m)^{-1}\,F_m^T = (F^T M^T M F)^{-1}\,F^T M^T \tag{86}$$

Using Eq (86) in Eq (85)

$$\langle \hat{a} \rangle = (F^T M^T M F)^{-1} F^T M^T M \, z \rangle \tag{87}$$

or by Eq (47)

$$\underset{d\times k}{w} = (F^T R^{-1} F)^{-1} F^T R^{-1} \tag{88}$$

One can now set

$$M^T M = R^{-1} \tag{89}$$

and

$$\underset{d\times k}{w} = \underset{d\times k}{F_m^*} \underset{k\times k}{M} \tag{90}$$

which are the factors of w.

the classical approach to weighted least squares minimizes

$$\langle \tilde{z} \, M^T M \, \tilde{z} \rangle = \sigma_{\tilde{z}} \tag{91}$$

or

$$\langle \frac{\partial}{\partial \hat{a}} \sigma_z = \langle 0 \tag{92}$$

which yields the same answer as Eq (87)

SECTION 19
RECURSIVE WEIGHTED UNCONSTRAINED PARAMETER ESTIMATION (INCREASING SPAN)

The optimal weighted estimate of the parameter vector is given by Eq (87) of Sec (17) as

$$\hat{a}\rangle = (F^T R^{-1} F)^{-1} F^T R^{-1} \, z\rangle \tag{1}$$

with

$$\underset{dxk}{W} = (F^T R^{-1} F)^{-1} F^T R^{-1} = F_m^{*} M \tag{2}$$

If we now have one more measurement z_{k+1} (a scalar)

$$\begin{bmatrix} z\rangle \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} F \\ \langle f(k+1) \end{bmatrix} \hat{a}(k+1)\rangle + z\rangle \tag{3}$$

or

$$\hat{a}(k+1)\rangle = \underset{dx(k+1)}{W} \begin{bmatrix} z\rangle \\ z_{k+1} \end{bmatrix} \tag{4}$$

The expanded matrix of Eq (2) is

$$\underset{dx(k+1)}{W} = \underset{d(k+1)d}{(F^T R^{-1} F)^{-1}} \; \underset{dx(k+1)}{F^T R^{-1}} \tag{5}$$

If the expanded variance matrix R is partitioned as

$$\underset{(k+1)(k+1)}{R^{-1}} = \begin{bmatrix} \underset{kxk}{R} & r\rangle \\ \langle r & r \end{bmatrix}^{-1} \tag{6}$$

then

$$\underset{(k+1)(k+1)}{R^{-1}} = \begin{bmatrix} R^{-1}\left[I - \dfrac{r\rangle\langle rR^{-1}}{d}\right] & \dfrac{-R^{-1}r\rangle}{d} \\ -\dfrac{\langle rR^{-1}}{d} & \dfrac{1}{d} \end{bmatrix} \tag{7}$$

with the denominator

$$d = r - \langle r R^{-1} r \rangle \quad , \tag{8}$$

and the cross correlation terms are in the vector $\langle r$.

For the assumption of no serial correlation, that is

$$\langle r = \langle 0 \tag{9}$$

Eq (7) becomes

$$\underset{(k+1)(k+1)}{R^{-1}} = \begin{bmatrix} R^{-1} & 0 \\ 0 & r^{-1} \end{bmatrix} \tag{10}$$

The transpose of the expanded fitting functions matrix of Eq (3) is

$$\underset{dx(k+1)}{F^T} = [\underset{dxk}{F^T}, f(k+1)\rangle\ ] \tag{11}$$

and the product

$$[F^T, f(k+1)\rangle\ ] \begin{bmatrix} R^{-1} & 0\rangle \\ \langle 0 & r^{-1} \end{bmatrix} = [F^T R^{-1}, f(k+1)\rangle\ r^{-1}] \tag{12}$$

The inverse of the expanded $F^T R^{-1} F$ matrix is given by Eq (48) sec(13) as

$$\underset{d(k+1)d}{(F^T R^{-1} F)^{-1}} = (F^T R^{-1} F)^{-1} [I - \frac{f\rangle\langle f(F^T R^{-1} F) r^{-1}}{1 + r^{-1} \langle f(F^T R^{-1} F)^{-1} f\rangle}] \tag{13}$$

Multiplying Eq (12) and (13)

$$\underset{dx(k+1)}{W} = \Bigg[\underset{dxk}{W} - \frac{(F^T R^{-1} F)^{-1} f\rangle\langle f(F^T R^{-1} F)^{-1} F^T R^{-1} r^{-1}}{1 + r^{-1} \langle f(F^T R^{-1} F)^{-1} f\rangle},$$

$$(F^T R^{-1} F)^{-1} f\rangle\ r^{-1} - \frac{(F^T R^{-1} F)^{-1} f\rangle\langle f(F^T R^{-1} F)^{-1} f\rangle\ r^{-2}}{1 + r^{-1} \langle f(F^T R^{-1} F)^{-1} f\rangle} \Bigg] \tag{14}$$

Multiplying Eq (14) by the partitioned measurement vector

$$\underset{dxk+1}{W} \begin{pmatrix} z\rangle \\ z_{k+1} \end{pmatrix} = \hat{a}(k+1)\rangle \tag{15}$$

and simplifying the terms one obtains

$$\hat{a}(k+1)\rangle = \hat{a}(k)\rangle + w(k+1)\rangle \; \tilde{z}(k+1,k) \tag{16}$$

with

$$\tilde{z}(k+1,k) = z(k+1) - \hat{z}(k+1,k) \tag{17}$$

and

$$\hat{z}(k+1,k) = \langle f(k+1) \; \hat{a}(k)\rangle \tag{18}$$

and

$$w(k+1)\rangle = \frac{(F^T R^{-1} F)^{-1} f(k+1)\rangle \;\; r^{-1}}{1 + \underset{k+1}{\langle} f(F^T R^{-1} F)^{-1} f \rangle_{k+1}} \tag{19}$$

Section 20 HOMOGENEOUS DISCRETE DYNAMIC PROCESS

Consider a sequence of j vector trajectories

$$X(k+1)\rangle\rangle_j = \Phi(k+1, k)\ X(k)\rangle\rangle_j \tag{1}$$

for k = 0, 1, 2 ...

$$j \geq k$$

or packagewise

$$\underset{p\times j}{X(k+1)} = \Phi(k+1, k)\ \underset{p\times j}{X(k)} \tag{2}$$

where $\Phi(k+1, k)$ is the usual state transition (discrete) matrix.

Sequencing the time index k we obtain for Eq. (2)

$$\begin{aligned} X(1) &= \Phi(1, 0)\ X(0) \\ X(2) &= \Phi(2, 0)\ X(1) = \Phi(2, 1)\ \Phi(1, 0)\ X(0) \\ &\ \ \vdots \\ X(k) &= \Phi(k, k-1)\ X(k-1) = \Phi(k, k-1)\ \ldots\ \Phi(1, 0)\ X(0) \\ X(k+1) &= \Phi(k+1, k)\ X(k) = \Phi(k+1, k)\ \ldots\ \Phi(1, 0)\ X(0) \end{aligned} \tag{3}$$

or packagewise

$$\begin{bmatrix} X(1) \\ X(2) \\ X(3) \\ \vdots \\ X(k) \\ X(k+1) \end{bmatrix} = \begin{bmatrix} \Phi(1, 0) \\ \Phi(2, 1)\ \Phi(1, 0) \\ \Phi(3, 2)\ \Phi(2, 1)\ \Phi(1, 0) \\ \\ \Phi(k, k-1)\ \ldots\ldots\ldots\ldots\ \Phi(1, 0) \\ \Phi(k+1, k)\ \ldots\ldots\ldots\ldots\ldots\ldots\ \Phi(1, 0) \end{bmatrix} X(0) \tag{4}$$

or

$$\begin{bmatrix} X(1) \\ X(2) \\ \vdots \\ X(k) \\ X(k+1) \end{bmatrix} = \begin{bmatrix} \Phi(1, 0) \\ \Phi(2, 0) \\ \vdots \\ \Phi(k, 0) \\ \Phi(k+1, 0) \end{bmatrix} X(0) \tag{5}$$

The last equation in the batch of Eq. (3) was obtained by substituting in from the top down. One can also work from the bottom up, thus

$$
\begin{aligned}
X(k) &= \Phi^{-1}(k+1,\ k)\ X(k+1) \\
X(k-1) &= \Phi^{-1}(k,\ k-1)\ X(k) = \Phi^{-1}(k,\ k-1)\ \Phi^{-1}(k+1,\ k)\ X(k+1) \\
&\ \ \vdots \\
X(1) &= \Phi^{-1}(2,\ 1)\ X(2) = \Phi^{-1}(2,\ 1)\ \Phi^{-1}(3,\ 2)\ \ldots\ \Phi^{-1}(k+1,\ k)\ X(k+1) \\
X(0) &= \Phi^{-1}(1,\ 0)\ X(1) = \Phi^{-1}(1,\ 0)\ \Phi^{-1}(2,\ 1)\ \ldots\ \Phi^{-1}(k+1,\ k)\ X(k+1)
\end{aligned}
\tag{6}
$$

and packagewise

$$
\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \cdot \\ \cdot \\ \cdot \\ X(k) \end{bmatrix} = \begin{bmatrix} \Phi^{-1}(1,\ 0)\ \Phi^{-1}(2,\ 1)\ \Phi^{-1}(3,\ 2)\ \ldots\ \Phi^{-1}(k+1,\ k) \\ \Phi^{-1}(2,\ 1)\ \Phi^{-1}(3,\ 2)\ \ldots\ \Phi^{-1}(k+1,\ k) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \Phi^{-1}(k+1,\ k) \end{bmatrix} X(k+1) \tag{7}
$$

or

$$
\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \cdot \\ \cdot \\ \cdot \\ X(k) \end{bmatrix} = \begin{bmatrix} \Phi^{-1}(k+1,\ 0) \\ \Phi^{-1}(k+1,\ 1) \\ \Phi^{-1}(k+1,\ 2) \\ \cdot \\ \cdot \\ \cdot \\ \Phi^{-1}(k+1,\ k) \end{bmatrix} \quad X(k+1) \tag{8}
$$

It can be seen by Eq. (5) the k+1 matrices are a function of the initial matrix. Likewise when one replaces the matrices with j=1, or a single vector, then the state vector at the k+1 time points is a function only of the initial vector $x(0)\rangle$.

If now Φ is a constant for all k, that is

$$\Phi(k+1,\ k) = \Phi(1,\ 0) = \Phi \qquad \forall k$$

then

$$\Phi(k+1,\ 0) = \Phi^{(k+1)} \tag{9}$$

and

$$\Phi^{-1}(k+1,\ 0) = \Phi^{-(k+1)}\ . \tag{10}$$

Using Eq. (10) in Eq. (8)

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \cdot \\ \cdot \\ \cdot \\ X(k) \end{bmatrix} = \begin{bmatrix} \Phi^{-(k+1)} \\ \Phi^{-k} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \Phi^{-1} \end{bmatrix} X(k+1) \tag{11}$$

Using Eq. (9) in Eq. (5)

$$\begin{bmatrix} X(1) \\ X(2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X(k) \\ X(k+1) \end{bmatrix} = \begin{bmatrix} \Phi \\ \Phi^{2} \\ \Phi^{3} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \Phi^{k+1} \end{bmatrix} X(0) \tag{12}$$

If we define the matrix of Eq. (12) as

$$\underset{(k+1)pxp}{\Theta(k+1,\ 0)} = \begin{bmatrix} \Phi \\ \Phi^{2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \Phi^{k+1} \end{bmatrix} = \begin{bmatrix} I \\ \Phi \\ \Phi^{2} \\ \cdot \\ \cdot \\ \cdot \\ \Phi^{k} \end{bmatrix} \Phi \tag{13}$$

$$= \Theta_I(k)\Phi \tag{14}$$

and seek the psuedo inverse of Eq. (13).

The matrix Φ is always full rank, hence

$$\Theta^*(k+1,\ 0) = \Phi^{-1}\ \Theta^*_I \tag{15}$$

The powers matrix

$$\Theta_I = \begin{bmatrix} I \\ \Phi \\ \Phi^{2} \\ \Phi^{k} \end{bmatrix} = \Phi\rangle \tag{16}$$

has psuedo inverse

$$\underset{px(k+1)p}{\Theta_I^*} = (\langle \Phi^T \Phi \rangle)^{-1} \langle \Phi^T \tag{17}$$

where

$$\langle \Phi^T = (I,\ \Phi^T,\ \Phi^{2T},\ \ldots\ \Phi^{kT}) \tag{18}$$

and the full rank Grammian is

$$\langle \Phi^T \Phi \rangle = I + \Phi^T + \Phi^{T2}\ \Phi^2 + \ldots\ 1 + \Phi^{Tk}\ \Phi^k \tag{19}$$

whose inverse is needed

$$(\langle \Phi^T \Phi \rangle)^{-1} = \left[I + \Phi^T \Phi + \ldots + \Phi^{Tk}\ \Phi^k \right]^{-1} . \tag{20}$$

Eq. (20) appears horrendous, however, we shall "construct" a psuedo inverse.

By Eq. (12)

$$X(0) = \Phi^{-1}\ X(1) \tag{21}$$

$$X(0) = \Phi^{-2}\ X(2)$$

or packagewise

$$X(0) = \frac{1}{2} \left[\Phi^{-1},\ \Phi^{-2} \right] \begin{pmatrix} X(1) \\ X(2) \end{pmatrix} \tag{22}$$

or

$$X(0) = \frac{1}{2}\ \Phi^{-1} \left[I,\ \Phi^{-1} \right] \begin{pmatrix} X(1) \\ X(2) \end{pmatrix} \tag{23}$$

or if

$$\Theta(2,\ 0) = \begin{pmatrix} I \\ \Phi \end{pmatrix} \Phi = \Theta_I(2,\ 0)\ \Phi \tag{24}$$

$$\Theta^*(2,\ 0) = \Phi^{-1} \left[(I,\ \Phi^T) \begin{pmatrix} I \\ \Phi \end{pmatrix} \right]^{-1} \left[I,\ \Phi^T \right] \tag{25}$$

$$= \frac{1}{2}\ \Phi^{-1}\ (I,\ \Phi^{-1}) \tag{26}$$

Note also

$$\Theta_I^*\ (2,\ 0) = \frac{1}{2}\ (I,\ \Phi^{-1}) \tag{27}$$

By definition of the psuedo inverse

$$\Theta^*_I (2, 0) = \left[(I, \Phi^T) \begin{pmatrix} I \\ \Phi \end{pmatrix}\right]^{-1} [I, \Phi^T]$$

$$\frac{1}{2} \left[(I, \Phi^{-1})\right] \equiv (I + \Phi^T \Phi)^{-1} (I, \Phi^T) \tag{28}$$

Multiply Eq. (28) by $(I, \Phi^{-T})^T$ and

$$\frac{1}{2} (I, \Phi^{-1}) \begin{pmatrix} I \\ \Phi^{-T} \end{pmatrix} = (I + \Phi^T \Phi)^{-1} \, 2I \tag{29}$$

or

$$(I + \Phi^T \Phi)^{-1} = \frac{1}{4} \left[I + (\Phi^T \Phi)^{-1}\right] \tag{30}$$

The first three elements of Eq. (12) are

$$X(0) = \Phi^{-1} X(1) = \Phi^{-2} X(2) = \Phi^{-3} X(3) \tag{31}$$

or

$$X(0) = \frac{1}{3} (\Phi^{-1}, \Phi^{-2}, \Phi^{-3}) \begin{pmatrix} X(1) \\ X(2) \\ X(3) \end{pmatrix} \tag{32}$$

or

$$X(0) = \frac{1}{3} \Phi^{-1} \left[I, \Phi^{-1}, \Phi^{-2}\right] \begin{pmatrix} X(0) \\ X(1) \\ X(2) \end{pmatrix} \tag{33}$$

hence by Eq. (12) the first three elements are

$$\begin{bmatrix} X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} I \\ \Phi \\ \Phi^2 \end{bmatrix} \Phi X(0) = \Theta(3, 0) X(0) \tag{34}$$

and

$$\Theta^* (3, 0) = \frac{1}{3} \Phi^{-1} \left[I, \Phi^{-1}, \Phi^{-2}\right] \tag{35}$$

also

$$\Theta_I (3, 0) = \begin{pmatrix} I \\ \Phi \\ \Phi^2 \end{pmatrix} \tag{36}$$

and

$$\Theta_I^* (3, 0) = \frac{1}{3} (I, \Phi^{-1}, \Phi^{-2}) \tag{37}$$

for by Eq. (36) and Eq. (37)

$$\Theta_I^* (3, 0)\ \Theta_I \equiv I \tag{38}$$

By definition of $\Theta_I^* (3, 0)$

$$\left[(I, \Phi^T, \Phi^{2T}) \begin{pmatrix} I \\ \Phi \\ \Phi^2 \end{pmatrix} \right]^{-1} (I, \Phi^T, \Phi^{2T}) = \frac{1}{3} \left[I\ \Phi^{-1}\ \Phi^{-2} \right] \tag{39}$$

Multiply Eq. (39) by $\left[I, \Phi^{-1}, \Phi^{-2} \right]^T$

and

$$\left[I + \Phi^T\Phi + (\Phi^T\Phi)^2 \right]^{-1} = \frac{1}{9} \left[I + (\Phi^T\Phi)^{-1} + \Phi^{-2}\ \Phi^{-2T} \right] \tag{40}$$

but

$$\Phi^{-2}\ \Phi^{-2T} = (\Phi^{2T}\ \Phi^2)^{-1} \tag{41}$$

hence

$$\left[I + \Phi^T\Phi + (\Phi^T\Phi)^2 \right]^{-1} = \frac{1}{3^2} \left[I + (\Phi^T\Phi)^{-1} + (\Phi^{T2}\ \Phi^2)^{-1} \right] \tag{42}$$

Define the Grammian

$$\Phi^T\Phi = G \tag{43}$$

then

$$\left[I + G + G^2 \right]^{-1} = \frac{1}{3^2} \left[I + G^{-1} + G^{-2} \right] \tag{44}$$

Continuing in the above manner one can show that (by Eq. (2))

$$\Theta(k+1, 0) = \Theta_I(k, 0)\ \Phi = \begin{bmatrix} I \\ \Phi \\ \cdot \\ \cdot \\ \cdot \\ \Phi^k \end{bmatrix} \Phi \tag{45}$$

and

$$\Theta^*(k+1,\ 0) = \Phi^{-1}\ \Theta^*_I(k,\ 0) = \frac{\Phi^{-1}}{k+1}\left[I,\ \Phi^{-1},\ \Phi^{-2},\ \ldots\ \Phi^{-k}\right] \tag{46}$$

Consider next Eq. (1)

$$\begin{bmatrix} X(0) \\ X(1) \\ \cdot \\ \cdot \\ \cdot \\ X(k) \end{bmatrix} = \begin{bmatrix} \Phi^{-(k+1)} \\ \Phi^{-k} \\ \cdot \\ \cdot \\ \cdot \\ \Phi^{-1} \end{bmatrix} X(k+1) = \begin{bmatrix} \Phi^{-k} \\ \Phi^{-(k-1)} \\ \cdot \\ \cdot \\ \cdot \\ I \end{bmatrix} \Phi^{-1}\ X(k+1) \tag{47}$$

or

$$\begin{bmatrix} X(0) \\ X(1) \\ \cdot \\ \cdot \\ \cdot \\ X(k) \end{bmatrix} = \underset{(k+1)pxp}{\Psi(k,\ k+1)}\ X(k+1) \tag{48}$$

$$= \Psi_I(k,\ k+1)\ \Phi^{-1}\ X(k+1) \tag{49}$$

One can construct the psuedo-inverse as before, for example

$$\begin{pmatrix} X(0) \\ X(1) \end{pmatrix} = \begin{pmatrix} \Phi^{-1} \\ I \end{pmatrix} \Phi^{-1}\ X(2) \tag{50}$$

and

$$X(2) = \frac{1}{2}\ \Phi\left[\Phi,\ I\right] \begin{pmatrix} X(0) \\ X(1) \end{pmatrix} \tag{51}$$

and in general if

$$\Psi_I\ (k,\ k+1) = \begin{bmatrix} \Phi^{-k} \\ \Phi^{-(k-1)} \\ \cdot \\ \cdot \\ \cdot \\ I \end{bmatrix} \tag{52}$$

$$\Psi^*_I(k,\ k+1) = \left[\Phi^k,\ \Phi^{k-1},\ \ldots\ \Phi,\ I\right]\frac{1}{k+1} \tag{53}$$

and

$$\Psi^*(k,\ k+1) = \Psi^*_I(k,\ k+1)\ \Phi \tag{54}$$

Section 21 NON-HOMOGENEOUS DISCRETE DYNAMICAL PROCESS

This section considers the non-homogeneous dynamical process

$$x(k+1)\rangle = \Phi(k+1, k)\, x(k)\rangle + B(k)f(k)\rangle \tag{1}$$

with Φ a pxp matrix and B a pxg matrix. For a sequence of j trajectories, we have a rectangular matrix relation

$$\underset{pxj}{X}(k+1) = \underset{pxp}{\Phi}(k+1, h)\, \underset{pxj}{X}(k) + \underset{pxg}{B}(k)f(k)\rangle\langle j)\ 1 \tag{2}$$

$$F(k) = f(k)\rangle\langle 1 \tag{3}$$

for k = 0, 1, 2, ... and $j \geq p$

Chasing the time-points (index k) as before we obtain

$$\begin{cases} X(1) = \Phi(1, 0)\, X(o) + B(o)F(o) \\ X(2) = \Phi(2, 1)\, X(1) + B(1)F(1) \\ X(2) = \Phi(2, 1)\ \ \Phi(1, 0)X(o) + B(o)F(o)\ \ + B(1)F(1) \\ X(2) = \Phi(2, 1)\Phi(1, 0)X(o) + \Phi(2, 1)B(o)F(o) + B(1)F(1) \\ X(2) = \Phi(2, 0)X(o) + \ B(1), \Phi(2, 1)B(o)\ \begin{pmatrix} F(1) \\ F(0) \end{pmatrix} \\ X(3) = \Phi(3, 0)X(o) + \ B(2), \Phi(3, 2)B(1), \Phi(3, 1)B(o)\ \begin{pmatrix} F(2) \\ F(1) \\ F(0) \end{pmatrix} \\ \cdot \\ \cdot \\ X(k) = \Phi(k, o)X(o) + \ B(k-1), \Phi(k, k-1)B(k-2), \ldots \Phi(k, o)B(o)\ \begin{pmatrix} F(k-1) \\ \cdot \\ \cdot \\ \cdot \\ F(0) \end{pmatrix} \\ X(k+1) = \Phi(k+1, o)X(o) + \ B(k), \Phi(k+1, k)B(k-1), \ldots \Phi(k+1, 1)B(o)\ \begin{pmatrix} F(k) \\ \cdot \\ \cdot \\ \cdot \\ F(0) \end{pmatrix} \end{cases} \tag{4}$$

Packaging the above equations

$$\begin{bmatrix} X(1) \\ X(2) \\ X(3) \\ \cdot \\ \cdot \\ \cdot \\ X(k) \\ X(k+1) \end{bmatrix} = \begin{bmatrix} \Phi(1, 0) \\ \Phi(2, 0) \\ \Phi(3, 0) \\ \cdot \\ \cdot \\ \cdot \\ \Phi(k, 0) \\ \Phi(k+1, 0) \end{bmatrix} X(0) \tag{5}$$

$$+ \begin{bmatrix} 0 \ldots & & 0 & B(0) \\ 0 \ldots & 0 & B(1) & \Phi(2, 1)B(0) \\ 0 \ldots 0 & B(2) & \Phi(3, 2)B(1) & \Phi(3, 1)B(0) \\ & & & \cdot \\ & & & \cdot \\ & & & \cdot \\ 0 & B(k-1) & \ldots & \Phi(k, 1)B(0) \\ B(k) & \Phi(k+1, k)B(k-1) & \ldots & \Phi(k+1, 1)B(0) \end{bmatrix} \begin{bmatrix} F(k) \\ F(k-1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ F(1) \\ F(0) \end{bmatrix}$$

Using the linear convolution matrix

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ \cdot \\ \cdot \\ \cdot \\ F(k) \end{bmatrix} = \begin{bmatrix} & & & I \\ & 0 & I & \\ & & \cdot & \\ & & \cdot & \\ & \cdot & & \\ & I & 0 & \\ I & & & \end{bmatrix} \begin{bmatrix} F(k) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ F(1) \\ F(0) \end{bmatrix} \tag{6}$$

Equation (7) becomes

$$\begin{bmatrix} X(1) \\ X(2) \\ \cdot \\ \cdot \\ X(k+1) \end{bmatrix} = \begin{bmatrix} \Phi(1, 0) \\ \cdot \\ \cdot \\ \cdot \\ \Phi(k+1, 0) \end{bmatrix} X(0) \tag{7}$$

$$+ \begin{bmatrix} B(0) & & & \\ \Phi(2, 1)B(0) & B(1) & & \\ \Phi(3, 1)B(0) & \Phi(3, 2)B(1) & B(2) & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \Phi(k+1, 1)B(0) & \ldots & & B(k) \end{bmatrix} \begin{bmatrix} F(0) \\ F(1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ F(k) \end{bmatrix}$$

$$= \underset{p(k+2)xp}{\Phi} \; \underset{pxj}{X(0)} + \beta\Gamma \tag{8}$$

The k^{th} state of Eq. (4) can be written in convolved form as

$$X(k) = \Phi(k, 0)X(0) + \sum_{\beta=0}^{k-1} \Phi(k, \beta+1)B(\beta)F(\beta) \tag{9}$$

and the $(k+1)^{th}$ state as

$$X(k+1) = \Phi(k+1, 0)X(0) + \sum_{\beta=0}^{k} \Phi(k+1, \beta+1)B(\beta)F(\beta) \tag{10}$$

If one is at the k^{th} stage and wants to advance m steps ahead

$$X(k+m) = \Phi(k+m, k)X(k) + \sum_{\beta=k}^{k+m-1} \Phi(k+m, \beta+1)B(\beta)F(\beta) \tag{11}$$

Equation (9) is the discrete analog of the continuous convolution integral.

If we lump the product terms

$$B(k)F(k) = G(k) \tag{12}$$

then

$$\begin{bmatrix} G(k) \\ G(k-1) \\ \cdot \\ \cdot \\ \cdot \\ G(1) \\ G(0) \end{bmatrix} = \begin{bmatrix} B(k) & & & \\ & B(k-1) & & \\ & & \ddots & \\ & & & B(0) \end{bmatrix} \begin{bmatrix} F(k) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ F(0) \end{bmatrix} \tag{13}$$

and Eq. (1) is (in open form)

$$X(k+m) = \Phi(k+m, k)X(k) + \left[\Phi(k+m, k+1), \Phi(k+m, k+2), \ldots \Phi(k+m, k+m-1), I\right] \begin{bmatrix} G(k) \\ G(k+1) \\ \cdot \\ \cdot \\ \cdot \\ G(k+m+1) \end{bmatrix} \tag{14}$$

Multiply Eq. (14) by $\Phi^{-1}(k+m, k)$ and obtain

$$\Phi^{-1}(k+m, k)X(k+m) = X(k) + \Phi^{-1}(k+m, k)\left[\Phi(k+m, k+1), \ldots\right] \begin{bmatrix} G(k) \\ \cdot \\ \cdot \\ G(k+m-1) \end{bmatrix} \tag{15}$$

Now

$$\Phi^{-1}(k+m,\ k)\Phi(k+m,\ k+1) = \Phi(k,\ k+m)\Phi(k+m,\ k+1)$$
$$= \Phi(k,\ k+1) = \Phi^{-1}(k+1,\ k) \qquad (16)$$
$$\vdots$$
$$\Phi^{-1}(k+m,\ k)\Phi(k+m,\ k+m-1) = \Phi(k,\ k+m)\Phi(k+m,\ k+m-1)$$
$$= \Phi(k,\ k+m-1) = \Phi^{-1}(k+m-1,\ k)$$

One can also view the first equation of the set (16) as

$$\Phi(k+m,\ k+1) = \Phi(k+m,\ k)\Phi(k,\ k+1) \qquad (17)$$

hence

$$\Phi^{-1}(k+m,\ k)\Phi(k+m,\ k+1) = \Phi^{-1}(k+m,\ k)\Phi(k+m,\ k)\Phi(k,\ k+1) \qquad (18)$$

etc.

Using the relations of Eq. (16) in Eq. (15)

$$X(k) = \Phi^{-1}(k+m,\ k)X(k+m) - \left[\Phi^{-1}(k+1,\ k),\ \Phi^{-1}(k+2,\ k)\ \ldots\ \Phi^{-1}(k+m-1,\ k),\ \Phi^{-1}(k+m,\ k)\right] \begin{bmatrix} G(k) \\ \cdot \\ \cdot \\ \cdot \\ G(k+m-1) \end{bmatrix} \qquad (19)$$

or index-wise

$$X(k) = \Phi^{-1}(k+m,\ k)X(k+m) - \sum_{\beta=k}^{k+m-1} \Phi^{-1}(\beta+1,\ k)G(\beta)\ . \qquad (20)$$

Observe that by the $x(k)\rangle$ term of Eq. (4) for the $j = 1$ or vector case and for $B = p\times1$ a constant column vector and for zero initial state, one has for Φ a constant matrix,

$$X(k)\rangle = \left[b\rangle,\ \Phi b\rangle,\ \Phi^2 b\rangle,\ \ldots\ \Phi^{k-1} b\rangle\right] \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_{K-1} \end{bmatrix} \qquad (21)$$

and if the kxk matrix of Eq. (21) is full rank, one can invert the matrix and solve for the sequence of scalar inputs.

By Eq. (9) of the previous section one can write Eq. (7) as

$$\begin{bmatrix} X(1) \\ X(2) \\ \cdot \\ \cdot \\ \cdot \\ X(k) \\ X(k+1) \end{bmatrix} = \begin{bmatrix} \Phi^{-1}(k+1,\ 0) \\ \Phi^{-1}(k+1,\ 1) \\ \cdot \\ \cdot \\ \cdot \\ \Phi^{-1}(k+1,\ k) \\ I \end{bmatrix} X(k+1) + \beta\Gamma \tag{22}$$

where for Φ a constant matrix

$$\beta = \begin{bmatrix} I & 0 & & \\ \Phi & I & & \\ \Phi^2 & \Phi & I & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \Phi^k, & \Phi^{k-1}, & \ldots & I \end{bmatrix} \tag{23}$$

and

$$\Gamma = \begin{bmatrix} G(0) \\ G(1) \\ \cdot \\ \cdot \\ \cdot \\ G(k) \end{bmatrix} \tag{24}$$

One can "construct" the inverse of Eq. (23) for example assume X(0)=0 and one has

$$\begin{bmatrix} X(1) \\ X(2) \\ \cdot \\ \cdot \\ \cdot \\ X(k+1) \end{bmatrix} = \begin{bmatrix} I & & & \\ \Phi & I & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \Phi^k, & \ldots & \Phi & I \end{bmatrix} \begin{bmatrix} G(0) \\ G(1) \\ \cdot \\ \cdot \\ \cdot \\ G(k) \end{bmatrix} \tag{25}$$

and

$$
\begin{aligned}
G(0) &= X(1) \\
G(1) &= (-\Phi,\ I) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\
&\ \vdots \\
G(k) &= (-\Phi,\ I) \begin{pmatrix} X(k) \\ X(k+1) \end{pmatrix}
\end{aligned}
\tag{26}
$$

since

$$
X(k+1) = \Phi X(k) + G(k) \tag{27}
$$

or

$$
\begin{bmatrix} G(0) \\ G(1) \\ G(2) \\ \cdot \\ \cdot \\ \cdot \\ G(k) \end{bmatrix} = \begin{bmatrix} I & 0 & & & & \\ -\Phi & I & 0 & & & \\ 0 & -\Phi & I & 0 & & \\ \cdot & & & \ddots & & \\ \cdot & & & & \ddots & \\ \cdot & & & & & \\ 0 & 0 & & & -\Phi & I \end{bmatrix} \begin{bmatrix} X(1) \\ X(2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X(k+1) \end{bmatrix} \tag{28}
$$

or

$$
\begin{bmatrix} I & 0 & 0 & & \\ \Phi & I & 0 & & \\ \Phi^2 & \Phi & I & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \Phi^k & \Phi^{k-1}, & \ldots & \Phi & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 & & & \\ -\Phi & I & & & \\ 0 & -\Phi & I & & \\ \cdot & & & \ddots & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & \ldots & -\Phi & I \end{bmatrix} \tag{29}
$$

Multiply Eq. (22) by Eq. (29) solves for the sequence of output forces and is called "deconvolution".

SECTION 22
STOCHASTIC DISCRETE DYNAMICAL SYSTEM

This section considers the non-homogenous dynamical process of the previous section but with a stochastic term added, that is, the sequence j

$$x(k+1)\rangle_j = \phi(k+1,k)x(k)\rangle_j + B(k)f(k)\rangle + u(k)\rangle_j \tag{1}$$

where without too much loss of generality

$$\phi(k+1, k) = \phi \quad \forall k \tag{2}$$

and

$$B(k) = B \tag{3}$$

For the package

$$\underset{pxj}{X(k+1)} = \phi\ X(k) + B\ f(k)\rangle\langle 1 + \underset{pxj}{U(k)} \tag{4}$$

The mean is

$$X(k+1)\ 1^*(j\rangle = \mu(k+1)\rangle \tag{5}$$

or

$$\mu(k+1)\rangle = \phi\ \mu(k)\rangle + B\ f(k)\rangle + \mu(k)\rangle_u \tag{6}$$

and

$$x(k)\rangle_j = \mu(k)\rangle + \tilde{x}(k)\rangle_j$$

or for the package

$$X(k) = \mu(k)\rangle\langle 1 + \tilde{X} \tag{7}$$

using Eq (6) (for k) in Eq (7)

$$X(k) = X(k)\ 1^*(j\rangle\langle 1 + \tilde{X} \tag{8}$$

or

$$\tilde{X}(k) = X(k)\ [I_j - 1^*\rangle\langle 1] \tag{9}$$

or

$$\tilde{X}(k) = X(k)\ \tilde{P}_{11} \tag{10}$$

By Eq (10) we see that the orthogonal complement projector $\tilde{P}_{11}$ maps Eq (4) onto the "residuals" that is

$$X(k+1)\tilde{P}_{11} = \phi\ X(k)\ \tilde{P}_{11} + B\ f\rangle\langle 1\ \tilde{P}_{11} + U\tilde{P}_{11} \tag{11}$$

and

$$\langle j\rangle\ 1\ \tilde{P}_{11} = \langle 0 \tag{12}$$

hence

$$\tilde{X}(k+1) = \phi\ \tilde{X}(k) + \tilde{U}(k) \tag{13}$$

which is the recursive dynamics for package of errors.

Transposing Eq (13)

$$\tilde{X}^T(k+1) = \tilde{X}^T(k)\ \phi^T + \tilde{U}^T(k) \tag{14}$$

The inner-Grammian of Eq (13) and Eq (14) is

$$\begin{aligned} \tilde{X}(k+1)\tilde{X}^T(k+1) &= \phi\ \tilde{X}(k)\ \tilde{X}^T(k)\ \phi^T \\ &+ \phi\ \tilde{X}(k)\ \tilde{U}^T(k) + \tilde{U}(k)\ \tilde{X}^T(k)\ \phi^T \\ &+ \tilde{U}(k)\ \tilde{U}^T(k) \end{aligned} \tag{15}$$

Consider next the "cross" terms of Eq (15); by Eq (13) and Eq (4) of the previous section.

$$\tilde{X}(k) = \phi^k\ \tilde{X}(o) + [I, \phi, \phi^2, \ldots \phi^{k-1}] \begin{bmatrix} \tilde{U}(o) \\ \tilde{U}(1) \\ \tilde{U}(2) \\ \cdot \\ \cdot \\ \cdot \\ \tilde{U}(k-1) \end{bmatrix} \tag{16}$$

and

$$\tilde{X}(k)\ \tilde{U}^T(k) = [I,\phi,\phi^2,\ldots \phi^{k-1}] \begin{bmatrix} \tilde{U}(o)\ \tilde{U}^T(k) \\ \tilde{U}(1)\ \tilde{U}^T(k) \\ \cdot \\ \cdot \\ \cdot \\ \tilde{U}(k-1)\ \tilde{U}^T(k) \end{bmatrix} \tag{17}$$

where it has been assumed that

$$\tilde{X}(o)\ \tilde{U}^T(k) = 0 \tag{18}$$

that is the initial states are independent of the process noise.

Using the countably infinite set of discrete trajectories concept, that is

$$\lim_{j\to\infty} \left\{ \underset{p\times j}{\tilde{X}(k+1)}\ \underset{j\times p}{\tilde{X}^T(k+1)}\ \left[\frac{1}{j}\right] \right\} \equiv P(k+1) \tag{19}$$

Equation (15) can now be written as

$$\begin{aligned} P(k+1) &= \phi\ P(k)\ \phi^T + Q(k,k) \\ &+ \phi[I,\phi,\phi^2,\ldots\ \phi^{k-1}] \begin{bmatrix} Q(o,k) \\ Q(1,k) \\ \cdot \\ \cdot \\ \cdot \\ Q(k-1,k) \end{bmatrix}^T \\ &+ [Q(o,k),\ Q(1,k)\ \ldots\ Q(k-1,k)] \begin{bmatrix} I \\ \phi^T \\ \phi^{T2} \\ \cdot \\ \cdot \\ \cdot \\ \phi^{Tk-1} \end{bmatrix} \phi^T \end{aligned} \tag{20}$$

If the process noise vectors are serially uncorrelated, that is

$$Q(i,k) = Q(i,i)\ \delta(i,k) \tag{21}$$

then Eq 920) reduces to the more familiar discrete variance dynamics

$$P(k+1) = \phi\ P(k)\ \phi^T + Q(k) \tag{22}$$

Observe that the discrete case is much simpler to deal with than the continuous case where the dirac delta function under the integral sign is needed as in Eq (73) sec (38).

By Eq (5) of the previous section for the package

$$\begin{bmatrix} \tilde{X}(1) \\ \tilde{X}(2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \tilde{X}(k) \end{bmatrix} = \begin{bmatrix} \phi \\ \phi^2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \phi^k \end{bmatrix} \tilde{X}(o) + \begin{bmatrix} I & & & \\ \phi & I & & \\ \phi^2 & \phi & I & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \phi^{k-1}, & & & I \end{bmatrix} \begin{bmatrix} \tilde{U}(o) \\ \tilde{U}(1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \tilde{U}(k-1) \end{bmatrix} \qquad (23)$$

also by Eq (22) of the previous section

$$\begin{bmatrix} \tilde{X}(1) \\ \tilde{X}(2) \\ \cdot \\ \cdot \\ \cdot \\ \tilde{X}(k) \\ \tilde{X}(k+1) \end{bmatrix} = \begin{bmatrix} \phi^{-(k+1)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \phi^{-1} \\ I \end{bmatrix} \tilde{X}(k+1) + \begin{bmatrix} I & 0 & & \\ \phi & I & & \\ \phi^2 & \phi & I & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \phi^k, & \cdots & & I \end{bmatrix} \begin{bmatrix} \tilde{U}(o) \\ \tilde{U}(1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \tilde{U}(k) \end{bmatrix} \qquad (24)$$

Transpose Eq (23) and take the "outer product"

$$\begin{bmatrix} \tilde{X}(1) \\ \tilde{X}(2) \\ \cdot \\ \cdot \\ \cdot \\ \tilde{X}(k) \end{bmatrix} [\tilde{X}^T(1), \tilde{X}^T(2), \ldots \tilde{X}^T(k)]$$

$$= \begin{bmatrix} \phi \\ \phi^2 \\ \cdot \\ \cdot \\ \cdot \\ \phi^k \end{bmatrix} \tilde{X}(o)\, \tilde{X}^T(o)\, [\phi^T, \ldots \phi^{kT}] \qquad (25)$$

$$+ B \begin{bmatrix} \tilde{U}(o)\, \tilde{U}^T(o) & & \tilde{U}(o)\, \tilde{U}^T(k-1) \\ \tilde{U}(1)\, \tilde{U}^T(o) & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \tilde{U}(k-1)\, \tilde{U}(o) & \ldots & \tilde{U}(k-1)\, \tilde{U}^T(k-1) \end{bmatrix} B^T$$

since

$$\begin{bmatrix} \tilde{U}(o) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \tilde{U}(k-1) \end{bmatrix} \tilde{X}(o) = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \qquad (26)$$

at the variance - Covariance level

$$\begin{bmatrix} P(1,1) & P(1,2) \ . \ . \ . & P(1,k) \\ P(2,1) & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ P(k,1) & & P(k,k) \end{bmatrix}$$

$$= \begin{bmatrix} \phi\ P(o,o)\ \phi^{T} & \phi\ P(o,o)\ \phi^{T2} \ . \ . \ . & \phi\ P(o,o)\ \phi^{Tk} \\ \phi^{2}\ P(o,o)\ \phi^{T} & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \phi^{k}\ P(o,o)\ \phi^{T} & & \phi^{k}\ P(o,o)\ \phi^{kT} \end{bmatrix}$$

$$+ B \begin{bmatrix} Q(o,o) & Q(o,1) \ . \ . \ . & Q(o,k-1) \\ Q(1,o) & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ Q(k-1,0) & & Q(k-1,k-1) \end{bmatrix} B^{T} \qquad (27)$$

The last term of Eq (27) is quite messy; just consider the diagonal matrix case for Q(i,k) (no serial correlation), then the last term alone is

$$B \begin{bmatrix} Q(o,o) & 0 & & \\ 0 & Q(1,1) & & \\ & & \ddots & \\ & & & Q(k-1,k-1) \end{bmatrix} B^{T} \qquad (28)$$

$$
= \begin{bmatrix}
Q(o,o) & Q(o,o)\phi^T & Q(o,o)\phi^{2T} & \ldots & Q(o,o)\phi^{(k-1)^T} \\
\phi Q(o,o) & \phi Q(o,o)\phi^T + Q(1,1) & \ldots & \ldots & \ldots \\
\phi^2 Q(o,o) & . & . & . & . \\
. & & & & . \\
. & & & & . \\
. & & & & . \\
\phi^{(k-1)} Q(o,o) & . & . & . & \alpha_{kk}
\end{bmatrix}
$$

with the last row last column element looking like

$$
\alpha_{kk} = \phi^{k-1} Q(o,o)\phi^{(k-1)^T} + \phi^{(k-2)} Q(1,1)\phi^{(k-2)^T} +
$$

$$
+ \ldots + \phi\ Q(k-2),k-2)\phi^T + Q(k-1,k-1). \tag{29}
$$

Observe by Eq (28) that if the process noise is serially uncorrelated that the ϕ matrix provides serial correlation on the states. Also even if the process noise is zero at all time points k, the random initial condition variance is seen by Eq (27) to provide serial correlation on all P(i,k) terms- Note further by Eq (25) the rank of the matrix of P(i,k) is reduced when Q(i,k) are all zero to rank equal p.

SECTION 23
ADDITIVE NOISY MEASUREMENTS OF THE DISCRETE DYNAMICAL PROCESS

Many real-world measurement processes are approximately described as the known measurement function plus additive noise; that is

$$z(k)\rangle_j = \underset{m\times p}{H(k)}\, x(k)\rangle_j + V(k)\rangle_j \tag{1}$$

Clearly for many physical and engineering applications the measurements are non-linear functions of the states, and the m x p matrix H of Eq (1) is a matrix of partials. In the same context as the previous section assume that H is a constant matrix (this assumption reduces carrying many integer-tags and greatly simplifies the typing.

The sequence of trajectories of Eq (1) is

$$\underset{m\times j}{Z(k)} = HX(k) + V(k) \tag{2}$$

As before defining each measurement error as

$$z(k)\rangle_j - \mu_z(k)\rangle = \tilde{z}(k)\rangle_j \tag{3}$$

operate on Eq (2) by $\tilde{P}_{11}$ in j-space

$$\tilde{Z}(k) = H\,\tilde{X}(k) + \tilde{V}(k) \tag{4}$$

Transposing Eq (4)

$$\tilde{Z}(k)^T = \tilde{X}^T(k)\,H^T + \tilde{V}(k)^T \tag{5}$$

and forming the inner-Grammian

$$\tilde{Z}(k)\tilde{Z}^T(k) = H\,\tilde{X}(k)\,\tilde{X}^T(k)\,H^T$$

$$+ H\tilde{X}(k)\tilde{V}^T(k) + \tilde{V}(k)\,\tilde{X}^T(k)\,H^T$$

$$+ \tilde{V}(k)\,\tilde{V}(k)^T \tag{6}$$

If the state errors are independent of, that is serially uncorrelated with the measurement noise, then

$$\tilde{X}(k)\,\tilde{V}^T(k) = 0 \tag{7}$$

and Eq (22) for the uncorrelated case of the previous section in Eq (6) yields

$$\tilde{Z}(k)\,\tilde{Z}^T(k) = H\,P(k)\,H^T + R(k,k) \tag{8}$$

or

$$\tilde{Z}(k)\tilde{Z}^T(k) = H\phi\ P(k-1)\phi^T H^T + HQ(k)H^T + R(k) \tag{9}$$

When the process noise and the measurement noise are correlated, one obtains by Eq (17) of the previous section

$$\tilde{X}(k)\tilde{V}^T(k) = [I,\phi,\phi^2,\ldots\phi^{k-1}] \begin{bmatrix} \tilde{U}(o)\ \tilde{V}^T(k) \\ \tilde{U}(1)\ \tilde{V}^T(k) \\ \cdot \\ \cdot \\ \cdot \\ \tilde{U}(k-1)\ \tilde{V}^T(k) \end{bmatrix} \tag{10}$$

and Eq (6) becomes at the variance level

$$\begin{aligned} \Sigma_{\tilde{z}}(k) &= HP(k)H^T + R(k) \\ &+ H[I,\phi,\phi^2,\ldots\phi^{k-1}] \begin{bmatrix} \Sigma_{uv}(o,k) \\ \Sigma_{uv}(1,k) \\ \cdot \\ \cdot \\ \cdot \\ \Sigma_{uv}(k-1,k) \end{bmatrix} \\ &+ [\Sigma_{u,v}^T(o,k),\ldots \Sigma_{u,v}^T(k-1,k)] \begin{bmatrix} I \\ \phi^T \\ \cdot \\ \cdot \\ \cdot \\ \phi^{(k-1)^T} \end{bmatrix} H^T \end{aligned} \tag{11}$$

By Eq (4)

$$\begin{bmatrix} \tilde{Z}(1) \\ \tilde{Z}(2) \\ \cdot \\ \cdot \\ \cdot \\ \tilde{Z}(k) \end{bmatrix} = \begin{bmatrix} H & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & H \end{bmatrix} \begin{bmatrix} \tilde{X}(1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \tilde{X}(k) \end{bmatrix} + \begin{bmatrix} \tilde{V}(1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \tilde{V}(k) \end{bmatrix} \tag{12}$$

By Eq (22) sec (21)

$$\begin{bmatrix} \tilde{X}(1) \\ \tilde{X}(2) \\ \cdot \\ \cdot \\ \cdot \\ \tilde{X}(k) \end{bmatrix} = \begin{bmatrix} \phi^{-k} \\ \cdot \\ \cdot \\ \cdot \\ \phi^{-1} \\ I \end{bmatrix} \tilde{X}(k) + \begin{bmatrix} I & 0 \ldots 0 \\ \phi & I \quad 0 \ldots \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \phi^{k-1}, \ldots & \phi \quad I \end{bmatrix} \begin{bmatrix} \tilde{U}(o) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \tilde{U}(k-1) \end{bmatrix} \quad (13)$$

or

$$\begin{bmatrix} \tilde{Z}(1) \\ \cdot \\ \cdot \\ \tilde{Z}(k) \end{bmatrix} = (I \otimes H) \left\{ \begin{bmatrix} \phi^{-k} \\ \cdot \\ \cdot \\ \phi^{-1} \\ I \end{bmatrix} \tilde{X}(k) + B \begin{bmatrix} \tilde{U}(o) \\ \cdot \\ \cdot \\ \tilde{U}(k) \end{bmatrix} \right\}$$

By Eq (14) one can obtain the matrix of Cross Covariances

$$\begin{bmatrix} \tilde{Z}(1) \; \tilde{X}^T(k) \\ \tilde{Z}(2) \; \tilde{X}^T(k) \\ \cdot \\ \cdot \\ \tilde{Z}(k) \; \tilde{X}^T(k) \end{bmatrix} = \begin{bmatrix} \Sigma(1,k)_{\tilde{Z}\tilde{X}} \\ \cdot \\ \cdot \\ \cdot \\ \Sigma(k,k)_{\tilde{Z}\tilde{X}} \end{bmatrix}$$

24. DISCRETE KALMAN FILTER DERIVATION

The derivation of the estimation equations about a single-state variable process $x_j(k)$ satisfying

$$x_j(k+1) = \phi(k+1, k)\ x_j(k) + f(k) + u_j(k) \tag{1}$$

can be made without drawing too heavily on vector-space theory. The multi-variable case in the following section does require the reader to be, or to become, familiar with matrix-analysis methods.

Henceforth we shall assume that at each stage k, the observation on the process state-variable $x_j(k)$ is contaminated by noise, hence the states $x_j(k)$ cannot be known for any k including k = 1 because the observation has additive noise on it

$$z_j(k) = h(k)\ x_j(k) + v_j(k) \tag{2}$$

Note that the permutation on the class of initial conditions has been dropped. We assume that it is merely one value of the unknown, $x_j(1)$.

We shall develop the mathware for a computable model of a system of equations which will compute at each stage the best estimate (in some mathematical sense) of where the state variable $x_j(k)$ is when we know only $z_j(k)$, the noisy instrument output.

The state variable x(k) of the process driven by the random unknown variable $u_j(k)$ would be known at each stage k if the sensor were noise-free, that is, if $v_j(k)$ were identically zero. With such an ideal instrument (or high precision device) there would be no doubt as to the trajectory traversed by $x_j(k)$. In fact we could even compute the values of the random force

$$u_j(1),\ u_j(2)\ .\ .\ .\ u_j(k).$$

However, when the measurement $z_j(k)$ is uncertain due to $v_j(k)$, we cannot truly know the states $x_j(k)$ since we do not know the instrument noise $v_j(k)$, hence cannot compute the process noise $u_j(k)$.

The next question is how to obtain an estimate at each stage k of the unknown state variable $x_j(k)$. The estimator will use the known dynamics $\phi(k+1, k)$ of the process, the known instrument function h(k) and other things as needed in the development.

The following mathematical developments are stage-wise tutorial, that is at stage k = 1, stage k = 2, and then finally at any arbitrary stage k, the procedures are developed for the scalar case and then for the vector case.

Those readers already familiar with the recursive relations of Kalman may proceed to the general scalar case at stage k in section II or the general vector case at stage k in section III.

Scalar Case

Stage k = 1.

At stage k = 1, the process is in some unknown initial state $x_j(1)$. The noisy instrument output $z_j(1)$ is available if we want it. Since we want to obtain an error signal based on the difference between the observation and some guess about what the instrument should measure then suppose we guess or estimate what the initial state variable is (based on no observations) and call this $\hat{x}_j(1, 0)$.

Since we know the instrument function h(k), then we can estimate that the initial instrument output should be

$$\hat{z}_j(1, 0) = h(1)\, \hat{x}_j(1, 0). \tag{3}$$

We can now compute an error signal.

$$\tilde{z}_j(1, 0) = z_{jn}(1) - \hat{z}_j(1, 0) \tag{4}$$

which is the difference between the first instrument reading and our estimate of what it should read.

We have now estimated what the initial process variable and instrument variable should be, have computed an error equation (4) and may now correct our estimate of the initial-state with no observation and obtain an estimate of the initial value of the state variable after having used or received the first measurement, or

$$\hat{x}_j(1, 1) = \hat{x}(1, 0) + w(1)\tilde{z}_j(1, 0). \tag{5}$$

We now seek a w(1) among an infinity of different "corrector weights" or "feed back gains" at stage 1. We want the w(1) to be used at stage one everytime we recycle the process (or experiment) regardless of what the random vectors u and v are. w(1) should be a function of some statistical

measures on u and v, however

The first noisy observation is

$$z_j(1) = h(1)\, x_j(1) + v_j(1) \tag{6}$$

Using (3) and (6) in (4)

$$\tilde{z}_j(1, 0) = h(1)[x_j(1) - \hat{x}_j(1, 0)] + v_j(1). \tag{7}$$

Using (7) in (5)

$$\hat{x}_j(1, 1) = \hat{x}_j(1, 0) + w(1)h(1)[x_j(1) - \hat{x}_j(1, 0)] + w(1)v_j(1) \tag{8}$$

$$= [1 - w(1)h(1)]\, \hat{x}_j(1, 0) + w(1)h(1)x_j(1) + w(1)\, v_j(1).$$

We now define <u>two new errors</u> as differences of the previous variables.

$$x_j(1) - \hat{x}_j(1, 0) = \tilde{x}_j(1, 0) \tag{9}$$

and

$$x_j(1) - \hat{x}_j(1, 1) = \tilde{x}_j(1, 1). \tag{10}$$

The variable $x_j(1)$ is the actual (but unknown) process state variable that exists on trajectory j. The variable $\hat{x}_j(1,1)$ is the estimate of $x_j(1)$, having used the first noisy observation.

Hence $\tilde{x}_j(1, 1)$ is the error in our estimate of the state at stage 1 based on one measurement.

From equation (3) we see that if our initial guess $\hat{x}_j(1, 0)$ is wrong, then $\hat{z}_j(1, 0)$ will also be wrong and after our observation $z(1)$ comes in and our computed error $\tilde{z}_j(1, 0)$ of equation (4) is large, then by equation (5) we will have a correction term which is the product $(w(1)\tilde{z}_j(1, 1))$ of a large term times the weight value $w(1)$.

Continuing with the derivation of the weights, we obtain the error term by equation (8) in equation (10) as

$$\tilde{x}_j(1, 1) = x_j(1) - \hat{x}_j(1, 1) = x_j(1) - w(1)h(1)x_j(1) \tag{11}$$

$$- [(1 - w(1)h(1)]\, \hat{x}_j(1, 0) - w(1)v_j(1).$$

or

$$\tilde{x}_j(1, 1) = [1 - w(1)h(1)][x_j(1) - \hat{x}_j(1, 0)] - w(1)v_j(1). \tag{12}$$

Using equation (9) in equation (12)

$$\tilde{x}_j(1,1) = [1 - w(1)h(1)]\,\tilde{x}_j(1, 0) - w(1)\,v_j(1). \tag{13}$$

If we square the error of equation (13)

$$\tilde{x}_j^2(1, 1) = [1-w(1)h(1)]^2\tilde{x}_j^2(1, 0) - 2w(1)v_j(1)[1-w(1)h(1)]\tilde{x}_j(1, 0) + w^2(1)v_j^2(1). \tag{14}$$

If we take the partial derivative of equation (14) with respect to w(1) and equate to zero we obtain

$$\frac{\partial \tilde{x}_j^2(1, 1)}{\partial w(1)} = 2[1 - w(1)h(1)]\,(-h(1))\,\tilde{x}_j^2(1, 0) - 2v_j(1)[1 - w(1)h(1)]\tilde{x}_j(1, 0) - 2w(1)\,v_j(1)(-h(1))\,\tilde{x}_j(1,0) + 2w(1)\,v_j^2(1) = 0 \tag{15}$$

Taking the expected value over all experiments in which j is varying we obtain

$$E\left\{\frac{\partial \tilde{x}_j^2(1,1)}{\partial w(1)}\right\} = -[1 - w(1)h(1)]\,E\left\{\tilde{x}_j^2(1,0)\right\}h(1) + w(1)\,E\left\{v_j^2(1)\right\} = 0 \tag{16}$$

Define the variances

$$E\left\{\tilde{x}_j^2(1, 0)\right\} = \sigma_{\tilde{x}\tilde{x}}(1, 0) = p(1, 0) \tag{17}$$

$$E\left\{v_j^2(1)\right\} = \sigma_{vv}(1) \tag{18}$$

We follow a number of current Kalman papers and papers about Kalman Estimation and designate the variance in state as p(1, 0).

The cross term of equation (15) under the expectation operator is zero when the following conditions hold,

$$E\left\{[1 - w(1)h(1)]\,v_j(2)\,\hat{\tilde{x}}_j(1, 0)\right\} = [1 - w(1)h(1)]\,E\left\{v_j(1)\,\hat{\tilde{x}}_j(1, 0)\right\} \tag{19}$$

We now assume that our initial guess of $\hat{x}(1,0)$ is independent of $v_j(1)$ as j ranges over all allowable values, that is

$$E\left\{v_j(1)\,\hat{\tilde{x}}_j(1, 0)\right\} = 0. \tag{20}$$

Under these assumptions equation (16) and (18) become

$$0 = -[1 - w(1)h(1)]h(1)\ p(1, 0) + w(1)\sigma_{vv}(1) \tag{21}$$

or

$$-h(1)p(1, 0) + w(1)h^2(1)\ p(1, 0) + w(1)\sigma_{vv}(1) = 0 \tag{22}$$

or

$$w(1)[h^2(1)\ p(1, 0) + \sigma_{vv}(1)] = h(1)\ p(1, 0) \tag{23}$$

$$\boxed{w(1) = h(1)\ p(1, 0)[h^2(1)\ p(1, 0) + \sigma_{vv}(1)]^{-1}} \tag{24}$$

The value of p(1, 0) which by equation (17) is

$$p(1, 0) = E\left\{[x_j(1) - \hat{x}_j(1, 0)]^2\right\} = \text{guess} \tag{25}$$

Initially

can be obtained by guess or by a "learning process", or by a-priori knowledge about the system.

The variance of the estimate of state at stage 1 based on the observation at stage one can also be obtained by equation (14) as

$$E\left\{\tilde{x}_j^2\ (1, 1)\right\} = [1 - w(1)h(1)]^2\ E\left\{\tilde{x}_j^2(1, 0)\right\} + w^2(1)\ E\left\{v_j^2(1)\right\} \tag{26}$$

when the cross terms ~~and~~ are zero, that is

$$E\left\{v_j(1)\ \hat{x}_j(1, 0)\right\}\ 2w(1)[1 - w(1)h(1)] = 0. \tag{27}$$

The variance terms of equation (26) will be denoted as

$$\sigma_{\tilde{x}\tilde{x}}(1, 1) = E\left\{\tilde{x}_j^2\ (1, 1)\right\} \equiv p\ (1, 1) \tag{28}$$

$$E\left\{v_j^2(1)\right\} = \sigma_{vv}(1) \tag{29}$$

The p(1, 1) designation is in keeping with the now near classical notation.

Expanding the terms of equation (26)

$$\begin{aligned} p(1, 1) &= [1 - w(1)h(1)]^2\ p(1, 0) + w^2(1)\sigma_{vv}(1) \\ &= [1 - 2w(1)h(1) + w^2(1)h^2(1)]\ p(1, 0) + w^2(1)\ \sigma_{vv}(1) \\ &= p(1, 0) - 2w(1)h(1)\ p(1, 0) \\ &\quad + w^2(1)[h^2(1)\ p(1, 0) + \sigma_{vv}(1)]. \end{aligned} \tag{30}$$

Consider the last term in equation (30) and use equation (24) for $w^2(1)$, then

$$w^2(1)[h^2(1)\ p(1,0) + \sigma_{vv}(1)] \tag{31}$$
$$= h^2(1)\ p^2(1,0)[h^2(1)\ p(1,0) + \sigma_{vv}(1)]^{-2}\ [h^2(1)\ p(1,0) + \sigma_{vv}(1)]$$
$$= h^2(1)\ p^2(1,0)[h^2(1)\ p(1,0) + \sigma_{vv}(1)]^{-1}$$
$$= h(1)\ p(1,0)\left\{h(1)\ p(1,0)\ [h^2(1)\ p(1,0) + \sigma_{vv}(1)]^{-1}\right\}$$

Replacing the bracket term of equation (31) by w(1) in equation (24) we obtain

$$w^2(1)[h^2(1)\ p(1,0) + \sigma_{vv}(1)] = h(1)\ p(1,0)\ w(1) \tag{32}$$

Using (32) in equation (30) we obtain

$$p(1,1) = p(1,0) - 2w(1)\ h(1)\ p(1,0) + h(1)\ p(1,0)\ w(1) \tag{33}$$
$$= p(1,0) - w(1)\ h(1)\ p(1,0).$$

$$\boxed{p(1,1) = p(1,0)[1 - w(1)h(1)]}$$

also using equation (24) in equation (33) we can write p(1, 1) as

$$\boxed{p(1,1) = p(1,0) - p(1,0)h(1)[h(1)p(1,0)h(1) + \sigma_{vv}(1)]^{-1}p(1,0)h(1)} \tag{34}$$

Equation (34) is the scalar variance (one dimensional uncertainty ellipsoid) of the estimate of the state using all of the current observations.

We can now predict what the next state variable value should be by propogating forward with the known dynamics as

$$\hat{x}(2,1) = \Phi(2,1)\ \hat{x}(1,1) + f(1). \tag{35}$$

The prediction of the next observation is

$$\hat{z}(2,1) = h(2)\ \hat{x}(2,1). \tag{36}$$

The elliposoid of uncertainty of the predicted state is

$$p(2,1) = E\left\{\tilde{x}^2(2,1)\right\} \tag{37}$$

where

$$\tilde{x}(2,1) = x(2) - \hat{x}(2,1) \tag{38}$$

Using equation (1) and equation (35) in equation (38)

$$\tilde{x}(2,1) = \phi(2, 1)\, \tilde{x}(1, 1) + u(1) \tag{39}$$

$$\tilde{x}^2(2, 1) = \phi(2, 1)\, \tilde{x}^2(1, 1)\phi(2, 1) + 2\phi(2, 1)\, \tilde{x}(1, 1)\, u(1) + u^2(1) \tag{40}$$

The expected value over all experiments of equation (40) is

$$E\left\{\tilde{x}^2(2, 1)\right\} = p(2, 1) = \phi(2, 1)\, p(1, 1)\phi(2, 1) + \sigma_{uu}(1) \tag{41}$$

where

$$\sigma_{uu}(1) = E_n\left\{u_n^2(1)\right\} \tag{42}$$

Cross term statistical independence assumptions are

$$E\left\{\tilde{x}(1, 1)\, u(1)\right\} = 0. \tag{43}$$

etc.

In conclusion, at stage one, we do:

Given

$$\sigma_{vv}(1)$$

guess or estimate

$$p(1, 0) \text{ and } \hat{x}(1, 0)$$

compute

$$\hat{z}(1, 0) = h(1)\, \hat{x}(1, 0) \tag{44}$$

Measure and compute <u>error</u>

$$\tilde{z}(1, 0) = z(1) - \hat{z}(1, 0) \tag{45}$$

Compute weight w(1) equation (24)

$$w(1) = h(1)\, p(1, 0)[h(1)p(1, 0)h(1) + \sigma_{vv}(1)]^{-1}. \tag{46}$$

Update or correct state-estimate

$$\hat{x}(1, 1) = \hat{x}(1, 0) + w(1)\, \tilde{z}(1, 0) \tag{47}$$

Compute ellipsoid of uncertainty of state estimation (48)

$$p(1, 1) = p(1,0)[1 - w(1)h(1)]$$

(4)

Predict or propogate one stage into future with known dynamics the following three variables:

Predict next stage.

$$\hat{x}(2, 1) = \phi(2, 1)\,\hat{x}(1, 1) + f(1) \quad (49)$$

Predict next observation.

$$\hat{z}(2, 1) = h(2)\,\hat{x}(2, 1) \quad (50)$$

Predict ellipsoid of uncertainty in state

$$p(2, 1) = \phi(2, 1)\,p(1, 1)\,\phi(2, 1) + \sigma_{uu}(1). \quad (51)$$

Wait for stage k = 2 and second observation to come in.

Stage k = 2.

The second observation is taken at stage 2 and is

$$z_j(2) = h(2)\,x_j(2) + v_j(2) \quad (52)$$

where, as before, $z_j(2)$ is known, but $x_j(2)$ and $v_j(2)$ are unknown.

We compute the error signal

$$\tilde{z}_j(2, 1) = z_j(2) - \hat{z}_j(2, 1), \quad (53)$$

The sub-script j can now be dropped for simplicity of notation, where it is henceforth understood that $\hat{x}(2, 1)$ for example means the best estimate of the true jth trajectory based on the sequence of instrument noises $\langle v_j$.

Using equation (50) and equation (52) in equation (53)

$$\tilde{z}(2, 1) = h(2)[x(2) - \hat{x}(2, 1)] + v(2) \quad (54)$$

we now have computed the error in the observation, hence we can correct our estimate of what the state variable should be based on the use of the second observation, that is

$$\hat{x}(2, 2) = \hat{x}(2, 1) + w(2)\,\tilde{z}(2, 1). \quad (55)$$

As before, we now seek the "feed-back" or correction weight w(2) at stage 2. Before deriving the w(2) expression, use (54) in (55)

$$\hat{x}(2, 2) = \hat{x}(2, 1) + w(2)h(2)[x(2) - \hat{x}(2, 1)] + w(2)\,v(2) \quad (56)$$

$$\hat{x}(2, 2) = [1 - w(2)h(2)]\,\hat{x}(2, 1) + w(2)\,h(2)\,x(2) + w(2)\,v(2). \quad (57)$$

From the error in state estimation at stage 2 using the second observation as before

$$\tilde{x}(2,2) = x(2) - \hat{x}(2, 2) = [1-w(2)\ h(2)]\tilde{x}(2,1) + w(2)\ v(2) \tag{58}$$

Multiply equation (58) by itself to obtain the square of the error term as

$$\tilde{x}^2(2, 2) = [1 - w(2)\ h(2)]^2\ \tilde{x}^2(2, 1) \tag{59}$$

$$+ 2w(2)\ v(2)[1 - w(2)\ h(2)]\ \tilde{x}(2, 1)$$

$$+ w^2(2)\ v^2(2).$$

Observe that the error and the square of the error in equation (59) is a function of w(2).

In order to select a w(2) which will minimize the square of the error we take the partial derivative of equation (59) with respect to w(2) and equate this "gradient" term to zero, hence

$$\frac{\partial \tilde{x}^2(2, 2)}{\partial w(2)} = -2h(2)[1 - w(2)\ h(2)]\ \tilde{x}^2(2, 1) \tag{60}$$

$$+ 2v(2)[1 - w(2)h(2)]\ \tilde{x}(2, 1)$$

$$+ 2w(2)\ v(2)\ \left(- h(2)\right)\ \tilde{x}(2, 1)$$

$$+ 2w(2)\ v^2(2) = 0$$

Since we want w(2) to be the same regardless of how many times we repeat the experiment; or, from a "single-test" stand-point, w(2) should be selected to hold regardless of the unknown sequence driving the two variables at this stage $u_j(2)$ and $v_j(2)$. Consequently we take the expected value over all admissible vectors $\langle j)u$ and $\langle n)v$ and obtain

$$E\left\{\frac{\partial \tilde{x}^2(2,2)}{\partial w(2)}\right\} = [1 - w(2)\ h(2)]\ \left(-h(2)\right) p(2, 1) + w(2)\sigma_{vv}(2) = 0. \tag{61}$$

The assumption of statistical independence of the variables $\tilde{x}(2, 1)$ and v(2) is again assumed

$$E\ \{\tilde{x}(2, 1)\ v(2)\} = 0 \tag{62}$$

also the assumption about $\langle j)v$

$$E_n \left\{ \begin{pmatrix} v_j(1) \\ v_j(2) \end{pmatrix} [v_j(1),\ v_j(2)] \right\} \tag{63}$$

$$= E_n \left\{ \begin{bmatrix} v_j(1)\ v_j(1) & v_n(1)\ v_n(2) \\ v_j(2)\ v_j(1) & v_n(2)\ v_n(2) \end{bmatrix} \right\} = \begin{bmatrix} \sigma_{vv}(1) & 0 \\ 0 & \sigma_{vv}(2) \end{bmatrix}$$

The assumptions of equation (63) can be relaxed but then one needs a deeper knowledge of multi-linear algebras, matrix-packaging and partitioning, matrix psuedo-inverse, etc. Hence such "highly-colored" or correlated noise cases will not be discussed in this paper. The majority of published papers on Kalman theory make the implied Gaussian assumptions implied by equation (63) for arbitrary large k. Computer storage problems for correlated noise for large k are also a problem.

Solving equation (6[illegible]) for w(2)

$$w(2)[h(2)\ p(2, 1)\ h(2) + \sigma_{vv}(2)] = h(2)\ p(2, 1) \tag{64}$$

or

$$\boxed{w(2) = h(2)\ p(2,1)\ [h(2)\ p(2, 1)\ h(2) + \sigma_{vv}(2)]^{-1}} \tag{65}$$

The second weight can be computed since $h(2)$, $\sigma_{vv}(2)$ are assumed known and $p(2, 1)$ was computed at stage 1.

We can now derive the expression for $p(2, 2)$, using equation (65) and taking the expected value over all experiments

$$p(2, 2) = E\ \{\tilde{x}^2(2, 2)\} = [1 - w(2)h(2)]^2\ p(2, 1) + w^2(2)\sigma_{vv}(2), \tag{66}$$

with assumptions

$$E\ \{v(2)\ \tilde{x}(2, 1)\} = 0. \tag{67}$$

Expanding equation (66)

$$p(2, 2) = [1 - 2w(2)\ h(2) + w^2(2)\ h^2(2)]\ p(2, 1) + w^2(2)\sigma_{vv}(2) \tag{68}$$

$$p(2, 2) = p(2, 1) - 2w(2)\ h(2)\ p(2, 1) + w^2(2)\ [h(2)\ p(2, 1)\ h(2) + \sigma_{vv}(2)] \tag{69}$$

Consider the last term of equation (69) using equation (65) for w(2)

$$w^2(2)\ [h(2)\ p(2, 1)\ h(2) + \sigma_{vv}(2)] \tag{70}$$

$$= h^2(2)\ p^2(2, 1)[h^2(2)\ p(2, 1) + \sigma_{vv}(2)]^{-2}[h^2(2)p(2,1) + \sigma_{vv}(2)]$$

$$= h^2(2)\ p^2(2, 1)\ [h^2(2)\ p(2, 1) + \sigma_{vv}(2)]^{-1}$$

and by equation (65) for w(2) we obtain from EQ (70)

$$w^2(2)[h(2)\ p(2,1)\ h(2) + \sigma_{vv}(2)] = h(2)\ p(2,1)\ w(2) \tag{71}$$

Using equation (71) in equation (69)

$$p(2,2) = p(2,1) - 2w(2)\ h(2)\ p(2,1) + h(2)\ p(2,1)\ w(2) \tag{72}$$

or

$$p(2,2) = p(2,1) - p(2,1)\ h(2)\ w(2) \tag{73}$$

$$p(2,2) = p(2,1)\ [1 - w(2)\ h(2)] \tag{74}$$

Equation (73) can also be written as by equation (65) in equation (73)

$$p(2,2) = p(2,1) - p(2,1)\, h(2)\, [h^2(2)p(2,1) + \sigma_{vv}(2)]^{-1}\ h(2)p(2,1) \tag{75}$$

k=3 We can now predict at stage k = 3 the state

$$\hat{x}(3,2) = \phi(3,2)\ \hat{x}(2,2) + f(2) \tag{76}$$

and the observation

$$\hat{z}(3,2) = h(3)\ \hat{x}(3,2) \tag{77}$$

and the state-variance term

$$p(3,2) = E\ \{\tilde{x}^2(3,2)\}. \tag{78}$$

By equation (1) and equation (76)

$$x(3) = \phi(3,2)\ x(2) + f(2) + u(2) \tag{79}$$

$$\hat{x}(3,2) = \phi(3,2)\ \hat{x}(2,2) + f(2) \tag{80}$$

and subtracting

$$x(3) - \hat{x}(3,2) = \tilde{x}(3,2) = \phi(3,2)\ \tilde{x}(2,2) + u(2) \tag{81}$$

Squaring equation (81)

$$\begin{aligned}\tilde{x}^2(3,2) &= \phi(3,2)\ \tilde{x}^2(2,2)\phi(3,2) \\ &+ 2\phi(3,2)\ \tilde{x}(2,2)\ u(2) \\ &+ u^2(2).\end{aligned} \tag{82}$$

Taking the expected value

$$E\{\tilde{x}^2(3, 2)\} = \phi(3, 2)\, E\{\tilde{x}^2(2, 2)\}\, \phi(3, 2) + E\{u^2(2)\} \tag{83}$$

or

$$p(3, 2) = \phi(3, 2)\, p(2,2)\, \phi(3, 2) + \sigma_{uu}(2) \tag{84}$$

Summarizing the steps at stage k = 2, then we:

measure and compute error

$$\tilde{z}(2, 1) = z(2) - \hat{z}(2, 1) \tag{85}$$

compute weight w(2) by equation (65)

$$w(2) = h(2)\, p(2, 1)[h(2)\, p(2, 1)\, h(2) + \sigma_{vv}(2)]^{-1} \tag{86}$$

compute corrected estimate of state

$$\hat{x}(2, 2) = \hat{x}(2, 1) + w(2)\, \tilde{z}(2, 1) \tag{87}$$

compute variance of state equation (74)

$$p(2, 2) = p(2, 1) - w(2)\, h(2)\, p(2, 1) \tag{88}$$

Predict (update) state via dynamics equation (80)

$$\hat{x}(3, 2) = \phi(3, 2)\, \hat{x}(2, 2) + f(2) \tag{89}$$

Predict observation at next stage

$$\hat{z}(3, 2) = h(3)\, \hat{x}(3, 2) \tag{90}$$

Predict next stage state-variance

$$p(3, 2) = \phi(3, 2)\, p(2, 2)\, \phi(3, 2) + \sigma_{uu}(2) \tag{91}$$

wait for next stage or third observation to arrive.

Stage k.

The derivations of the equations will not be repeated for stage k, the relations will be based on the mathematical process of reasoning by analogy. The treatment of the multi-variable or vector case will derive the relations at stage k, but will not develop the stage-wise logic at k = 1 and k = 2.

We have available from previous stage predictions

$\hat{x}(k, k-1)$
$\hat{z}(k, k-1)$
$p(k, k-1)$

and stored $\sigma_{vv}(k)$, $\sigma_{uu}(k)$

Measure and compute error

$$\tilde{z}(k, k-1) = z(k) - \hat{z}(k, k-1) \tag{92}$$

Compute w(k)

$$w(k) = h(k)\ p(k, k-1)[h^2(k)\ \ p(k, k-1) + \sigma_{vv}(k)]^{-1} \tag{93}$$

Compute corrected state estimate

$$\hat{x}(k, k) = \hat{x}(k, k-1) + w(k)\ \tilde{z}(k, k-1) \tag{94}$$

Compute variance

$$p(k, k) = p(k, k-1)\ [1 - w(k)\ h(k)] \tag{95}$$

Predict next state

$$\hat{x}(k + 1, k) = \phi(k + 1, k)\ \hat{x}(k, k) + f(k) \tag{96}$$

Predict next observation

$$\hat{z}(k + 1, k) = h(k+1)\ \hat{x}(k+1, k) \tag{97}$$

Predict variance of state

$$p(k+1, k) = \phi(k+1, k)\ p(k, k)\ \phi(k+1, k) + \sigma_{uu}(k). \tag{98}$$

We define the noise variances in the notation of the many Kalman oriented papers, that is

$$\sigma_{uu}(k) = q(k) \tag{99}$$

$$\sigma_{vv}(k) = r(k) \tag{100}$$

The three familiar equations can be written as

$$\begin{aligned} \hat{x}(k+1, k+1) = {} & \phi(k+1, k)\ \hat{x}(k, k) \\ & + p(k+1, k)\ h(k+1)[h(k+1)\ p(k+1, k)\ h(k+1) + r(k)]^{-1} \\ & [z(k+1) - h(k+1)\ \phi(k+1, k)\ \hat{x}(k, k)] \end{aligned} \tag{101}$$

$$p(k+1, k) = \phi(k+1, k)\ p(k, k)\ \phi(k+1, k) + q(k) \tag{102}$$

$$p(k+1, k+1) = p(k+1, k) - p(k+1, k)\ h(k+1) \tag{103}$$
$$[h(k+1)\ p(k+1, k)\ h(k+1) + r(k)]^{-1} h(k+1)\ p(k+1, k)$$

We shall now define in words the meanings at stage k of the variables and rewrite the equations using the distinguishing j

$$x_j(k+1) = \phi(k+1, k)\ x_j(k) + f(k) + u_j(k) \tag{104}$$

$$z_{jj}(k) = h(k)\ x_j(k) + v_j(k) \tag{105}$$

$$\hat{x}_{jj}(k+1, k+1) = \phi(k+1, k)\ \hat{x}_{jj}(k, k)$$
$$+ p(k+1,k)h(k)[h^2(k+1)\ p(k+1) + r(k)]^{-1}$$
$$[z_j(k+1) - h(k+1)\phi(k+1,k)\ \hat{x}_j(k, k)] \tag{106}$$

$$p(k+1, k) = \phi(k+1, k)\ p(k, k)\phi(k+1, k) \tag{107}$$

$$p(k+1, k+1) = p(k+1, k) - p(k+1, k)\ h(k+1) \tag{108}$$
$$[h^2(k+1)\ p(k+1, k) + r(k)]^{-1} h(k+1)\ p(k+1, k)$$

$x(k) = x_j(k)$ is the true (unknown) value of the process state at stage k as a result of the unknowns $u_j(1), \ldots v_j\ k)$ forcing the system.

$z_j(k) = z_j(k)$ is measurement of the true noise process $x_j(k)$ with additive unknown measurement noise $v(k)$.

$\hat{x}_j(k, k) = \hat{x}(k, k)$ is the best estimate of the state at stage k of the jth trajectory based on past observations up to stage k, that is recursively we have used noisy

$z_j(1), z_j(2) \ldots z_j(k)$

made noisy by $v(1), \ldots u(k)$.

$\hat{x}_j(k+1, k) = \hat{x}(k+1, k)$ is the best estimate of the state of the jth trajectory at stage k+1, based on observations only up to k. Also interpreted as the prediction of the state at next stage k+1, based on current stage k and past measurements.

VECTOR ESTIMATION EQUATIONS

This section derives the Kalman Estimation Equations for the multivariable case using matrix analysis methods. The derivation techniques are the same as used in the previous scalar case. The essential difference lies in the minimization methods. The variance of the estimate in state for the scalar case is a scalar valued function of a scalar argument $w(k)$. For the vector case, the trace of the variance matrix of the estimate of state is likewise a scalar-valued function, but a function of a matrix of p rows and m columns $W(k)$. The minimization of a scalar-valued function with respect to this matrix.

One can arrive at the equations via strictly algebraic concepts of orthogonal projection matrices etc., in which one does not have to enter into discussions of partial derivaties, continuity of continous variables and gradients. Since the majority of expected readers are assummed to be more familiar with the least-squares criterion via gradients, this report will stick strictly with this method.

The general linearized vector equations are

$$x(k+1)\rangle_{p} = \Phi(k+1,\ k)\ x(k)\rangle_{p} + \underset{pxg}{B(k)} f(k)\rangle_{g} + \underset{pxq}{N(k)} u(k)\rangle_{q} \qquad (1)$$

$$z(k)\rangle_{m} = \underset{mxp}{H(k)}\ x(k)\rangle_{p} + v(k)\rangle_{m} \qquad (2)$$

The deterministic k-varying vector $f(k)\rangle_{g}$ is of dimension, g less than or equal to p, and gets distributed or cross-coupled into all p state variables $x(k+1)\rangle_{p}$ via the functional relations of $\underset{pxg}{B(k)}$.

The same statements apply to the noise input vector $u(k)\rangle_{q}$.

The reader should keep in mind the families of trajectories accurately described by the j and n indices, that is

$$x(k+1)\rangle_{p,j} = \Phi(k+1,\ k)\ x(k)\rangle_{p,j} + B(k) f(k)\rangle + N(k) u(k)\rangle_{j} \qquad (3)$$

$$z_{j}(k)\rangle = H(k)\ x(k)\rangle_{j} + v(k)\rangle \qquad (4)$$

As before, the accurate descriptions designated by j and n will be dropped for simplicity of representation.

The equations are developed as a "recursive process" or an "on-line" processor; that is, as the observations "role in" the mechanized computer-estimator utilizes the data, and discards it or stores it on tape or what have you. All past data is sequentially accumulated in the "memory of the math-ware" via up dated estimates and variance matrices etc.

Stage k.

Suppose we are at stage k and have computed during stage k-1, the following

$$\hat{x}(k-1, k-1)\rangle$$

$$P(k-1, k-1)$$

and predicted via dynamics

$$\hat{x}(k, k-1)\rangle = \Phi(k, k-1)\ \hat{x}(k-1, k-1)\rangle + B(k-1)\ f(k-1)\rangle \tag{5}$$

$$\hat{z}(k, k-1)\rangle = H(k)\ \hat{x}(k, k-1)\rangle \tag{6}$$

$$\hat{z}(k, k-1)\rangle = H(k)\ \phi(k, k-1)\ \hat{x}(k-1, k-1)\rangle + H(k)\ B(k-1)\ f(k-1)\rangle \tag{7}$$

$$P(k, k-1) = \Phi(k, k-1)\ P(k-1, k-1)\ \Phi^{T}(k, k-1) + N(k-1)\ Q(k-1)\ N^{T}(k-1). \tag{8}$$

We now receive the kth observation

$$z(k)\rangle = H(k)\ x(k)\rangle + v(k)\rangle \tag{9}$$

where $z(k)\rangle$ is known but $x(k)\rangle$ and $v(k)\rangle$ are unknown. We can compute an observation error vector by equation (7) and (9) as

$$\tilde{z}(k, k-1)\rangle = H(k)\ [x(k)\rangle - \hat{x}(k, k-1)\rangle] + v(k)\rangle \tag{10}$$

we now can correct the estimate in the state vector based on the observable and computable estimate in the observation vector as

$$\hat{x}(k, k)(p)\rangle = \hat{x}(k, k-1)(p)\rangle + W(k)\ \tilde{z}(k, k-1)(m)\rangle \tag{11}$$

where the weighting matrix W(k) at stage k has p rows and m columns.

We next seek a procedure for selecting at each stage a pxm weighting matrix W(k).

Using equation (10) in equation (11)

$$\hat{x}(k, k)\rangle = \hat{x}(k, k-1)\rangle + W(k)\ \{H(k)[x(k)\rangle - \hat{x}(k, k-1)\rangle] + v(k)\rangle\} \tag{12}$$

$$= [\underset{pxp}{I} - \underset{pxm}{W(k)}\ \underset{mxp}{H(k)}]\ \hat{x}(k, k-1)\rangle + W(k)H(k)x(k)\rangle + W(k)v(k)\rangle$$

If we now define the "unknown" error vectors

$$x(k)\rangle - \hat{x}(k, k)\rangle = \tilde{x}(k, k)\rangle \tag{13}$$

and

$$x(k)\rangle - \hat{x}(k, k-1)\rangle = \tilde{x}(k, k-1)\rangle \tag{14}$$

then theoretically equation (12) in (13) yields

$$\tilde{x}(k, k)\rangle = [I - W(k)H(k)]x(k)\rangle - [I - W(k)H(k)]\,\hat{x}(k, k-1)\rangle - W(k)v(k)\rangle \tag{15}$$

$$\tilde{x}(k, k)\rangle = [I - W(k)H(k)]\tilde{x}(k, k-1)\rangle - W(k)v(k)\rangle \tag{16}$$

Transposing (16) we obtain

$$\langle\tilde{x}(k, k) = \langle\tilde{x}(k, k-1)\,[I - H^T(k)\,W^T(k)] - \langle v(k)W^T(k) \tag{17}$$

The dyadic product of equation (16) and (17) yields

$$\begin{aligned}\tilde{x}(k, k)\rangle\langle\tilde{x}(k, k) &= \{[I - W(k)H(k)]\,\tilde{x}(k, k-1)\rangle - W(k)v(k)\rangle\}\\ &\quad\{\langle\tilde{x}(k, k-1)[I - H^T(k)\,W^T(k)] - \langle v(k)\,W^T(k)\}\\ &= [I - W(k)H(k)]\tilde{x}(k, k-1)\rangle\langle\tilde{x}(k, k-1)[I - H^T(k)W^T(k)]\\ &\quad - [I - W(k)H(k)]\tilde{x}(k, k-1)\rangle\langle v(k)W^T(k)\\ &\quad - W(k)v(k)\rangle\langle\tilde{x}(k, k-1)[I - H^T(k)W^T(k)]\\ &\quad + W(k)v(k)\rangle\langle v(k)W^T(k).\end{aligned} \tag{18}$$

The square of the magnitude of the error vector $\tilde{x}(k, k)\rangle$ is given as the inner-product of equation (16) and equation (17) or, as the trace of the outer-product of equation (18) as

$$\begin{aligned}\langle\tilde{x}(k, k)\,\tilde{x}(k, k)\rangle &= \langle\tilde{x}(k, k-1)\;\;\tilde{x}(k, k-1)\rangle - 2\langle\tilde{x}(k, k-1)W(k)H\tilde{x}(k,k-1)\rangle\\ &\quad + \langle\tilde{x}(k,k-1)H^T(k)W^TWH(k)\;\tilde{x}(k, k-1)\rangle\\ &\quad - 2\langle\tilde{x}(k, k-1)W(k)\,v(k)\rangle\\ &\quad + 2\langle\tilde{x}(k, k-1)H^T(k)\;W^T(k)\;W(k)\;v(k)\rangle\\ &\quad + \langle v(k)\;W^T(k)\;W(k)\;v(k)\rangle\end{aligned} \tag{19}$$

Equation (19) is a scalar valued function of a matrix argument W(k) of size pxm. We shall take the partial derivative of the scalar with respect to the matrix W(k),

$$\frac{\partial}{\partial W} \langle \tilde{x}(k, k)\, \tilde{x}(k, k)\rangle \quad \text{term by term.}$$

By equation (19), there are six additive terms, each term will be handled via the "gradient" methods in Appendix B.

The first term is not a function of

W(k), hence

$$\frac{\partial}{\partial W(k)} \langle \tilde{x}(k, k-1)\, \tilde{x}(k, k-1)\rangle = [0] \tag{20}$$

The second term is by equation

$$\frac{\partial}{\partial W(k)} \left\{-2 \langle \tilde{x}(k, k-1)\, W(k)\, H(k)\, \tilde{x}(k, k-1)\rangle\right\} = \underset{mkp}{-2H(k)}\tilde{x}(k,k-1)\rangle\langle\tilde{x}(k,k-1) \tag{21}$$

The third term is

$$\frac{\partial}{\partial W}\{\langle \tilde{x}(k, k-1)\, H^T(k)\, W^T(k)\, W(k)\, H(k)\, \tilde{x}(k, k-1)\rangle\} \tag{22}$$
$$= 2H(k)\, \tilde{x}(k, k-1)\rangle\langle\tilde{x}(k, k-1)\, H^T(k)\, W^T(k)$$

The latter derivation is based on

$$\langle c = \langle \tilde{x}(k, k-1)\, H^T(k) \tag{23}$$
$$b(m)\rangle = H(k)\, \tilde{x}(k, k-1)\rangle$$

The fourth term is by equation (24)

$$\frac{\partial}{\partial W}\{-2\langle \tilde{x}(k, k-1)\, W(k)\, v(k)\rangle\} = -2v(k)\rangle\langle\tilde{x}(k, k-1) \tag{25}$$

The fifth term by equation is

$$\frac{\partial}{\partial W}\{2\langle \tilde{x}(k, k-1)\, H^T(k)\, W^T(k)\, W(k)\, v(k)\rangle\} \tag{26}$$
$$= \{H(k)\, \tilde{x}(k, k-1)\rangle\langle v(k) + v(k)\rangle\langle\tilde{x}(k, k-1)H^T\}\, W^T$$

based on setting

$$\langle c = \langle \tilde{X} H^T \rangle \tag{27}$$

in equation (   )

The sixth term is

$$\frac{\partial}{\partial W} \{ \langle v(k)\ W^T(k)\ W(k)\ v(k) \rangle \} = 2v(k) \rangle\langle v(k)\ W^T(k) \tag{28}$$

Utilizing the above six expressions in equation (19) after the partial derivative has been taken

$$\begin{aligned}
&\frac{\partial}{\partial W} \left\{ \langle \tilde{x}(k, k)\ \tilde{x}(k, k) \rangle \right\} \\
&= -2H(k)\ \tilde{x}(k, k-1) \rangle\langle \tilde{x}(k, k-1) \\
&+ 2H(k)\ \tilde{x}(k, k-1) \rangle\langle \tilde{x}(k, k-1) H^T(k)\ W^T(k) \\
&- 2v(k) \rangle\langle \tilde{x}(k, k-1) \\
&+ \{H(k)\ \tilde{x}(k, k-1) \rangle\langle v(k) + v(k) \rangle\langle \tilde{x}(k, k-1) H^T(k)\}\ W^T(k) \\
&+ 2v(k) \rangle\langle v(k)\ W^T(k) = [0]_{mxp}
\end{aligned} \tag{29}$$

The expected value over all experiments and allowable values of j and n yields

$$\begin{aligned}
&E\left\{ \frac{\partial}{\partial W} \langle \tilde{x}(k, k)\ \tilde{x}(k\ k) \rangle \right\} = -2H(k)\ E\{\tilde{x}(k, k-1) \rangle\langle \tilde{x}(k, k-1)\} \\
&+2H(k)\ E\{\tilde{x}(k, k-1) \rangle\langle \tilde{x}(k, k-1)\}\ H^T(k)\ W^T(k) \\
&-2E\{v(k) \rangle\langle \tilde{x}(k, k-1)\} \\
&+ [H(k)\ E\{\tilde{x}(k, k-1) \rangle\langle v(k) + v(k) \rangle\langle \tilde{x}(k, k-1\}\ H^T(k)]\ W^T(k) \\
&+ 2E\{v(k) \rangle\langle v(k)\}\ W^T(k) = [0]
\end{aligned} \tag{30}$$

If we use the mtation

$$P(k, k) = \Sigma_{\tilde{x}\tilde{x}}(k, k) = E\{\tilde{x}(k, k) \times \tilde{x}(k, k)\} \tag{31}$$

$$P(k, k-1) = \Sigma_{\tilde{x}\tilde{x}}(k, k-1) = E\{\tilde{x}(k, k-1) \times \tilde{x}(k, k-1)\} \tag{32}$$

$$Q(k) = E\{u(k) \times u(k)\} \tag{33}$$

$$R(k) = E\{v(k) \times v(k)\} \tag{34}$$

and assume that the conventional statistical independence assumptions hold,

$$E\{\hat{x}(k, k-1) \times u(k)\} = [0] \tag{35}$$

$$E\{v(k) \times v(k-1)\} = [0] \tag{36}$$

$$E\{v(k) \times u(k)\} = [0], \text{ etc.} \tag{37}$$

By equation (31) through (37) in equation (30)

$$\begin{aligned} \frac{\partial\{tr\, P(k,k)\}}{\partial W(k)} = & -2H(k)\, P(k, k-1) \\ & + 2H(k)\, P(k, k-1)\, H^T(k)\, W^T(k) \\ & + 2\Sigma_{vv}(k)\, W^T(k) = [0] \end{aligned} \tag{38}$$

$$[H(k)\, P(k, k-1)\, H^T(k) + \Sigma_{vv}(k)]\, W^T(k) = H(k)\, P(k, k-1) \tag{39}$$

Transposing

$$W(k)[H(k)\, P(k, k-1)\, H^T(k) + \Sigma_{vv}(k)] = P(k, k-1)\, H^T(k) \tag{40}$$

Inverting

$$W(k) = P(k, k-1)\, H^T(k)\, [H(k)\, P(k, k-1)\, H(k) + \Sigma_{vv}(k)]^{-1} \tag{41}$$

or

$$\boxed{W(k) = P(k, k-1)\, H^T(k)[H(k)\, P(k, k-1)H^T(k) + R(k)]^{-1}} \tag{42}$$

$$W^T(k) = [H(k)\, P(k, k-1)\, H^T(k) + R(k)]^{-1}\, H(k)\, P(k, k-1) \tag{43}$$

The pxp matrix variance of the estimate of state (the p-space ellipsoid of uncertainty) can be obtained by taking the expected value over all experiments of the dyadic product of equation (18),

$$P(k, k) = E\{(\tilde{x}(k, k) \times \tilde{x}(k, k)\} \tag{44}$$

$$= [I - W(k)\ H(k)]\ P(k, k-1)[I - H^T(k)\ W^T(k)]$$

$$+ W(k)\ R(k)\ W^T(k)$$

Multiplying out the terms of equation (44) we obtain

$$P(k, k) = P(k, k-1) - P(k, k-1)\ H^T(k)\ W^T(k) \tag{45}$$

$$- W(k)\ H(k)\ P(k, k-1) + W(k)\ H(k)\ P(k, k-1)\ H^T(k)\ W^T(k)$$

$$+ W(k)\ R(k)\ W^T(k)$$

$$= P(k, k-1) - P(k, k-1)\ H^T(k)\ W^T(k)$$

$$- W(k)\ H(k)\ P(k, k-1)$$

$$+ W(k)[H(k)\ P(k, k-1)\ H^T(k) + R(k)]\ W^T(k)$$

Consider the last term of the above equation and equation (42) for W(k) with the transpose (43), then the last term becomes

$$W(k)[H(k)\ P(k, k-1)\ H^T(k) + R(k)]\ W^T(k) \tag{46}$$

$$= P(k, k-1)\ H^T(k)[H(k)\ P(k, k-1)H^T(k) + R(k)]^{-1}[H(k)P(k,k-1)H^T(k)+R(k)]$$

$$\times[H(k)\ P(k,k-1)\ H^T(k) + R(k)]^{-1}\ H(k)\ P(k, k-1)$$

$$= P(k,k-1)\ H^T(k)[H(k)\ P(k,k-1) + R(k)]^{-1}\ H(k)\ P(k, k-1)$$

$$= W(k)\ H(k)\ P(k, k-1).$$

Using the above expression for the last term in equation (45) we obtain

$$P(k, k) = P(k,k-1) - P(k, k-1)\ H^T(k)\ W^T(k) \tag{47}$$

$$- W(k)\ H(k)\ P(k, k-1) + W(k)\ H(k)\ P(k, k-1)$$

or

$$P(k,k) = P(k, k-1) - P(k, k-1)\ H^T(k)\ W^T(k) \tag{48}$$

$$P(k,k) = P(k, k-1)[\underset{pxp}{I} - \underset{pxm}{H^T(k)}\ \underset{mxp}{W^T(k)}] \tag{49}$$

The matrix P(k,k-1) was predicted and computed during stage k-1.

We can now predict the next stage (k+1) state vector via the deterministic process-dynamics

$$\hat{x}(k+1, k)\rangle = \Phi(k+1, k)\, \hat{x}(k, k)\rangle + B(k)\, f(k)\rangle . \tag{50}$$

The next stage prediction of the observation vector is

$$\hat{z}(k+1, k)\rangle = H(k+1)\, \hat{x}(k+1, k)\rangle . \tag{51}$$

The error in the state vector at stage k+1 based on the prediction of equation (50) is

$$\tilde{x}(k+1, k)\rangle = x(k+1)\rangle - \hat{x}(k+1, k)\rangle \tag{52}$$

where the unknown state vector is

$$x(k+1)\rangle = \Phi(k+1, k)x(k)\rangle + B(k)\, f(k)\rangle + N(k)\, u(k)\rangle \tag{53}$$

and the unknown error is by equation (50) and equation (53) in equation (52)

$$\begin{aligned} \tilde{x}(k+1, k)\rangle &= \Phi(k+1, k)x(k)\rangle + B(k)\, f(k)\rangle \\ &\quad + N(k)\, u(k)\rangle - \Phi(k+1, k)\hat{x}(k, k)\rangle - B(k)\, f(k)\rangle \\ &= \Phi(k+1, k)[x(k)\rangle - \hat{x}(k, k)\rangle] + N(k)\, u(k)\rangle \end{aligned} \tag{54}$$

Using equation in equation

$$\tilde{x}(k+1, k)\rangle = \Phi(k+1, k)\, \tilde{x}(k, k)\rangle + N(k)\, u(k)\rangle \tag{55}$$

The transpose of equation (55) is

$$\langle\tilde{x}(k+1, k) = \langle\tilde{x}(k, k)\, \Phi^T(k+1, k) + \langle u(k)\, N^T(k) \tag{56}$$

The dyadic product of equation (55) and equation (56) is

$$\begin{aligned} &\tilde{x}(k+1, k)\rangle\langle\tilde{x}(k+1, k) \\ &= \Phi(k+1, k)\tilde{x}(k, k)\rangle\langle\tilde{x}(k, k)\Phi^T(k+1, k) \\ &+ \Phi(k+1, k)\, \tilde{x}(k, k)\rangle\langle u(k)\, N^T(k) \\ &+ N(k)\, u(k)\rangle\langle\tilde{x}(k, k)\Phi^T(k+1, k) \\ &+ N(k)\, u(k)\rangle\langle u(k)\, N^T(k) \end{aligned} \tag{57}$$

The expectation over all experiments of equation (57) is

$$E\{\tilde{x}(k+1, k)\rangle\langle\tilde{x}(k+1, k)\} = P(k+1, k) \tag{58}$$

$$= \Phi(k+1, k)\, E\{\tilde{x}(k, k)\rangle\langle\tilde{x}(k, k)\}\, \Phi^T(k+1, k)$$

$$+\, N(k)\, E\{u(k)\rangle\langle u(k)\}\, N^T(k).$$

The statistical independence assumption was invoked:

$$E\{\tilde{x}(k, k)\rangle\langle u(k)\} = [0].$$

Define the process noise variance matrix

$$E\{u(k)\rangle\langle u(k)\} = Q(k) \tag{59}$$

and equation becomes

$$P(k+1, k) = \Phi(k+1, k)\, P(k, k)\, \Phi^T(k+1, k) + N(k)\, Q(k)\, N^T(k). \tag{60}$$

We may now summarize the equations and the computations to be performed at the kth stage as the <u>kth stage summary</u>.

We have available from stage k-1:

$$\hat{x}(k, k-1)\rangle$$

$$\hat{z}(k, k-1)\rangle = H(k)\, \hat{x}(k, k-1)\rangle$$

$$P(k, k-1).$$

Receive $z(k)\rangle$

Compute error in observation

$$\tilde{z}(k, k-1)\rangle = z(k)\rangle - \hat{z}(k, k-1)\rangle \tag{61}$$

Compute weight matrix $W(k)$ by equation (42)

$$W(k) = P(k, k-1)\, H^T(k)[H(k)\, P(k, k-1)\, H^T(k) + R(k)]^{-1} \tag{62}$$

Correct state estimate by equation (11)

$$\hat{x}(k, k)\rangle = \hat{x}(k, k-1)\rangle + W(k)\, \tilde{z}(k, k-1)\rangle \tag{63}$$

Compute new state variance by equation (49)

$$P(k, k) = P(k, k-1)[I - H^T(k)\, W^T(k)] \tag{64}$$

Predict to stage k+1, by equation (50)

$$\hat{x}(k+1, k)\rangle = \Phi(k+1, k)\, \hat{x}(k, k)\rangle + B(k)\, f(k)\rangle \tag{65}$$

Predict observation by equation (51)

$$\hat{z}(k+1, k)\rangle = H(k+1)\, \hat{x}(k+1, k)\rangle \tag{66}$$

Predict state variance by equation (60)

$$P(k+1, k) = \Phi(k+1, k)\, P(k, k)\, \Phi^T(k+1, k) + N(k)\, Q(k)\, N^T(k) \tag{67}$$

wait for stage k+1 and new measurement vector.

The equations can be substituted and juggled around to obtain alternate expressions, for example using equation (42) and (61) in (63) we obtain

$$\begin{aligned} \hat{x}(k, k)\rangle = {} & \hat{x}(k, k-1)\rangle \\ & + P(k, k-1)\, H^T(k)[H(k)\, P(k, k-1)\, H^T(k) + R(k)]^{-1} \\ & \times \{z(k)\rangle - H(k)\, \Phi(k, k-1)\, \hat{x}(k-1, k-1)\rangle - H(k) B(k-1) f(k-1)\rangle\} \end{aligned} \tag{68}$$

Many similar variations of the above systems of equations occur in the literature.

<u>Stage k = 1.</u>

The vector starting system of equations can be derived from equation for k = 1,

$$\hat{x}(1, 0)\rangle = \text{intelligent guess} \tag{69}$$

$$\hat{z}(1,0)\rangle = H(1)\, \hat{x}(1,0)\rangle \tag{70}$$

and

$$P(1, 0) = \text{intelligent guess based on experience about the process.} \tag{71}$$

Receive $z(1)\rangle$

Compute error

$$\tilde{z}(1, 0)\rangle = z(1)\rangle - \hat{z}(1, 0)\rangle \tag{72}$$

Compute first weight

$$W(1) = P(1, 0) \left[H^T(1) P(1, 0) H^T(1) + R(1) \right]^{-1} \tag{73}$$

Correct state

$$\hat{x}(1, 1)\rangle = \hat{x}(1, 0)\rangle + W(1) \tilde{z}(1, 0)\rangle \tag{74}$$

Compute state variance matrix by equation 49

$$P(1, 1) = P(1, 0) \left[I - H^T(1) W^T(1) \right] \tag{75}$$

Predict stage 2 by equation (50)

$$\hat{x}(2, 1)\rangle = \Phi(2, 1) \hat{x}(1, 1)\rangle + B(1) f(1)\rangle \tag{76}$$

Predict stage 2 observation

$$\hat{z}(2, 1)\rangle = H(2) \hat{x}(2, 1)\rangle \tag{77}$$

Predict state variance matrix by equation (67)

$$P(2, 1) = \Phi(2, 1) P(1, 1) \Phi^t(2, 1) + N(1)Q(1)N^T(1) \tag{78}$$

EQUATION SUMMARY FOR COMPUTER APPLICATION

This section summarizes the equations of the previous sections and points out how to compute mechanize the estimation equations to recursively estimate the state vector as the observations "roll into the computer". Precomputation of the sequence of weighting matrices for large dynamical systems is a necessity.

The dynamical process is described by the state vector equation

$$X(k+1)\rangle = \Phi(k+1, k)X(k)\rangle + Bf(k)\rangle + Bf(k)\rangle + N(k)\, U(k)\rangle \qquad (1)$$

and a system of noisy instruments whose outputs are functionally related to the states by the observation equation

$$Z(k)\rangle = H(k)X(k)\rangle + V(k)\rangle \ . \qquad (2)$$

The system of estimation equations are:

The state vector prediction equation

$$\hat{X}(k+1, K)\rangle = \Phi(k+1, k)\hat{X}(k,k)\rangle + B(k)f(k)\rangle . \qquad (3)$$

The observation prediction

$$\hat{Z}(k+1, k)\rangle = H(k+1)\hat{X}(k+1,k)\rangle \quad . \qquad (4)$$

The Observation error

$$\tilde{Z}(k+1, k)\rangle = Z(k+1)\rangle - \hat{Z}(k+1,k)\rangle \qquad (5)$$

and the correction to the predicted states at k+1 after the $(k+1)^{th}$ observation is available

$$\hat{X}(k+1, k+1)\rangle = \hat{X}(k+1, k)\rangle + W(k+1)\ \tilde{Z}(k+1,k)\rangle \quad . \qquad (6)$$

The sequence of weighting matrices W(k) can be precomputed and stored in memory. The weights are:

$$W(k+1) = P(k+1, k)H(k+1) \left[H(k+1)P(k+1, k)H^T(k+1) + R(k+1) \right]^{-1} \quad (7)$$

where

$$P(k+1, k) = \Phi(k+1,k)P(k,k)\Phi^T(k+1, k) + N(k+1)Q(k+1)N^T(k+1) \; . \quad (8)$$

By E(64) $P(k,k) = P(k,k-1)[I - H^T(k) W^T(k)]$ (9)

The block diagram is shown in Figure (1) as the conventional feedback (discreet) system.

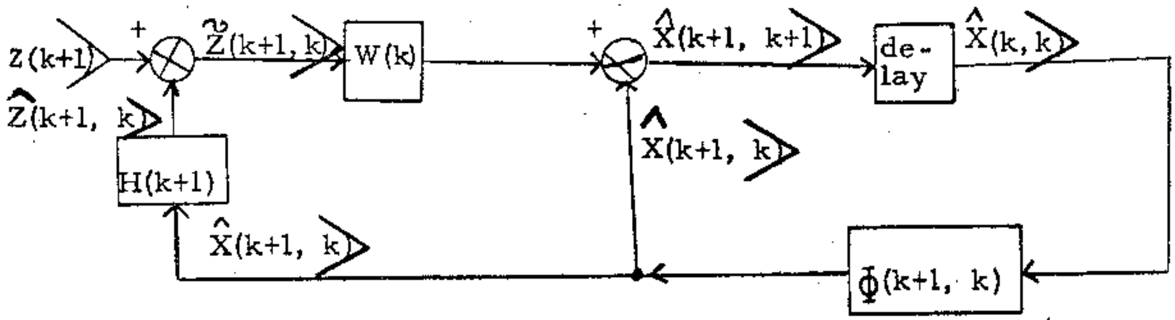

FIGURE (1) - DISCREET FEEDBACK BLOCK

By re-arranging the positions of the feedback blocks one can obtain a flow-block which looks more familiar to a digital programmer as shown in Figure (2).

Tests or applications in which one can plan or design the experiment and the times k, k+1, etc. at which instrument-data will be used to estimate appear to admit of pre-computing the weights. If the estimation times are not pre-designed one must compute the weights on-line.

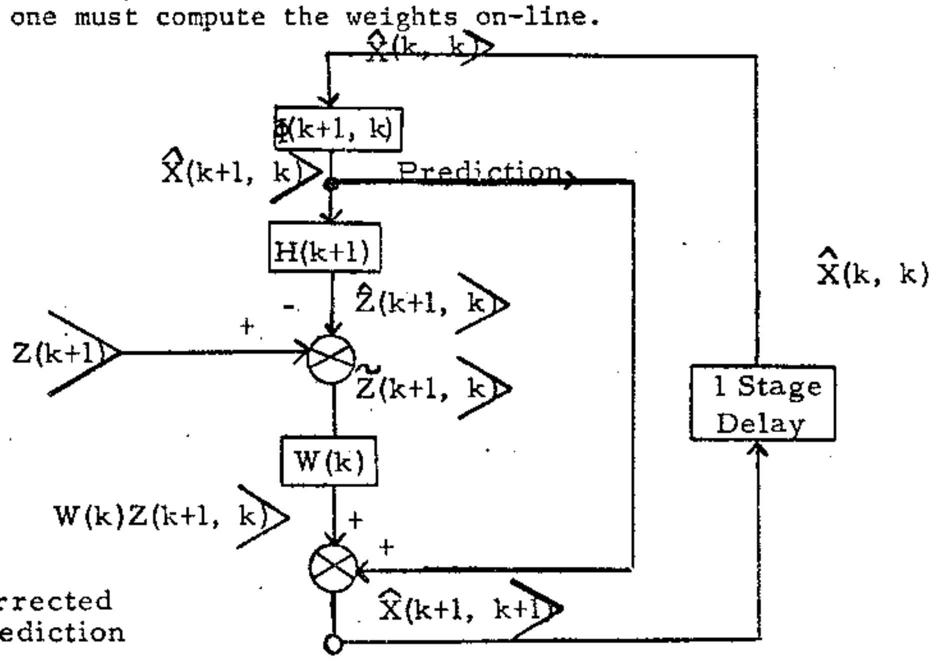

Figure (3) - Flow Block of Estimator

Section 25 WALSH FUNCTIONS

Walsh Functions are highly suitable for digital devices since they are square waves and have been increasingly successfully applied to digital communication and two dimensional imaging and filtering. Very few attempts thus far have been made to trajectory estimation. Dr. C. F. Chen published the first paper to my knowledge along these lines on October 1975 (reference 22) in a paper called "Design of Piecewise Constant Gains for Optimal Control via Walsh Functions". It is hoped that these techniques will soon be applied to optimal estimation.

This section presents some of the known and published properties of Walsh functions and adds a few new relations. Chen also in reference (20) refers to block pulse functions very closely related to the Walsh Functions, which he says in theory the block pulse functions do not form a complete set while the Walsh Functions do. Chens papers in reference (21) a excellent state-space treatments of this subject matter.

The Rademacker functions (not discussed in this report) are basically trigonemetric functions blocked off for a fundamental and harmonics as shown in Figure (1)

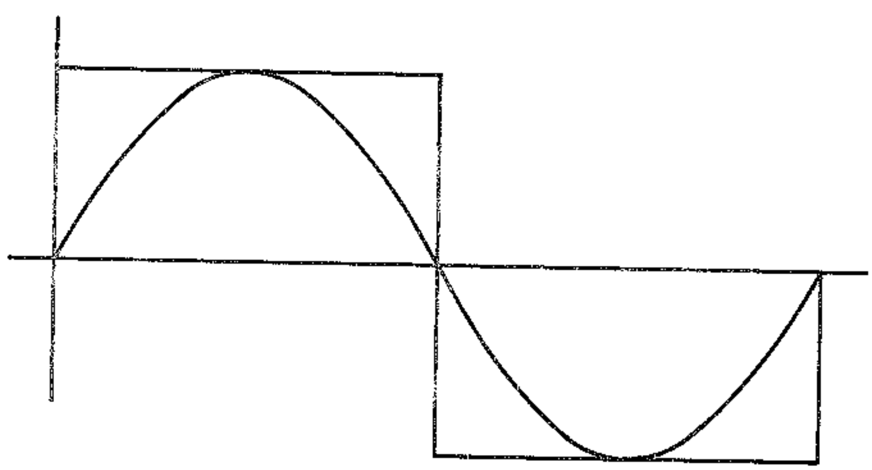

Figure 1 Rademecker and Sine Function

The Walsh function (first 10) are shown in Figure (2)

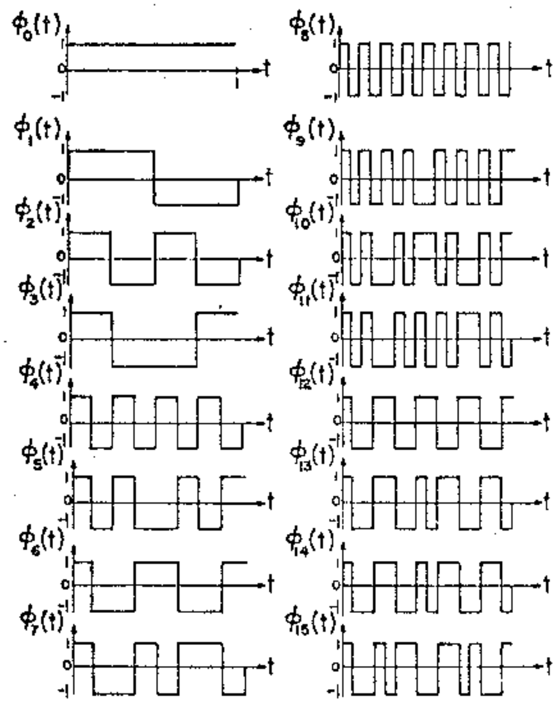

Figure 2. Walsh functions.

Continuous Time Functions.

The first four block pulse functions are shown in Figure (3)

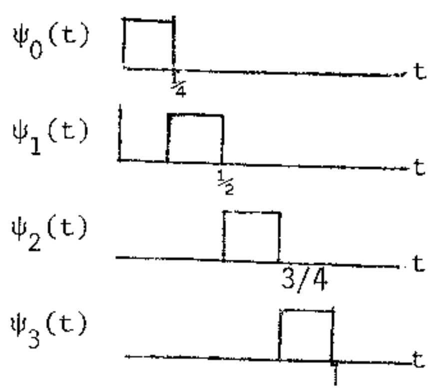

Figure (3) Block Pulse Functions

The block pulse functions are the "building blocks" for the Walsh functions and the relations are for the 4 x 4 case

$$\begin{bmatrix}\phi_0\\ \phi_1\\ \phi_2\\ \phi_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix} \begin{bmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{bmatrix} \qquad (1)$$

or

$$\phi(t)(4\rangle = \underset{4x4}{W}\ \psi(t)(\psi\rangle \qquad (2)$$

or for the m x m case

$$\phi(t)(m\rangle = \underset{mxm}{W}\ \psi(t)(m\rangle \qquad (3)$$

The inverse of the Walsh matrix is

$$W^{-1} = \frac{1}{m} W \qquad (4)$$

or

$$W^2 = mI \qquad (5)$$

$$W^T = W \qquad (6)$$

thus

$$\psi(t)\rangle = \frac{W}{m}\ \phi(t)\rangle \qquad (7)$$

By Figure (2) it is seen that continuous derivatives of the square wave functions do not exist at the break points, however integrations behave quite well, thus one should approximate the highest derivative of the function and integrate. Recall that with the polynomials the approximations were made at the position level and velocities and accelerations obtained by the derivatives of the fitting functions, that is

$$\langle\dot{f} = \langle f V_f \qquad (8)$$

Consider approximating the derivative of a function with the Walsh functions

$$\dot{x}(t) = \langle m)\phi(t)\ c(m\rangle + x_n(t)$$

or (9)

$$\dot{x}(t) = \sum_{i=0}^{\infty} \phi_i(t) c_i$$

where the c_i are constants.

The Walsh functions form an ortho-normal set that is

$$\int_0^1 \phi(t)\rangle\langle\phi(t)\,dt = I_{mxm} \tag{10}$$

For example the first two

$$\int_0^1 \phi\rangle\langle\phi = \int_0^1 \begin{bmatrix} \phi_0^2 & \phi_0\phi_1 \\ \phi_1\phi_0 & \phi_1^2 \end{bmatrix} dt = I \tag{11}$$

for clearly

$$\int_0^1 \phi_0^2 dt = \int_0^1 1\,dt \tag{12}$$

and

$$\int_0^1 \phi_0\phi_1 dt = \int_0^{1/2} 1\,dt + \int_{1/2}^1 (-1)dt = 0 \tag{13}$$

The coefficients of Eq. (8) can now be obtained

$$\int_0^1 \phi(t)\rangle\,\dot{x}(t)dt = c\rangle + \int_0^1 x\,dt \cong c\rangle \tag{14}$$

where $\cong$ is approximation.

Clearly the block pulse functions are an orthogonal set, that is

$$\int_0^1 \psi(t)\rangle\langle\psi(t)dt = I_{mxm}\left[\frac{1}{m}\right] \tag{15}$$

The matrix of inner-products between the two sets of functions of Eq. (1) is

$$\int_0^1 \phi(t)\rangle\langle\psi(t)dt = W\,I_m = W^{-1} \tag{16}$$

also by Eq. (7)

$$\int_0^1 \psi(t)\rangle\langle\phi(t)dt = \frac{W}{m} = W^{-1} \tag{17}$$

or the matrix of inner-products commutes, that is

$$\int_0^1 \psi(t)\rangle\langle\phi(t)dt = \int_0^1 \phi(t)\rangle\langle\psi(t)dt \tag{18}$$

By Eq. (9) we can approximate the function in block-pulse function base as (errors ommitted)

$$\dot{x}(t) \cong \langle\phi(t)c\rangle \cong \langle\psi(t)c^{\psi}\rangle \tag{20}$$

or by Eq. (16) and taking inner-product

$$\int_0^1 \phi(t)\rangle\, \dot{x}(t)dt \cong c\rangle \cong W^{-1}c^{\psi}\rangle \tag{21}$$

also

$$c^{\psi}\rangle \cong WC\rangle \tag{22}$$

thus a base change is simple.

One can also by approximate relations rather than equivalent relations obtain the monomial base in the Walsh base, that is if for example

$$\langle f(t) = (1,\ t,\ t^2) = \langle t \tag{23}$$

OBTAIN FOR FOURIER BASE

then

$$\langle t = \langle\phi(t)\underset{3x3}{C_{\phi t}} \tag{24}$$

that is

$$f(t)_i = \langle\phi(t)c\rangle_i \tag{25}$$

$$i = 0,\ 1,\ 2$$

Obviously

$$f_o(t) = \phi_o(t) = 1 \tag{26}$$

and

$$t = \langle\phi(t)c\rangle_1$$

or

$$\int_0 \phi(t)\rangle\, t\ dt = c(3\rangle_1 = \begin{pmatrix} c_0 \\ c1 \\ c2 \end{pmatrix}_1 \tag{27}$$

By Eq. (24)

$$C_{\phi t} = \left[\vec{c}_0, \vec{c}_1, \vec{c}_2 \right] = \int_0^1 \phi(t) \times t \; dt \tag{28}$$

$$C_{\phi t} = \int_0^1 \left[\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix} dt, \quad \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix} tdt, \quad \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix} t^2 dt \right] \tag{29}$$

The first column vector has values

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{30}$$

for

$$\int_0^1 \phi_0 dt = 1$$

$$\int_0^1 \phi_1 dt = \int_0^{1/2} 1dt + \int_{1/2}^1 (-1)dt = 0 \tag{31}$$

etc.

The second column vector has approximate values

$$\vec{c}_1 \cong \begin{pmatrix} 1/2 \\ -1/4 \\ -1/8 \end{pmatrix} \tag{32}$$

The approximation and equal signs will loosely by used so reader beware. For example

$$\int_0^1 \phi_0 tdt = \int_0^1 tdt = 1/2 \tag{33}$$

and

$$\int_0^1 \phi_1(t) tdt = \int_0^{1/2} tdt + \int_{1/2}^1 (-1)tdt = -1/4 \tag{34}$$

etc.

The third column vector of Eq. (28) is

$$c\rangle_2 = \begin{bmatrix} 1/3 \\ -1/3 \\ -1/8 \end{bmatrix} \tag{35}$$

By Eq. (30), (32) and (35) in Eq. (24)

$$\langle 3)t \cong \langle 3)\phi(t) \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & -1/4 & -1/3 \\ 0 & -1/8 & -1/8 \end{bmatrix} \tag{36}$$

Note that if one approximates the first three monomial base functions by a large number of Walsh functions say $m > 3$ then

$$\langle 3)f(t) = \langle m)\phi C_{\phi t \atop mx3} \tag{37}$$

$$\langle f(t)C^*_{3xm} = \langle m)\phi C_{\phi t} C^*_{\phi t} \tag{38}$$

The inverse of the matrix is

$$C^{-1}_{\phi t} = \begin{bmatrix} 1 & -2 & 8 \\ 0 & 12 & -4(8) \\ 0 & -12 & 3(8) \end{bmatrix} \tag{39}$$

By Eq. (24)

$$\int_0^1 \phi\rangle\langle t dt \cong C_{\phi t} \tag{40}$$

also

$$\int_0^1 t\rangle\langle t \, dt = H_{ill} = \int_0^1 t\rangle\langle \phi dt \; C_{\phi t} \tag{41}$$

or

$$H_{ill} \, C^{-1}_{\phi t} \cong \int_0^1 t\rangle\langle \phi dt \tag{42}$$

where H_{ill} is the 3 x 3 Hilbert matrix.

Transposing Eq. (24)

$$t\rangle = C^T_{\phi t}\phi\rangle \tag{43}$$

hence

$$\int_0^1 t\rangle\langle t\,dt = H_{ill} \cong C_{\phi t}^T C_{\phi t} \qquad (44)$$

and

$$H_{ill}^{-1} \cong C_{\phi t}^{-1} C_{\phi t}^{-T} \qquad (45)$$

Since the Walsh matrix is symmetric by Eq. (1)

$$\langle\phi = \langle\psi W \qquad (46)$$

and

$$\langle\psi = \langle\phi W \frac{1}{m} \qquad (47)$$

If we partition W into its column vectors

$$W = [w\rangle_0,\ w\rangle_1,\ w\rangle_2,\ w\rangle_3] \qquad (48)$$

then

$$(\phi_0,\ \phi_1,\ \phi_2,\ \phi_3) = [\langle\psi w\rangle_0,\ \langle\psi w\rangle_1, \langle\psi w\rangle_2\ \langle\psi w\rangle_3] \qquad (49)$$

or

$$\phi_i = \langle\psi w\rangle_i \qquad (50)$$

and the $w\rangle_i$ are the coordinates in the block-pulse base of each of the Walsh functions.

Likewise

$$\langle\psi = \langle\psi I = \langle\psi \left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right] \qquad (51)$$

The transition matrix on the column vectors of Eq. (51) is

$$e\rangle_{i+1} = S_d\ e\rangle_i \qquad (52)$$

$$i = 0,\ 1,\ 2,\ 3$$

where the permutation matrix or shift-down operator is

$$S_d = \begin{pmatrix}0 & 0 & 0 & 1\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{pmatrix} \qquad (53)$$

and

$$\underset{4\times 4}{I} = [e\rangle_0,\ e\rangle_1,\ e\rangle_2,\ e\rangle_3] \tag{54}$$

The meaning of Eq. (52) is, for example

$$e\rangle_1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{bmatrix}0 & 0 & 0 & 1\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix} \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = S_d\ e\rangle_0 \tag{55}$$

etc.

By Eq. (47)

$$\langle\psi = \langle\phi W^{-1} \tag{56}$$

or

$$\langle\psi\ [e\rangle_0,\ e\rangle_1,\ e\rangle_2,\ e\rangle_3] = \langle\phi\ [w\rangle_0,\ w\rangle_1,\ w\rangle_2,\ w\rangle_3]\ \frac{1}{m} \tag{57}$$

Eq. (56) in Eq. (57)

$$\langle\psi\ [e\rangle_0,\ e\rangle_1,\ e\rangle_2,\ e\rangle_3] = \langle\psi\ [W\rangle_0,\ w\rangle_1,\ w\rangle_2,\ w\rangle_3] \tag{58}$$

or

$$e\rangle_i = W\ w\rangle_i \tag{59}$$

Using the base-change relation of Eq. (59) in Eq. (52)

$$Ww\rangle_{i+1} = S_d\ Ww\rangle_i \tag{60}$$

or

$$w\rangle_{i+1} = W^{-1} S_d W w\rangle_i \tag{61}$$

or

$$w\rangle_{i+1} = \Phi_w\ w\rangle_i \tag{62}$$

which is the transition matrix on columns of w.

$$\Phi_w = W^{-1} S_d W = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & -1 & 0\\0 & -1 & 0 & 0\end{pmatrix} \tag{63}$$

By Eq. (63) the orthogonal matrix W can be written as

$$W = [1\rangle,\ \Phi_w 1\rangle,\ \Phi^2 1\rangle,\ \Phi^3 1\rangle] \tag{64}$$

where

$$1\rangle = w\rangle_0 \tag{65}$$

Note the cyclic properties

$$\underset{m\times m}{S_d^m} = I \tag{66}$$

$$S_d^{m+1} = S_d \tag{67}$$

and

$$\Phi_w = W^{-1} S_d W \tag{68}$$

$$\Phi_w^m = W^{-1} S_d^m W = I \tag{69}$$

$$\Phi_w^{m+1} = \Phi_w \tag{70}$$

we also have

$$\begin{aligned} W^0 &= I \\ W^1 &= W \\ W^2 &= mI \\ W^3 &= mW \\ W^4 &= m^2 I \\ &\vdots \\ W^5 &= m^2 W \\ W^6 &= m^3 I \\ &\vdots \\ W^m &= m^{m/2} I \end{aligned} \tag{71}$$

for m even.

<u>Integration of Walsh Functions and Block Pulse Functions.</u>

Chen in reference (21) presents the operational matrix for Walsh integration, and for the 4 x 4 case, the functions and their integrals look like

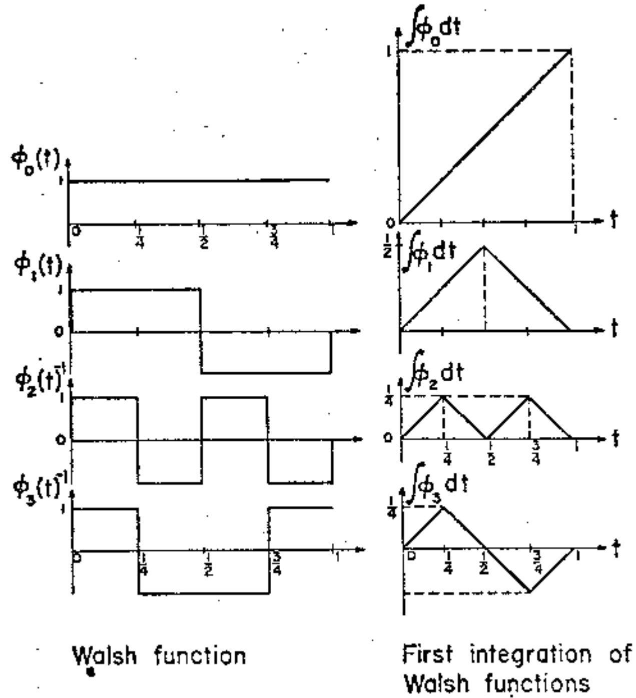

Figure 4 Walsh Integration

The Walsh functions are rectangular waves, their first integrals are triangular waves and the second integrals are quadratic curves. Chen approximated the triangular waves over the intervals as shown in Figure (4), to obtain

$$\int_0^t \langle\phi(t)\,dt \cong \langle\phi(t) \begin{bmatrix} 1/2 & 1/4 & 1/8 & 0 \\ -1/4 & 0 & 0 & 1/8 \\ -1/8 & 0 & 0 & 0 \\ 0 & -1/8 & 0 & 0 \end{bmatrix} \tag{72}$$

or

$$\int_0^t \langle\phi(t)\,dt \cong \langle\phi(t)\ P_w \tag{73}$$

where the integration matrix is a constant.

The 8 x 8 integration matrix is given by Chen as

$$\underset{8\times 8}{P_w} = \begin{bmatrix} 1/2 & 1/4 & 1/8 & 0 & 1/16 & 0 & 0 & 0 \\ -1/4 & 0 & 0 & 1/8 & 0 & 1/16 & 0 & 0 \\ -1/8 & 0 & 0 & 0 & 0 & 0 & 1/16 & 0 \\ 0 & -1/8 & 0 & 0 & 0 & 0 & 0 & 1/16 \\ -1/16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/16 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/16 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Chen also gives the 16 x 16 matrix which is the same as a table in Corringtons paper of reference (25).

The integration of the first four block pulse functions is obtained in reference (20) by Chen and they look like Figure (5).

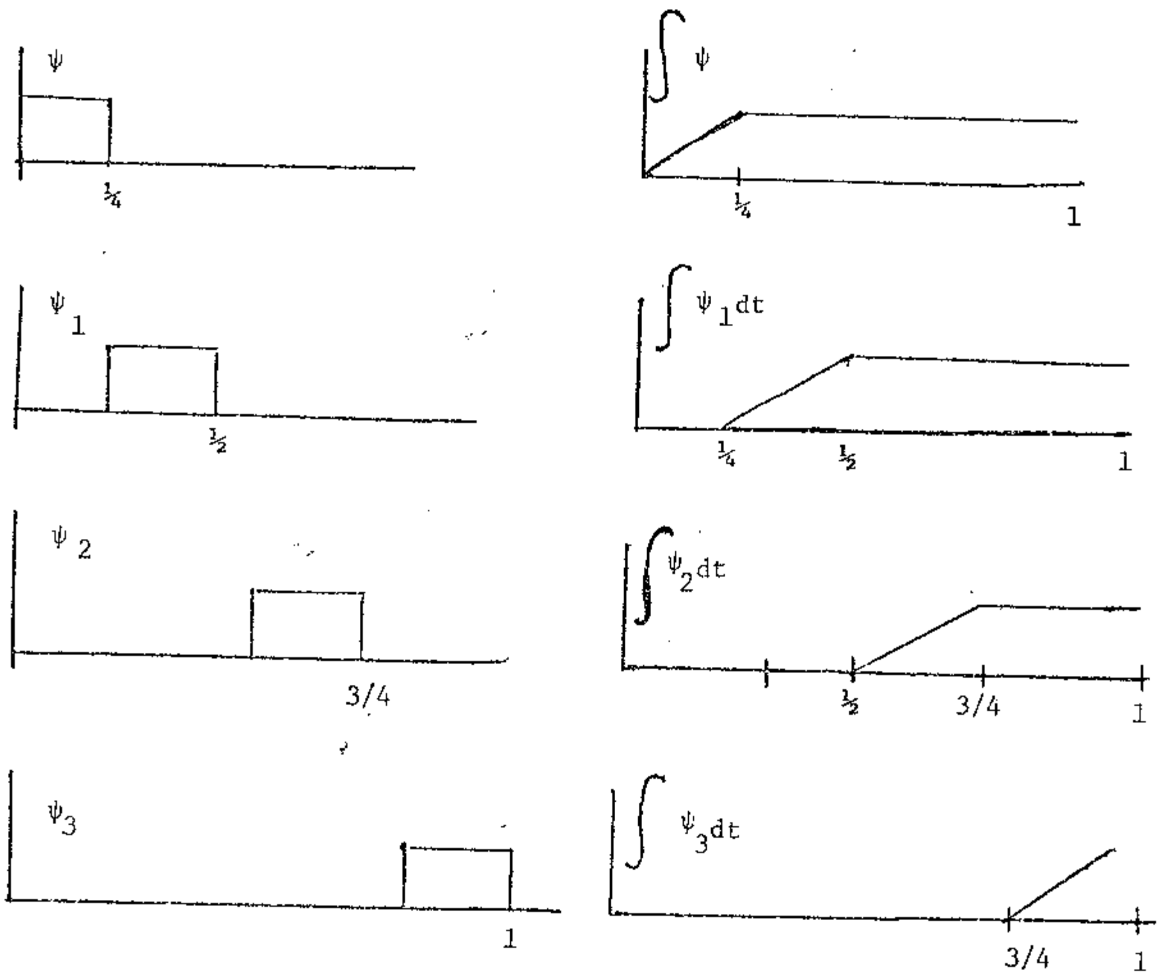

Figure 5 Block Pulse Integrals

The integral of the block pulse functions is given by Chen as

$$\int_0^t \psi(t)dt = \psi P_b \tag{75}$$

with

$$P_{b\,4\times4} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 1 & 1/2 & 0 & 0 \\ 1 & 1 & 1/2 & 0 \\ 1 & 1 & 1 & 1/2 \end{bmatrix} 1/4 \qquad (76)$$

The integration matrices P_b and P_w are related through a base change (or a similarity transformation as) by Eq. (46)

$$<\phi = <\psi W \qquad (77)$$

and integrating both sides of Eq. 77.

$$\int<\phi dt = \int<\psi dt\ W \qquad (78)$$

or

$$<\phi P_w = <\psi P_b W \qquad (79)$$

or Eq. (78) in Eq. (79)

$$<\psi W P_w = <\psi P_b W \qquad (80)$$

or

$$P_w = W^{-1} P_b W \qquad (81)$$

One can now easily invert the Walsh integration matrix for by Eq. (81)

$$P_w^{-1} = W^{-1} P_b^{-1} W = \frac{W P_b^{-1} W}{m} \qquad (82)$$

The block pulse matrix is lower triangular and easily invertable.

Also note that

$$P_b = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} - 1/2\ I \right\} 1/4 \qquad (83)$$

where it is know that

$$(I - S_{do})^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad (84)$$

where the shift-down and out matrix (shift the row 2 of the identify matrix down) is

$$S_{do} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad (85)$$

and lower triangular unit matrix can be written as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = I + S_{do} + S_{do}^2 + S_{do}^3 \qquad (86)$$

One can write the inverse of Eq. (84) as

$$(I-S_{do})^{-1} = (I-S_{do})^T \; [(I-S_{do})(I-S_{do})^T]^{-1} \qquad (87)$$

The symmetric matrix

$$(I-S_{do})(I-S_{do})^T = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \qquad (88)$$

with inverse given by Eq. (38) see (6) as the Frankel matrix, that is

$$[(I-S_{do})(I-S_{do})^T]^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \qquad (89)$$

and Eq. (89) used in Eq. (87) yields Eq. (84). One can now use the Householder Inversion Lemma to invert Eq. (83).

One can also use the relations (not derived here).

$$\begin{aligned} (I-S)^{-1} &= I + S + S^2 + S^3 \\ (I+S)^{-1} &= I - S + S^2 - S^3 \\ (I-S^2)^{-1} &= I + S^2 \\ (I+S^2)^{-1} &= I - S^2 \end{aligned} \qquad (90)$$

and for a scalar a

$$\begin{aligned} (I-aS)^{-1} &= I + aS + a^2S^2 + a^3S^3 \\ (I+as)^{-1} &= I - aS + a^2S - a^3S^3 \end{aligned} \qquad (91)$$

where

$$S^4 = 0 \qquad (92)$$

Note the class of matrices

$$C = C_0 I + C_1 S + C_2 S^2$$
$$D = d_o I + d_1 S + d_2 S^2 \tag{93}$$

commute, for $S^3 = 0$ and

$$CD = (c_0 I,\ c_1 I,\ c_2 I) \begin{pmatrix} I \\ S \\ S^2 \end{pmatrix} \begin{pmatrix} I, & S, & S^2 \end{pmatrix} \begin{pmatrix} d_0 I \\ d_1 I \\ d_2 I \end{pmatrix} \tag{94}$$

$$= (c_0 I,\ c_1 I,\ c_2 I) \begin{bmatrix} I & S & S^2 \\ S & S^2 & 0 \\ S^2 & 0 & 0 \end{bmatrix} \begin{pmatrix} d_0 I \\ d_1 I \\ d_2 I \end{pmatrix} \tag{95}$$

$$CD = c_0 d_0 I + (c_1 d_0 + c_0 d_1)\ S + (c_2 d_0 + c_1 d_1 + d_2 c_0) S^2 \tag{96}$$

This class of commuting matrices are to be contrasted with the commuting circulant matrices

$$C = a_0 I + a_1 P + a_2 P^2 \tag{97}$$

with the permutation matrix

$$P^3 = I \tag{98}$$

and Eq. (95) becomes

$$CD = (c_0,\ c_1,\ c_2,) \begin{bmatrix} I & P & P^2 \\ P & P^2 & I \\ P^2 & I & P \end{bmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \end{pmatrix} \tag{99}$$

Consider a linear constant coefficient differential equation

$$\dddot{x}(t) = \langle a\ x(3) \rangle_s = a_0 x + a_1 \dot{x} + a_2 \ddot{x} \tag{100}$$

approximating the highest derivative with Walsh functions or block pulse functions

$$x(t) = \langle ax(3) \rangle_s \cong \psi(t)c \tag{101}$$

Integrating (and remembering approximation errors)

$$\ddot{x}(t) = \ddot{x}(to) + \langle \psi(t) P_b c \rangle \tag{102}$$

now

$$\ddot{x}(to) = {}^{1}\langle e\,\psi(t)\rangle \ddot{x}(to) \tag{103}$$

or

$$\ddot{x}(t) = \langle \psi(t)[e\rangle_1 \ddot{x}(to) + P_b c\rangle] \tag{104}$$

Integrating again

$$\dot{x}(t) = \dot{x}(to) + \langle \psi(t) P_b [e\rangle_1 \ddot{x}(to) P_b c\rangle] \tag{105}$$

or

$$\dot{x}(t) = \langle \psi(t)\ [e\rangle_1 \dot{x}(to) + P_b e\rangle_1 \ddot{x}(to) + P_b^2 c\rangle] \tag{106}$$

and integrating again

$$x(t) = x(to) + \langle \psi(t)[P_b e\rangle_1 \dot{x}(to) + P_b^2 e\rangle_1 \ddot{x}(to) + P_b^3 c\rangle] \tag{107}$$

$$x(t) = \langle \psi(t)[e\rangle_1 x(to) + P_b e\rangle_1 \dot{x}(to) + P_b^2 \quad \ddot{x}(to) \quad + P_b^3 c\ ] \tag{108}$$

Packaging

$$\begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \\ x \end{pmatrix} \quad \begin{bmatrix} \psi P_b^3 \\ \psi P_b^2 \\ \psi P_b \\ \psi \end{bmatrix} \quad c\rangle + \begin{pmatrix} \text{I.C.} \\ \text{terms} \end{pmatrix} \tag{109}$$

hence powers of the integration matrix arise. Note that powers of P_b are related to powers of P_w since by Eq. (81)

$$\begin{aligned} P_w &= W^{-1} P_b W \\ P_w^2 &= W^{-1} P_b^2 W \\ &\vdots \\ P_w^m &= W^{-1} P_b^m W \end{aligned} \tag{110}$$

From the foregoing one would expect to exploit the cyclic nature of Walsh functions just as has been done with Fast Forrier Transforms.

Chen presents a method for solving a non-homogeneous differential equation

$$\dot{x}\rangle = \underset{mxm}{A}\ x\rangle + f(t)\rangle \tag{111}$$

and approximates the velocity vector with say block-pulse functions

$$\dot{x}\rangle = C\psi(t)\rangle + v_e(t) \tag{112}$$

where $V_e(t)$ is the approximating error and will be ignored in the derivations. Integrating Eq. (112)

$$x(t)\rangle = x(to)\rangle + C\int_0^t \psi(t)\rangle\, dt \tag{113}$$

or by Eq. (75)

$$x(t)\rangle = x(to)\rangle + CP_b\psi(t)\rangle \tag{114}$$

Note

$$x(to)\rangle\, 1 = x(to)(n\rangle\langle m)\quad \psi(t)\rangle \tag{115}$$

since

$$1 = \langle m)e\quad \psi(t)\rangle\, i \tag{116}$$

hence

$$x(t)\rangle = [x(to)\rangle\langle e + CP_b]\ \psi(t)\rangle \tag{117}$$

Using Eq (117) in Eq (112)

$$\{A[x\rangle_0\langle e + CP_b] + C_f\}\,\psi\rangle = C\psi\rangle \tag{118}$$

where the approximation of the forcing function in the block pulse base is

$$f(t)\rangle = C_f\psi(t)\rangle \tag{119}$$

operator Eq. (118) with inner-product, that is

$$\int_0^t C\psi\rangle\langle\psi dt = C\,\frac{1}{m} \tag{119}$$

hence

$$A[x\rangle_0\langle e + CP_b] = C \tag{120}$$

or

$$ACP_b - C = -Ax\rangle_0\langle e \tag{121}$$

Set

$$-Ax\rangle_0\langle e = B \tag{122}$$

then

$$ACP_b - C = B \tag{123}$$

One can solve Eq. 123 for the rectangular matrix C via the Kronker matrix product schemes of Eq. (48) Section (G).

$$\text{vec}\ ACP_b = (P_b^T \times A)\ \text{vec}\ C \tag{124}$$

Using Eq. (124) in Eq. (123)

$$[P_b^T \times A - I] \text{ vec } C = \text{vec } B \tag{125}$$

or

$$\text{vec } C = [P_b^T \times A - I]^{-1} \text{ vec } B \tag{126}$$

By Eq. (76) and Eq. (2) Section (G) for

$$P_b^T \times A = \frac{1}{m} \begin{pmatrix} A/2 & A & A \\ 0 & A/2 & A \\ 0 & 0 & A/2 \end{pmatrix} \tag{127}$$

and

$$P_b^T \times A - I = \begin{bmatrix} \frac{A}{m2}-I & A\frac{1}{m} & A\frac{1}{m} \\ 0 & \frac{A}{m2} - I & A\frac{1}{m} \\ 0 & 0 & \frac{A}{m2} - I \end{bmatrix} \tag{128}$$

Eq. (127) is Block upper-triangular and easy to invert. Note that it is a generalization of the matrix of Eq. (93), for one can express

$$P_b^T = (\tfrac{1}{2}I, I, I) \begin{pmatrix} I \\ S_{uo} \\ S_{uo}^2 \end{pmatrix} \tag{129}$$

where $S_{uo} = S_{do}^T$

and the (tensor product) Kronecker matrix product becomes

$$P_b^T \times A = (\tfrac{1}{2}I, I, I) \begin{pmatrix} I \\ S \\ S^2 \end{pmatrix} \times A \tag{130}$$

Thus one solves for the matrix C by unpacking or de-vecing Eq. (126).

The matrix of Eq. (128) is a special case of the block Toeplitz mattrix. The Toeplitz matrix is of the form for a 3 x 3.

$$\Upsilon = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_{-1} & a_0 & a_1 \\ a_{-2} & a_{-1} & a_0 \end{pmatrix} \tag{131}$$

$$= (a_{-2}I,\ a_{-1}I,\ a_0I,\ a_1I,\ a_2I) \begin{bmatrix} S_{uo}^{T2} \\ S_{uo}^{T} \\ I \\ S_{uo} \\ S_{uo}^{2} \end{bmatrix} \tag{132}$$

A matrix closely related to the Teoplitz matrix is the Hankel matrix whose diagonals run from top right to bottom left or

$$H = \begin{pmatrix} a_2 & a_1 & a_0 \\ a_1 & a_0 & a_{-1} \\ a_0 & a_{-1} & a_{-2} \end{pmatrix} \tag{133}$$

$$= (a_2,\ a_1,\ a_0,\ a_{-1},\ a_{-2}) \begin{bmatrix} L_cS_{uo}^{T2} \\ L_cS_{uo}^{T} \\ L_c \\ L_cS_{uo} \\ L_cS_{uo}^{2} \end{bmatrix} \tag{134}$$

with the linear convolution matrix

$$L_c = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{135}$$

One example previously encountered is the block bi-diagonal matrix and its inverse

$$\begin{bmatrix} \Lambda & I & 0 & 0 \\ 0 & \Lambda & I & 0 \\ 0 & 0 & \Lambda & I \\ 0 & 0 & 0 & \Lambda \end{bmatrix}^{-1} = \begin{bmatrix} \Lambda^{-1} & -\Lambda^{-2} & \Lambda^{-3} & -\Lambda^{-4} \\ 0 & \Lambda^{-1} & -\Lambda^{-2} & \Lambda^{-3} \\ 0 & 0 & \Lambda^{-1} & -\Lambda^{-2} \\ 0 & 0 & 0 & \Lambda^{-1} \end{bmatrix} \tag{136}$$

Another example if A and B commute and

$$T = \begin{pmatrix} A & 0 & 0 & 0 \\ -B & A & 0 & 0 \\ 0 & -B & A & 0 \\ 0 & 0 & -B & A \end{pmatrix} \tag{137}$$

then

$$T^{-1} = \begin{bmatrix} I & 0 & 0 & 0 \\ A^{-1}B & I & 0 & 0 \\ (A^{-1}B)^2 & A^{-1}B & I & 0 \\ (A^{-1}B)^3 & (A^{-1}B)^2 & A^{-1}B & I \end{bmatrix} \qquad (138)$$

In general if A and B (and A^{-1} exists) commute and the special Teoplitz matrix is

$$T = \begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 \\ B & A & 0 & 0 & & \\ B & B & A & 0 & & \\ B & B & B & A & & \\ \cdot & & & & & \\ \cdot & & & & & 0 \\ \cdot & & & & & \\ B & B & \cdots & \cdots & B & A \end{bmatrix} \qquad (139)$$

then

$$T^{-1} = A^{-1} \begin{bmatrix} I & 0 & & 0 \\ C & I & & 0 \\ CE & C & \cdots & \\ CE^2 & CE & & \\ CE^3 & CE^2 & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ CE^{m-2} & CE^{m-3} & \cdots & I \end{bmatrix} \qquad (140)$$

with

$$C = A^{-1}B$$

$$E = I - A^{-1}B$$

It is seen by Eq. (128) that the block pulse integration matrix can be put into the transposed form of Eq. (139), that is

$$P_b^T \times A - I = \frac{-1}{2m} \begin{bmatrix} 2mI-A & -2A & -2A \\ 0 & 2mI-A & -2A \\ 0 & 0 & 2mI-A \end{bmatrix} \qquad (141)$$

and that the matrices commute.

Set the diagonal matrix of Eq. (141)

$$L = 2mI-A \qquad (142)$$

and

$$B = -2L^{-1}A \tag{143}$$

then Eq. (141) becomes

$$P_b^T \times A\text{-}I = \frac{-L}{2m} \begin{bmatrix} I & B & B \\ 0 & I & B \\ 0 & 0 & I \end{bmatrix} \tag{144}$$

Consider the inverse of the unit-diagonal upper-triangular matrix of Eq. (144)

$$\begin{pmatrix} I & B & B \\ 0 & I & B \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & 0 & R_{33} \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{145}$$

Multiply out the elements of Eq. (145) to solve for

$$\begin{pmatrix} I & B & B \\ 0 & I & B \\ 0 & 0 & I \end{pmatrix}^{-1} = \begin{bmatrix} I & -B & -B(I-B) \\ 0 & I & -B \\ 0 & 0 & I \end{bmatrix} \tag{146}$$

and using Eq. (146) in Eq. (144)

$$[P_b^T \times A\text{-}I]^{-1} = (-2m) \begin{bmatrix} L^{-1} & -BL^{-1} & -B(I-B)L^{-1} \\ 0 & L^{-1} & -BL^{-1} \\ 0 & 0 & L^{-1} \end{bmatrix} \tag{147}$$

How one applies Walsh functions to budding digital filters is a wide open untouched area as far as trajectory estimation is concerned. Clearly the cyclic-nature of the functions as well as their pulse or digital-device nature excites the imagination in the area of "real-time" computations. The cyclic nature of the trigonemetric functions have served continuous analog devices in computing and communications for years; perhaps these functions will do the same for all phases of trajectory estimation on digital computers.

I. HOMOGENEOUS DYNAMICAL SYSTEM. Consider the homogeneous dynamical system

$$\dot{x}\rangle = Ax\rangle \tag{1}$$

where A is a constant p×p matrix.

In general when one is studying the effects of a linear transformation on a vector it is fruitful to study the effect of A on a linearly independent set of $x\rangle_j$ vectors (that is any basis set); for example if

$$y\rangle = Ax\rangle \tag{2}$$

where $x\rangle$ is a p-dimensional column vector then for a package of p vectors

$$\left[y\rangle_1, y\rangle_2, \cdots y\rangle_p\right] = \left[Ax\rangle_1, Ax\rangle_2, \cdots Ax\rangle_p\right] \tag{3}$$

or

$$\underset{p\times p}{Y} = \underset{p\times p}{A}\ \underset{p\times p}{X} \tag{4}$$

If the background base vectors have coordinates

$$e\rangle_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \text{ i}^{\text{th}} \text{ position} \tag{5}$$

then

$$I = \left[e\rangle_1, e\rangle_2, \cdots e\rangle_p\right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & \cdot & \cdot & 1 \end{bmatrix} \tag{6}$$

and if

$$X = I \tag{7}$$

then

$$Y = AI \tag{8}$$

or the base vector I are mapped by A to vector Y whose coordinates are the columns of A.

If we ask what linearly independent set of vectors X are mapped by A to the base vectors I, that is find the X such that

$$Y = I \tag{9}$$

then

$$I = AX \tag{10}$$

or

$$X = A^{-1} \tag{11}$$

for full rank A.

In general if Y and X of Equation (4) are given then we can solve for A as

$$A = YX^{-1} \tag{12}$$

The inverse of a square matrix (non-singular) can be expressed as

$$X^{-1} = X^T (XX^T)^{-1} \tag{13}$$

Likewise, if we have an over-specified system (that is more equations than unknowns) or more than a linearly independent set of

$$X\rangle_i \quad i=1,2\cdots p$$

then

$$\underset{p\times j}{Y} = \underset{p\times p}{A} \; \underset{p\times j}{X} \tag{14}$$

and for rank of X=p

$$\underset{p\times p}{A} = \underset{p\times j}{Y} \; \underset{j\times p}{X^*} \tag{15}$$

where the pseudo-inverse can be written as

$$\underset{j\times p}{X^*} = X^T (XX^T)^* = (X^T X)^* X^T \tag{16}$$

and for full rank case

$$(XX^T)^* = (XX^T)^{-1} \tag{17}$$

Likewise we shall consider a package or sequence of p vectors (or trajectories) propogating under A of Equation (1) such that at time zero

$$X(0) = \left[x(0)\rangle_1, \cdots x(0)\rangle_p \right] \tag{18}$$

or each trajectory $x(t)\rangle_j$ originates at

$x(0)\rangle_j$; $j=1,2,\cdots p$

The dynamics of each trajectory or the velocity of each of the p vectors is

$$\dot{x}(t)\rangle_j = A\, x(t)\rangle_j \tag{19}$$

and the dynamics of the package is the matrix differential equation

$$\dot{X} = AX \tag{20}$$

One normally assumes the solutions to (20) to be

$$X(t) = e^{At} X(0) = \Phi(t) X(0) \tag{21}$$

or the solution to Equation (19) to be

$$x(t)\rangle_j = e^{At} x(0)\rangle_j \tag{22}$$

and takes the derivative

$$\dot{X} = A\ e^{At}\ X(0) \tag{23}$$

$$\dot{X} = A\ X(t) \tag{24}$$

where

$$\Phi(t) = e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots \tag{25}$$

By inspection of Equation (21) if

$$X(0) = I \tag{26}$$

then the solution of Equation (21) is a very special set of time varying base vectors called the fundamental basis and designated as

$$\Phi(t) = e^{At} \tag{27}$$

The time varying column vector of $\Phi(t)$ start with initial conditions on the fixed background bases and propogate under A, for example

$$\phi(0)\rangle_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{28}$$

and at any time the first column vector has velocity

$$\dot{\phi}\rangle_1 = A\ \phi(t)\rangle_1 \tag{29}$$

etc.

It can be proven but will not be shown here that $\Phi(t)$ remains non-singular, that is the vectors in the column space of Φ remain linearly independent over the finite time trajectory, see Equation (25). For example consider the scalar (one dimensional system)

$$\ddot{x} = a_1\ x + a_2\ \dot{x} + f(t) \tag{30}$$

where

$$a_1 = a_2 = 0$$

or

$$\ddot{x} = f(t) \tag{31}$$

the state space formulation is

$$x_1 = x$$

$$x_2 = \dot{x}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \tag{32}$$

or the homogeneous system is

$$\dot{x}\rangle = A\ x\rangle$$

where A is rank one and is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0,1) = e\rangle_1\ {}^2\langle e \tag{33}$$

that is

$${}_1\langle e = (1,0)$$

$${}_2\langle e = (0,1)$$

note

$$A^2 = e\rangle_1{}^2\langle e \; e\rangle_1{}^2\langle e = [0] \tag{34}$$

$$\vdots \qquad\qquad \vdots$$

$$A^n \qquad\qquad [0]$$

Hence

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

or

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \Phi(t) \tag{35}$$

and

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0) \tag{36}$$

Clearly the system dynamic matrix A is rank one (a dyad) by Equation (33) and singular, by Equation (35) we see that Φ is non-singular (invertible).

Since the columns of Φ form a base, each $\phi(t)\rangle_j$ obeys Equation (1), or

$$\dot{\Phi} = A\,\Phi \tag{37}$$

where

$$\Phi(0) = I \tag{38}$$

Fundamental Solutions and Arbitrary Initial Condition Solution as Linear Combination of Fundamental Basis. This section obtains the homogeneous solution for an arbitrary initial condition as a linear combination of the time-varying fundamental base vectors which start with initial conditions "on" the "fixed" base vectors (non-time varying background bases).

Consider a sequence of $j \geq p$ trajectories generated by Equation (19)

$$\dot{X} = AX \tag{39}$$

with arbitrary initial conditions

$$X(0) = X \tag{40}$$

Since $\Phi(t)$ is a set of base vectors (columns form a base) the j^{th} vector has coordinates in the Φ base as

$$x(t)\rangle_j = \Phi(t)\ x^{\phi}_{(t)}\rangle_j \tag{41}$$

Taking the derivative of Equation (41)

$$\dot{x}\rangle_j = \dot{\Phi}\ x^{\phi}(t)\rangle_j + \Phi\dot{x}^{\phi}\rangle_j \tag{42}$$

Using Equation (19) and Equation (37) in Equation (42)

$$Ax\rangle_j = A\Phi x^{\phi}(t)\rangle_j + \dot{x}^{\phi}\rangle_j \tag{43}$$

and by Equation (41) in Equation (43)

$$A\Phi x^{\phi}\rangle_j = A\Phi x^{\phi}\rangle_j + \Phi\dot{x}^{\phi}\rangle_j \tag{44}$$

or

$$\dot{x}^{\phi}\rangle = 0\rangle \tag{45}$$

which implies that $x^{\phi}\rangle$ is a constant vector; that is the coordinates of the vector $x(t)\rangle_j$ in the fixed background base has constant coordinates in the fundamental base; furthermore by Equation (41) at time zero

$$x(0)\rangle = Ix^{\phi}(0)\rangle \tag{46}$$

For the non-homogeneous case the coordinates will be shown to be time varying in the Φ base in a later section.

Consider next the dynamics of the inverse fundamental system of Equation (37), where

$$\Phi\Phi^{-1} = I \tag{47}$$

or taking the derivative

$$\dot{\Phi}\Phi^{-1} + \Phi\dot{\Phi}^{-1} = 0 \tag{48}$$

or

$$\frac{d}{dt}(\Phi^{-1}) = - \Phi^{-1}\ \dot{\Phi}\Phi^{-1} \tag{49}$$

by Equation (37)

$$\dot{\Phi} = A\Phi \tag{50}$$

or

$$\dot{\Phi}\Phi^{-1} = A \tag{51}$$

and Equation (51) in Equation (49) yields

$$\frac{d}{dt}\ \Phi^{-1} = - \Phi^{-1}A \tag{52}$$

Transposing Equation (52) one obtains

$$\frac{d}{dt}\ \Phi^{-T} = - A^{T}\ \Phi^{-T} \tag{53}$$

where

$$\Phi^{-T} = (\Phi^{-1})^{T} \tag{54}$$

Packaging Equations (50) and (53)

$$\begin{bmatrix} \dot{\Phi} \\ \dot{\Phi}^{-T} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & -A^{T} \end{bmatrix} \begin{bmatrix} \Phi(t) \\ \Phi^{-T}(t) \end{bmatrix} \tag{55}$$

The fundamental inverse can be written as two ways

$$\Phi^{-1} = (\Phi^T\Phi)^{-1}\,\Phi^T = \Phi^T(\Phi\Phi^T)^{-1} \tag{56}$$

or

$$\Phi^{-T} = \Phi(\Phi^T\Phi)^{-1} = (\Phi\Phi^T)^{-1} \tag{57}$$

Define the two Grammian matrices

$$\Phi\Phi^T = G_{\phi i}(t) \tag{58}$$

and

$$\Phi^T\Phi = G_{\phi 0}(t) \tag{59}$$

if the two Grammians are equal Φ is said to be normal.

Using Equation (58) in Equation (57)

$$\Phi = G_{\phi i}\,\Phi^{-T} \tag{60}$$

Taking the time derivative

$$\dot{\Phi} = \dot{G}_{\phi i}\,\Phi^{-T} + G_{\phi i}\,\dot{\Phi}^{-T} \tag{61}$$

By Equation (55) in Equation (61)

$$A\Phi = \dot{G}_{\phi i}\,\Phi^{-T} + G_{\phi i}(-A^T\Phi^{-T}) \tag{62}$$

or

$$A\Phi\Phi^T = \dot{G}_{\phi i} + G_{\phi i}\,(-A^T) \tag{63}$$

and solving for $\dot{G}_{\phi i}$

$$\dot{G}_{\phi i} = A\,G_{\phi i} + G_{\phi i}\,A^T \tag{64}$$

which is a special case of the martix Riccatti, called the Lypunov relation.

The dynamics of the inverse Grammian $G_{\phi i}$ is obtained from

$$G_{\phi i}\, G_{\phi i}^{-1} = I \tag{65}$$

$$\dot{G}_{\phi i}\, G_{\phi i}^{-1} + G_{\phi i}\, \dot{G}_{\phi i}^{-1} = 0 \tag{66}$$

or

$$\dot{G}_{\phi i}^{-1} = - G_{\phi i}^{-1}\, \dot{G}_{\phi i}\, G_{\phi i}^{-1} \tag{67}$$

Using Equation (64) in Equation (67)

$$\dot{G}_{\phi i}^{-1} = - G_{\phi i}^{-1}\, [AG_{\phi i}+G_{\phi i}A^T]\, G_{\phi i}^{-1} \tag{68}$$

or

$$\boxed{\dot{G}_{\phi i}^{-1} = - G_{\phi i}^{-1}\, A - A^T\, G_{\phi i}^{-1}} \tag{69}$$

and by Equation (64)

$$\boxed{\dot{G}_{\phi i} = A\, G_{\phi i} + G_{\phi i}\, A^T} \tag{70}$$

By Equation (59) the time derivative is

$$\dot{G}_{\phi o}(t) = \dot{\Phi}^T\Phi + \Phi^T\dot{\Phi} \tag{71}$$

and by Equation (50)

$$\dot{G}_{\phi o} = \Phi^T\, A^T\Phi + \Phi^T A\Phi \tag{72}$$

$$\dot{G}_{\phi o} = \Phi^T(A^T+A)\Phi \tag{73}$$

The dynamics of $G_{\phi o}^{-1}$ is obtained from

$$G_{\phi o} G_{\phi o}^{-1} = I \tag{74}$$

or

$$\dot{G}_{\phi o} G_{\phi o}^{-1} + G_{\phi o} \dot{G}_{\phi o}^{-1} = 0 \tag{75}$$

or

$$\dot{G}_{\phi o}^{-1} = - G_{\phi o}^{-1} \dot{G}_{\phi o} G_{\phi o}^{-1} \tag{76}$$

and by Equation (59)

$$G_{\phi o}^{-1} = \Phi^{-1} \Phi^{-T} \tag{77}$$

and Equations (73) and (77) in Equation (76) yields

$$\dot{G}_{\phi o}^{-1} = - \Phi^{-1} (A^T+A) \Phi^{-T} \tag{78}$$

a congruent transformation.

Summarizing for the outer-Grammian dynamics by Equation (73) and Equation (78)

$$\begin{aligned} \dot{G}_{\phi o} &= \phi^T (A+A^T)\phi \\ \dot{G}_{\phi o}^{-1} &= - \Phi^{-1} (A+A^T) \Phi^{-T} \end{aligned} \tag{79}$$

Returning now to the arbitrary initial conditions for the homogeneous case of Equation (20) and Equation (21). The pseudo-inverse or dual system to Equation (21) is given by Equation (16) where we have two Gram matrices, again one p×p in size the other j×j in size, or

$$\underset{p\times j}{X} \; \underset{j\times p}{X^T} = G_{xxi}(t) \tag{80}$$

and

$$\underset{j\times j}{X^T X} = G_{xxo}(t) \tag{81}$$

One expression of Equation (16) is

$$X^*(XX^T) = X^T(XX^T)^*(XX^T) = X^T \tag{82}$$

since the projector $(XX^T)^*(XX^T)$ acts like the identity for the vector in the row space of X^T, hence transposing Equation (82).

$$X = (XX^T)X^{*T} = G_{xxi}X^{*T} \tag{83}$$

Taking the derivative of Equation (83)

$$\dot{X} = \dot{G}_{xxi}X^{*T} + G_{xxi}\dot{X}^{*T} \tag{84}$$

Full Rank X(t). The solution to the matrix equation

$$\dot{X} = AX(t) \tag{85}$$

is

$$X(t) = \Phi(t)\ X(o) \tag{86}$$

Since $\Phi(t)$ is full rank, the rank of $X(t)$ is determined by the rank of the initial condition matrix $X(o)$. If $X(o)$ has full rank, then Equation (86) is full rank factors of $X(t)$, hence the pseudo-inverse is

$$X^*_{(t)} = X^*(o)\ \Phi^{-1}(t) \tag{87}$$

Taking the derivative of Equation (87)

$$\dot{X}^*(t) = X^*(o)\ \dot{\Phi}^{-1}(t) \tag{88}$$

By Equation (52) in Equation (88)

$$\dot{X}^* = -\ X^*(o)\ \Phi^{-1}A \tag{89}$$

and by Equation (87) in Equation (89)

$$\dot{X}^* = - X^*(t)A \tag{90}$$

or

$$\dot{X}^{*T} = - A^T X^{*T}_{(t)} \tag{91}$$

The dynamics of the dual (pseudo-inverse transpose) can be obtained for the full rank case from the relation

$$\underset{p\times j}{X} \; \underset{j\times p}{X^*} = \underset{p\times p}{I} \tag{92}$$

or

$$\dot{X}X^* + X\dot{X}^* = 0 \tag{93}$$

or

$$X\dot{X}^* = - \dot{X}X^* \tag{94}$$

Multiply Equation (94) on left by X*

$$X^*X\dot{X}^* = - X^*\dot{X}X^* \tag{95}$$

Using Equation (85) in the right side of Equation (95) with Equation (92)

$$P_{xo}\dot{X}^* = - X^*AXX^* = - X^*A \tag{96}$$

By Equation (90)

$$P_{xo}\dot{X}^* = \dot{X}^* = - X^*A \tag{97}$$

or

$$\dot{X}^* = - X^*A \tag{98}$$

or transposing

$$\dot{X}^{*T} = -A^T X^{*T} \tag{99}$$

Packaging Equation (18) and Equation (99)

$$\underset{2p\times j}{\begin{bmatrix} \dot{X} \\ \dot{X}^{*T} \end{bmatrix}} = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} \underset{2p\times j}{\begin{bmatrix} X(t) \\ X^{*T}_{(t)} \end{bmatrix}} \tag{100}$$

The inner-Grammian by Equation (19) in Equation (80) is

$$G_{xxi}(t) = XX^T = \Phi(t)\ X(o)\ X^T(o)\ \Phi^T(t) \tag{101}$$

or

$$G_{xxi}(t) = \Phi(t)\ G_{xxi}(o)\ \Phi^T_{(t)} \tag{102}$$

The dynamics of the inner-Grammian can be obtained easily from

$$\dot{G}_{xxi}(t) = \dot{X}(t)\ X^T + X(t)\ \dot{X}^T \tag{103}$$

or

$$\dot{G}_{xxi}(t) = A\ G_{xxi}(t) + G_{xxi}(t)\ A^T \tag{104}$$

For the full-rank case

$$G_{xxi}\ G^{-1}_{xxi} = I \tag{105}$$

and one obtains

$$\dot{G}^{-1}_{xxi} = [G^{-1}_{xxi}A + A^T G^{-1}_{xxi}] \tag{106}$$

Consider next the commute of Equation (92), the outer projector

$$\dot{P}_{xo} = \frac{d}{dt}(X^*X) = \dot{X}^*X + X^*\dot{X} \tag{107}$$

By Equation (100)

$$\dot{P}_{xo} = - X^{*}AX + X^{*}AX = 0 \qquad (108)$$

Thus the time-derivative of the time varying outer-projector is zero when X is full rank p. Also it is obvious that for full rank p

$$\ddot{P}_{xo} = 0 \qquad (109)$$

<u>Homogeneous System Means Averages</u>. The dynamics of the unweighted mean of the j-trajectories of Equation (39) is

$$\underset{p\times j}{\dot{X}} \; 1^{*} = AX1^{*}\rangle \qquad (110)$$

where

$$1^{*}(j\rangle = 1(j\rangle \frac{1}{j} \qquad (111)$$

that is the pseudo-inverse of the sum vector $\langle j)1$ where

$$\langle j)\, 1 = \underset{i\times j}{(1,1,\cdots 1)} \qquad (112)$$

Define the mean velocity vector as

$$\dot{X}1^{*}\rangle = \dot{\mu}_x(t)(p\rangle \qquad (113)$$

and

$$X1^{*}\rangle = \mu_x(t)(p\rangle \qquad (114)$$

hence Equation (101) becomes

$$\dot{\mu}_x(t)(p\rangle = A\, \mu_x(t)\rangle \qquad (115)$$

Partition Equation (114) into its row space and we have

$$X1^*\rangle = \begin{bmatrix} {}^1\langle j)x \\ {}^2\langle j)x \\ \vdots \\ {}^p\langle j)x \end{bmatrix} 1^*(j\rangle = \begin{bmatrix} {}^1\langle j)x\ 1^*(j\rangle \\ \vdots \\ {}^p\langle j)x\ 1^*\rangle \end{bmatrix} = \begin{bmatrix} \mu_x^1(t) \\ \\ \mu_x^p(t) \end{bmatrix} \tag{116}$$

or in its column space

$$X1^*\rangle = \begin{bmatrix} x(p\rangle_1 \cdots X(p\rangle_j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \frac{1}{j} = \begin{bmatrix} \sum_{j=1}^{j\ max} x(p\rangle_j \end{bmatrix} \frac{1}{j\ max} = \tag{117}$$

$$\mu_x(t)(p\rangle$$

Subtracting the mean vectors from each of the j-trajectories, that is Equation (19) minus Equation (115).

$$\dot{x}\rangle_j - \dot{\mu}_x\rangle = \dot{\tilde{x}}\rangle_j = A\left(x\rangle_j - \mu_x\rangle \right) \tag{118}$$

Define the error vector

$$\tilde{x}\rangle_j = x\rangle_j - \mu_x\rangle \tag{119}$$

$$\dot{\tilde{x}}\rangle_j = \dot{x}\rangle_j - \dot{\mu}_x\rangle \tag{120}$$

or

$$\dot{\tilde{x}}\rangle_j = A\tilde{x}\rangle_j \tag{121}$$

or package-wise

$$\dot{\tilde{X}} = A\tilde{X} \tag{122}$$

Consider the package $\tilde{X}$ by Equation (119)

$$\tilde{X} = X - \mu_x\rangle\langle j)1 \qquad (123)$$

or

$$X = \mu_x\rangle\langle 1 + \tilde{X} \qquad (124)$$

The orthogonal projection in j space of the p-row vectors of X onto the $\langle j)1$ vector is shown in Figure (1)

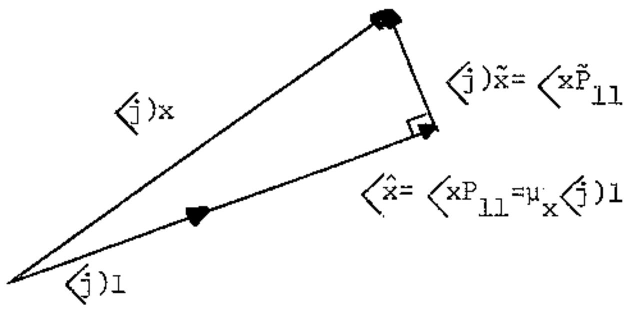

FIGURE (1)
PROJECTION IN j-SPACE

The projector P_{11} (rank-one)

$$P_{11} = \frac{1(j\rangle\langle j)1}{\langle 11\rangle} = \frac{1\rangle\langle 1}{j} = 1{*}\rangle\langle 1 \qquad (125)$$

applied to $\langle j)x$ yields for the p^{th} vector

$$\begin{aligned} {}^{p}\langle x\, P_{11} &= {}^{p}\langle x\; 1{*}\rangle\langle 1 \qquad (126) \\ &= \mu_x^p \langle 1 \\ &= {}^{p}\langle j)\hat{x} \end{aligned}$$

or by Figure (1)

$${}^{p}\langle j)x = {}^{p}\langle j)\hat{x} + {}^{p}\langle j)\tilde{x} \qquad (127)$$

or

$$ {}^{p}\langle\tilde{x} = {}^{p}\langle x - {}^{j}\langle\hat{x} = {}^{p}\langle x - {}^{p}\langle x P_{11} \tag{128} $$

or

$$ {}^{p}\langle j)\tilde{x} = {}^{p}\langle x\,[I-P_{11}] = {}^{p}\langle x\tilde{P}_{11} \tag{129} $$

where the orthogonal complement projector is

$$ \tilde{P}_{11} = I - P_{11} \tag{130} $$

Clearly by Equation (114) in Equation (124)

$$ \tilde{X} = X - X^{*}1\rangle\langle 1 \tag{131} $$

$$ \tilde{X} = X(I-1^{*}\rangle\langle 1) \tag{132} $$

$$ \tilde{X} = X\tilde{P}_{11} \tag{133} $$

Thus the $\hat{X}$ and $\tilde{X}$ imply orthogonal and orthogonal-complement projectors respectively, that is

$$ X = \hat{X} + \tilde{X} \tag{134} $$

where

$$ \hat{X} = XP_{11} = \mu_x(p\rangle\langle j)1 \tag{135} $$

is a rank-one dyad.

Take the derivative of Equation (135)

$$ \dot{\hat{X}} = \dot{X}P_{11} = AXP_{11} = A\hat{X} \tag{136} $$

Thus we see that X, $\hat{X}$ and $\tilde{X}$ all have the same velocity matrix A.

By Equation (136)

$$ \dot{\hat{X}} = \dot{\mu}_x\rangle\langle 1 \tag{137} $$

hence Equations (135) and (136) in Equation (137) yields

$$\dot{\mu}_x \rangle\langle 1 = A\ \mu_x \rangle\langle 1 \tag{138}$$

The dynamics of the error vectors of Equation (131) or the package (aggragate or ensemble of trajectories) is the same as Equation (136) except for the wiggle (~), hence the dynamics of the duals of the errors, the error Grammians, etc., are the same except for notation change, hence we have via Equation (100)

$$\begin{bmatrix} \dot{\tilde{X}} \\ \dot{\tilde{X}}^{*T} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \tilde{X}^{*T} \end{bmatrix} \tag{139}$$

If we define the average inner-Grammian as a finite variance matrix as

$$\underset{p\times p}{\dot{\Sigma}_{\tilde{x}}(t)} = \frac{1}{j_m} \tilde{X}\tilde{X}^T = \frac{1}{j_m} G_{i\tilde{x}} = \left[\sum_{j=1}^{j_m} \tilde{x}\rangle_j {}^j\langle \tilde{x} \right] \frac{1}{j_m} \tag{140}$$

The dynamics of the error variance is given by Equation (104) as

$$\dot{\Sigma}_{\tilde{x}} = A\ \Sigma_{\tilde{x}} + \Sigma_{\tilde{x}}\ A^T \tag{141}$$

and

$$\dot{\Sigma}_{\tilde{x}}^{-1} = -\ \Sigma_{\tilde{x}}^{-1}\ A - A^T\ \Sigma_{\tilde{x}}^{-1} \tag{142}$$

Note that in the above

$$\frac{d}{dt}(\Sigma^{-1}) = \dot{\Sigma}^{-1} \neq (\dot{\Sigma})^{-1} \tag{143}$$

By Equation (133)

$$\tilde{X} = X\tilde{P}_{11} \tag{144}$$

we see that the rank of $\tilde{X}$ is $j-1$ since

$$\rho \ \tilde{P}_{11} = j - 1 = \rho \ \underset{j\times j}{I} - P \tag{145}$$

The inner-Grammian is p×p and for the rank p case we have

$$\underset{p\times p}{\ddagger_{\tilde{x}}} = \frac{1}{j} \underset{2p\times j}{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}} \underset{j\times 2p}{\begin{bmatrix} X_1^T, X_2^T \end{bmatrix}} = \begin{bmatrix} \ddagger_{11} & \ddagger_{12} \\ \ddagger_{21} & \ddagger_{22} \end{bmatrix} \tag{146}$$

and in partitioned form the dynamics of Equation (141) is

$$\begin{bmatrix} \dot{\ddagger}_{11} & \dot{\ddagger}_{12} \\ \dot{\ddagger}_{21} & \dot{\ddagger}_{22} \end{bmatrix} = \begin{bmatrix} A_{11}\ddagger_{11}+A_{12}\ddagger_{21} & A_{11}\ddagger_{12}+A_{12}\ddagger_{22} \\ A_{21}\ddagger_{11}+A_{22}\ddagger_{21} & A_{21}\ddagger_{12}+A_{22}\ddagger_{22} \end{bmatrix} + \begin{bmatrix} \ddagger_{11}A_{11}^T+\ddagger_{12}A_{12}^T & \ddagger_{11}A_{21}^T+\ddagger_{12}A_{22}^T \\ \ddagger_{21}A_{11}^T+\ddagger_{22}A_{12}^T & \ddagger_{21}A_{21}^T+\ddagger_{22}A_{22}^T \end{bmatrix} \tag{147}$$

Note that $\dot{\ddagger}_{11}$ etc., are not simple matrix Riccatti equations like $\dot{\ddagger}$ of Equation (141).

The pseudo-inverse of Equation (144) can be written as

$$\underset{j\times p}{\tilde{X}^*} \quad \underset{j\times p}{\tilde{X}^T} \underset{p\times p}{(\tilde{X}\tilde{X}^T)^{-1}} = j_m \tilde{X}^T \underset{p\times p}{\ddagger_{\tilde{x}}^{-1}} \tag{148}$$

The inverse can be obtained by partitioning in reference (39) as

$$\begin{bmatrix} \ddagger_{11} & \ddagger_{12} \\ \ddagger_{21} & \ddagger_{22} \end{bmatrix}_{\tilde{x}}^{-1} \quad \begin{bmatrix} \ddagger_{11}^*(1,2) & \ddagger_{12}^*(1,2) \\ \ddagger_{21}^*(1,2) & \ddagger_{22}^*(1,2) \end{bmatrix} \tag{149}$$

where

$$\Phi^*_{11}(1,2) = \Phi^{-1}_{11} + \Phi^{-1}_{11}\,\Phi_{12}\,(\Phi_{22}-\Phi_{21}\Phi^{-1}_{11}\Phi_{12})^{-1}\,\Phi_{21}\,\Phi^{-1}_{11}$$

$$\Phi^*_{12}(1,2) = -\,\Phi^{-1}_{11}\,\Phi_{12}\,(\Phi_{22}-\Phi_{21}\Phi^{-1}_{11}\Phi_{12})^{-1}$$

$$\Phi^*_{21}(1,2) = -\,(\Phi_{22}-\Phi_{21}\Phi^{-1}_{11}\Phi_{12})^{-1}\,\Phi_{21}\,\Phi^{-1}_{11} \qquad (150)$$

$$\Phi^*_{22}(1,2) = (\Phi_{22}-\Phi_{21}\Phi^{-1}_{11}\Phi_{12})^{-1}$$

If we partition the full rank pseudo-inverse of Equation (148) as

$$\underset{j\times p}{\tilde{X}^*} = \begin{bmatrix} \underset{j\times n}{\tilde{X}^*_1(1,2)}, & \underset{j\times n}{\tilde{X}^*_2(1,2)} \end{bmatrix} \qquad (151)$$

where

$$p = 2n$$

then

$$\underset{p\times p}{\tilde{X}\tilde{X}^*} = \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} \begin{bmatrix} \tilde{X}^*_1(1,2), & \tilde{X}^*_2(1,2) \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{X}_1\tilde{X}^*_1(1,2) & \tilde{X}_1\tilde{X}^*_2(1,2) \\ \tilde{X}_2\tilde{X}^*_1(1,2) & \tilde{X}_2\tilde{X}^*_2(1,2) \end{bmatrix} \qquad (152)$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

or index-wise

$$\tilde{X}_i\,\tilde{X}^*_k(1,2) = I\,\delta_{ik} \qquad (153)$$

for i,k=1,2 or the biorthogonal conditions hold.

Equation (148) computes the package pseudo-inverse. Each of the partitioned matrix elements also has a pseudo-inverse computed as

$$\underset{j\times n}{\tilde{X}_1^*} = \underset{j\times n}{\tilde{X}_1^T} \underset{n\times n}{(\tilde{X}_1\tilde{X}_1^T)^{-1}}$$
$$\underset{j\times n}{\tilde{X}_2^*} = \tilde{X}_2^T (\tilde{X}_2\tilde{X}_2^T)^{-1} \qquad (154)$$

or in terms of the Grammians the package becomes

$$(\tilde{X}_1^*,\tilde{X}_2^*) = (\tilde{X}_1^T,\tilde{X}_2^T)\begin{bmatrix} G_{11}^{-1} & 0 \\ 0 & G_{22}^{-1} \end{bmatrix} \qquad (155)$$

The matrix product of Equation (155) with $\tilde{X}$ partitioned is

$$\begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} (\tilde{X}_1^*,\tilde{X}_2^*) = \begin{bmatrix} \underset{n\times n}{I} & \tilde{X}_1\tilde{X}_2^* \\ \tilde{X}_2\tilde{X}_1^* & \underset{n\times n}{I} \end{bmatrix} \qquad (156)$$

Note that

$$\underset{j\times n}{\tilde{X}_1^*} \neq \tilde{X}_1^*(1,2)$$
$$\tilde{X}_2^* \neq \tilde{X}_2^*(1,2) \qquad (157)$$

The package of <u>singleton-pseudos</u> will be designated as

$$\underset{j\times p}{\tilde{X}_s^*} = \begin{bmatrix} \tilde{X}_1^*,\tilde{X}_2^* \end{bmatrix} \qquad (158)$$

Equation (156) can also be written in terms of the Grammians via Equation (155) in Equation (156) as

$$\tilde{X}\tilde{X}_s^* = \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} \begin{bmatrix} \tilde{X}_1^T, \tilde{X}_2^T \end{bmatrix} \begin{bmatrix} G_{11}^{-1} & 0 \\ 0 & G_{11}^{-1} \end{bmatrix}$$

or

$$\tilde{X}\tilde{X}_s^* = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} G_{11}^{-1} & 0 \\ 0 & G_{22}^{-1} \end{bmatrix} = \begin{bmatrix} I & G_{12}G_{22}^{-1} \\ G_{21}G_{11}^{-1} & I \end{bmatrix} \tag{159}$$

Equation (154) can be written in terms of variances as

$$\begin{aligned} \tilde{X}_1^* &= \frac{\tilde{X}_1^T}{j_m} \left[\frac{\tilde{X}_1 \tilde{X}_1}{j_m} \right]^{-1} = \frac{\tilde{X}_1^T}{j_m} \Sigma_{11}^{-1} \\ \tilde{X}_2^* &= \frac{\tilde{X}_2^T}{j_m} \left[\frac{\tilde{X}_2 \tilde{X}_2}{j_m} \right]^{-1} = \frac{\tilde{X}_2^T}{j_m} \Sigma_{22}^{-1} \end{aligned} \tag{160}$$

and hence the package

$$\begin{bmatrix} \tilde{X}_1^*, \tilde{X}_2^* \end{bmatrix} = \frac{1}{j_m} \begin{bmatrix} \tilde{X}_1^T, \tilde{X}_2^T \end{bmatrix} \begin{bmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{bmatrix} \tag{161}$$

Using Equation (161) in Equation (159) we obtain the product as a function of the variances

$$\begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} \begin{bmatrix} \tilde{X}_1^*, \tilde{X}_2^* \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{bmatrix}$$

or

$$\tilde{X}\tilde{X}_s^* = \begin{bmatrix} I & \Phi_{12}\Phi_{22}^{-1} \\ \Phi_{21}\Phi_{11}^{-1} & I \end{bmatrix} \tag{162}$$

The off-diagonal terms are

$$\underset{n\times j}{\tilde{X}_1}\ \tilde{X}_2^* = \Phi_{12}\Phi_{22}^{-1} \tag{163}$$

and

$$\tilde{X}_2\tilde{X}_1^* = \Phi_{21}\Phi_{11}^{-1} \tag{164}$$

can be related to <u>correlation matrices</u> in very simple manner. Consider the 2n linearly independent vectors in

$$\underset{2n\times j}{\tilde{X}}$$

or the 2n row vectors in j-space, we can say for the first n

$$\underset{n\times j}{\tilde{X}_1} \ \varepsilon\ L^{(j,n)}\ [\tilde{X}]$$

where $L^{(j,n)}$ is an n-dimensional sub-space or linear manifold of j-space and [ ] implies <u>spanned-by</u>.

Like-wise for $\tilde{X}_2$. Decompose the n vector of $\tilde{X}_1$ into components lying in the subspace spanned by $\tilde{X}_2$ and components out of $\tilde{X}_2$ as

$$\tilde{X}_1 = C_{12}\tilde{X}_2 + \tilde{X}_{1\text{ out }2} \tag{165}$$

and like-wise

$$\tilde{X}_2 = C_{21}\ \tilde{X}_1 + \tilde{X}_{2\text{ out }1} \tag{166}$$

Multiply Equations (165) and (166) on right by X_2^* and X_1^* respectively

$$\tilde{X}_1\tilde{X}_2^* = C_{12} + \tilde{X}_{1\ out\ 2}\ \tilde{X}_2^* \tag{167}$$

$$\tilde{X}_2\tilde{X}_1^* = C_{21} + \tilde{X}_{2\ out\ 1}\ \tilde{X}_1^* \tag{168}$$

Equation (162) can be written via Equations (167) and (168) as

$$\tilde{X}\tilde{X}_s^* = \begin{bmatrix} I & C_{12} \\ C_{21} & I \end{bmatrix} + \begin{bmatrix} 0 & \tilde{X}_{1\ out\ 2}\tilde{X}_2^* \\ \tilde{X}_{2\ out\ 1}\tilde{X}_1^* & 0 \end{bmatrix} \tag{169}$$

If the n vectors of $\tilde{X}_1$ are orthogonally projected onto $\tilde{X}_2$ as shown in Figure [2]

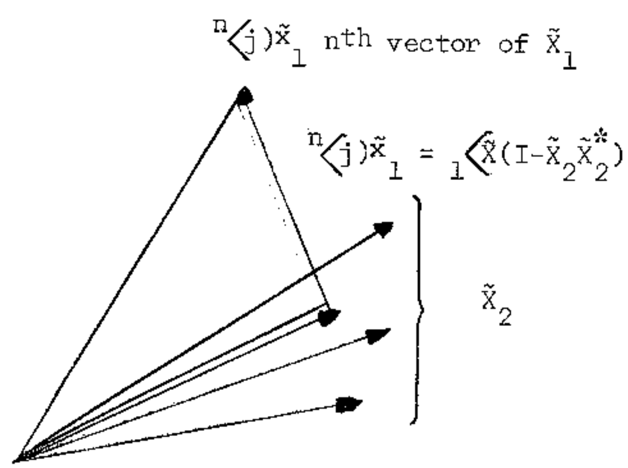

FIGURE [2]
ORTHOGONAL PROJECTIONS OF $\tilde{X}_1$ ONTO $\tilde{X}_2$

then for the package of vectors

$$\underset{n\times j}{\tilde{X}_1} = \tilde{X}_1 \underset{j\times j}{\left[\tilde{X}_2^{*}\tilde{X}_2\right]} + \tilde{X}_{1\text{ out }2\perp} \tag{170}$$

and likewise for $\tilde{X}_2$

$$\tilde{X}_2 = \tilde{X}_2 \underset{j\times j}{\left[\tilde{X}^{*}\tilde{X}_1\right]} + \tilde{X}_{2\text{ out }1\perp} \tag{171}$$

By the associative property for matrix products one obtains

$$\tilde{X}_1 = C_{12\perp}\,\tilde{X}_2 + \tilde{X}_{1\text{ out }2\perp} \tag{172}$$

$$\tilde{X}_2 = C_{21\perp}\,\tilde{X}_1 + \tilde{X}_{2\text{ out }1\perp} \tag{173}$$

where

$$C_{12\perp} = \tilde{X}_1\tilde{X}_2^{*} \tag{174}$$

$$C_{21\perp} = \tilde{X}_2\tilde{X}_1^{*} \tag{175}$$

One can normalize the vectors and come-up with the standard correlation coefficients as direction cosines between unit vectors or as angles between sub-spaces etc., but these geometrical concepts will not be pursued further here.

We can now obtain the dynamics of the <u>singleton-duals</u> $\dot{X}_s^{*}$ by Equation (161)

$$\dot{X}_s^{*} = \left[\dot{X}_1^{*},\dot{X}_2^{*}\right] = \frac{1}{j_m}\left[\dot{\tilde{X}}_1^T,\dot{\tilde{X}}_2^T\right]\begin{bmatrix} \Phi_{11}^{-1} & 0 \\ 0 & \Phi_{22}^{-1} \end{bmatrix}$$

$$+ \frac{1}{\dot{j}_m}\left[\tilde{X}_1^T,\tilde{X}_2^T\right]\begin{bmatrix} \dot{\Phi}_{11}^{-1} & 0 \\ 0 & \dot{\Phi}_{22}^{-1} \end{bmatrix} \tag{176}$$

By Equation (139)

$$\ddot{\tilde{X}}^T = \tilde{X}^T A^T \tag{177}$$

and by Equation (147) we see that $\dot{\Phi}_{11}$ and $\dot{\Phi}_{22}$ are nastily coupled relations with the needed dynamics $\dot{\Phi}_{11}^{-1}$ being difficult to come by.

The transformation between the singleton-duals and the package-duals can be obtained from Equation (161) and Equation (148) where

$$\frac{1}{J}\left[\tilde{X}_1^T, \tilde{X}_2^T\right] = \left[\tilde{X}_1^*(1,2), \tilde{X}_2^*(1,2)\right]\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \tag{178}$$

Using Equation (178) in Equation (161)

$$\left[\tilde{X}_1^*, \tilde{X}_2^*\right] = \left[\tilde{X}_1^*(1,2), \tilde{X}_2^*(12)\right]\begin{bmatrix} I & \Phi_{12}\Phi_{22}^{-1} \\ \Phi_{21}\Phi_{11}^{-1} & I \end{bmatrix} \tag{179}$$

or

$$X_s^* = X^*\begin{bmatrix} I & \Phi_{12}\Phi_{22}^{-1} \\ \Phi_{21}\Phi_{11}^{-1} & I \end{bmatrix} \tag{180}$$

which yields

$$XX_s^* = XX^*\begin{bmatrix} I & \Phi_{12}\Phi_{22}^{-1} \\ \Phi_{21}\Phi_{11}^{-1} & I \end{bmatrix} = \begin{bmatrix} I & \Phi_{12}\Phi_{22}^{-1} \\ \Phi_{21}\Phi_{11}^{-1} & I \end{bmatrix} \tag{181}$$

which agrees with Equation (159).

One can invert the matrix of Equation (181) to obtain the inverse maps of Equation (180)

The dynamics of the singleton-duals and the dynamics of the correlation matrices will not be obtained at this time. An indication of dynamics of the correlation matrices will be indicated, by Equation (172)

$$\begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & C_{21\perp} \\ C_{21\perp} & 0 \end{bmatrix} \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} + \begin{bmatrix} \tilde{X}_{1 \text{ out } 2\perp} \\ \tilde{X}_{2 \text{ out } 1\perp} \end{bmatrix} \tag{182}$$

or

$$\dot{\tilde{X}} = A\tilde{X} = \begin{bmatrix} 0 & \dot{C}_{12\perp} \\ \dot{C}_{21\perp} & 0 \end{bmatrix} \tilde{X} + \begin{bmatrix} 0 & C_{12\perp} \\ C_{21\perp} & 0 \end{bmatrix} A\tilde{X} + \begin{bmatrix} \dot{\tilde{X}}_{1 \text{ out } 2\perp} \\ \dot{\tilde{X}}_{2 \text{ out } 1\perp} \end{bmatrix} \tag{183}$$

or

$$A = \dot{C} + CA + \begin{bmatrix} \dot{\tilde{X}}_{1 \text{ out } 2\perp} \\ \dot{\tilde{X}}_{2 \text{ out } 1\perp} \end{bmatrix} \tag{184}$$

where

$$C = \begin{bmatrix} 0 & C_{12\perp} \\ C_{21\perp} & 0 \end{bmatrix} \tag{185}$$

The inverse of the matrix of Equation (181) can be written as

$$\begin{bmatrix} I & \Phi_{12}\Phi_{22}^{-1} \\ \Phi_{21}\Phi_{11}^{-1} & I \end{bmatrix}^{-1} = \begin{bmatrix} I+\Phi_{12}\Phi_{22}^{-1}\Phi_{21}\Phi_{11}^{-1} & -\Phi_{12}\Phi_{22}^{-1} \\ \Phi_{21}\Phi_{11}^{-1} & (I-\Phi_{21}\Phi_{11}^{-1}\Phi_{12}\Phi_{22}^{-1})^{-1} \end{bmatrix} \quad (186)$$

Consider an additional relation involving sequences of sequences, that is let

$$\underset{p\times j}{\dot{\tilde{X}}(t)_k} = A \underset{p\times j}{\dot{\tilde{X}}_k(t)} \quad (187)$$

for $k=1,2,\cdots n$, where

$$\tilde{X}_k(t) = \Phi(t)\tilde{X}_k(0) \quad (188)$$

Assume the initial conditions for two packages are related as

$$\underset{p\times j}{\tilde{X}(0)_{k+1}} = \underset{p\times p}{B} \underset{p\times j}{\tilde{X}(0)_k} \quad (189)$$

Where B is a constant matrix

$$\dot{\tilde{X}}_{k+1} = A \dot{\tilde{X}}_{k+1}(t) \quad (190)$$

and

$$\tilde{X}_{k+1}(t) = \Phi\tilde{X}_{k+1}(0) \quad (191)$$

by Equation (189) in Equation (191)

$$\tilde{X}_{k+1}(t) = \Phi B\tilde{X}(0)_k \quad (192)$$

By Equation (188)

$$\tilde{X}_k(0) = \Phi^{-1}\tilde{X}_k(t) \quad (193)$$

or

$$\tilde{X}_{k+1}(t) = \Phi(t)\ B\Phi^{-1}(t)\ \tilde{X}_k(t) \tag{194}$$

For the case of $\tilde{X}$ having rank n, the $\tilde{X}_{1\ \text{out}\ 2}$ of Equation (170) and Equation (171) are equal to zero and we see by Equation (172) and Equation (173) that

$$C_{12\perp} = C_{21\perp}^{-1} = T(t) \tag{195}$$

and the full rank factor of $\tilde{X}$ are

$$\underset{p\times j}{\tilde{X}} = \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} = \begin{bmatrix} I \\ T(t) \end{bmatrix} \tilde{X}_1(t) \tag{196}$$

The dynamics of T or $\dot{T}$ is discussed in the next section via Riccatti relations.

27. PARTITIONED HOMOGENEOUS DYNAMICAL SYSTEMS RICCATTI DIFFERENTIAL EQUATIONS AND ADJOINT SYSTEMS. Consider a sequence of linear coupled homogeneous differential equations

$$\dot{x}(p_1\rangle_j = \underset{p_1 x p_1}{A_{11}} x(p_1\rangle_{1j} + \underset{p_1 x p_2}{A_{12}} x(p_2\rangle_{2j} \qquad (1)$$

$$\dot{x}(p_2\rangle_{2j} = \underset{p_2 x p_1}{A_{21}} x(p_1\rangle_{1j} + \underset{p_2 x p_2}{A_{22}} x(p_2\rangle_{2j} \qquad (2)$$

where the index j generates the sequence $j \geq p$ and

$$p = p_1 + p_2$$

The package of j trajectories can be written as

$$\begin{bmatrix} \underset{p_1 x j}{\dot{X}_1} \\ \underset{p_2 x j}{\dot{X}_2} \end{bmatrix} = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \qquad (3)$$

or

$$\underset{pxj}{\dot{X}} = A \underset{pxj}{X} \qquad (4)$$

In this section many relations are established so in order to keep the notation simple we assume p1=p2=n, hence p=2n.

The solution of Equation (4) is

$$\underset{2nxj}{X(t)} = \underset{2nx2n}{\phi(t)} \underset{2nxj}{X(o)} \qquad (5)$$

Since ϕ has rank 2n, the rank of X is determined by the rank of X(o). Suppose X(o) has rank n, then full rank factors are

$$\underset{2nxj}{X(o)} = \underset{2nxn}{B_o} \underset{nxj}{C_o} \qquad (6)$$

Partition $\phi(t)$ as

$$\underset{2n \times 2n}{\phi(t)} = \begin{bmatrix} \underset{n \times 2n}{\phi_1} \\ \underset{n \times 2n}{\phi_2} \end{bmatrix} \tag{7}$$

then Equation (5) can be written as

$$\underset{2nxj}{X\ (t)} = \begin{bmatrix} \underset{nxj}{X_1\ (t)} \\ \underset{nxj}{X_2\ (t)} \end{bmatrix} = \begin{bmatrix} \underset{nx2n}{\phi_1}\ \underset{2nxj}{X\ (o)} \\ \underset{nx2n}{\phi_2}\ \underset{2nxj}{X\ (o)} \end{bmatrix} \tag{8}$$

Using Equation (6) in Equation (8)

$$X\ (t) = \begin{bmatrix} \phi_1\ B_o\ C_o \\ \phi_2\ B_o\ C_o \end{bmatrix} = \begin{bmatrix} Y_1\ C_o \\ \underset{nxn}{Y_2}\ \underset{nxj}{C_o} \end{bmatrix} \tag{9}$$

where

$$\underset{nxn}{Y_k} = \underset{nx2n}{\phi_k}\ \underset{2nxn}{B_o} \quad k=1,\ 2 \tag{10}$$

Equating elements of Equation (9) by proper selection of X_o

$$X_k\ (t) = Y_k\ (t)\ C_o \tag{11}$$

or

$$Y_1^{-1}\ X_1\ (t) = C_o = Y_2^{-1}\ X_2 \tag{12}$$

or

$$X_2 = Y_2\ Y_1^{-1}\ X_1 \tag{13}$$

or

$$\underset{nxj}{X_2} = \underset{nxn}{T\ (t)}\ \underset{nxj}{X_1} \tag{14}$$

where

$$T\ (t) = Y_2\ Y_1^{-1} \tag{15}$$

and

$$T^{-1}(t) = Y_1 \, Y_2^{-1} \tag{16}$$

Note also that

$$Y(t) = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \phi B_o \tag{17}$$

and

$$\dot{Y} = \dot{\phi} B_o = A \phi B_o \tag{18}$$

or

$$\dot{Y} = AY \tag{19}$$

Using Equation (14) in Equation (18)

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \underset{2n \times n}{\begin{pmatrix} I \\ T \end{pmatrix}} \underset{n \times j}{X_1} \tag{20}$$

or rank n factors which is the same as Equation (190) in the previous section.

Consider the dynamics of the time varying transformation between the matrices of Equation (14), that is

$$\underset{n \times j}{X_2} = T(t) \, X_1(t) \tag{21}$$

If X_1 is a full rank sequence then by Equation (21) and Equation (15)

$$X_2 \, X_1^{*} = T = Y_2 \, Y_1^{-1} \tag{22}$$

or

$$X_2 \, X_1^{t} (X_1 \, X_1^{t})^{-1} = T \tag{23}$$

also

$$G_{21x} \, G_{11x}^{-1} = T \tag{24}$$

where the inner-cross Grammian is

$$G_{21x} = X_2 \, X_1^{t} \tag{25}$$

Equation (22) and Equation (23) express the matrix factors of T having rank n.

The package of states by Equation (21) is

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} I_n \\ \underset{nxn}{T} \end{bmatrix} \underset{nxj}{X_1} \tag{26}$$

or the 2nxj matrix X(t) has rank n factors.

Taking the derivative of Equation (26) and using Equation (3)

$$\dot{X} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{T} \end{bmatrix} X_1 + \begin{bmatrix} I \\ T \end{bmatrix} \dot{X}_1 \tag{27}$$

By Equation (3)

$$\dot{X}_1 = (A_{11}, A_{12}) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = (A_{11}, A_{12}) \begin{bmatrix} I \\ T \end{bmatrix} X_1 \tag{28}$$

Using Equations (26) and (28) in Equation (27)

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I \\ T \end{bmatrix} X_1 = \begin{bmatrix} 0 \\ \dot{T} \end{bmatrix} X_1 + \begin{bmatrix} I \\ T \end{bmatrix} (A_{11}, A_{12}) \begin{bmatrix} I \\ T \end{bmatrix} X_1 \tag{29}$$

Multiply on right of Equation (29) with X_1^* and use full rank property

$$X_1 X_1^* = I \tag{30}$$

hence

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I \\ T \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{T} \end{bmatrix} + \begin{bmatrix} I \\ T \end{bmatrix} (A_{11} \; A_{12}) \begin{bmatrix} I \\ T \end{bmatrix} \tag{31}$$

or

$$\begin{bmatrix} A_{11} + A_{12} T \\ A_{21} + A_{22} T \end{bmatrix} = \begin{bmatrix} A_{11} + A_{12} T \\ \dot{T} + TA_{11} + TA_{12} T \end{bmatrix} \tag{32}$$

or

$$\dot{T} = -TA_{11} + A_{22}\,T - TA_{12}T + A_{21} \tag{33}$$

which is the Matrix Ricatti Differential Equation for the system. The initial condition for Equation (33) is given by Equation (22), it is quadratic and non-homogeneous

$$T\,(o) = X_2\,(o)\,X_1^{*}\,(o) = Y_2(o)\,Y_1^{-1}(o) \tag{34}$$

or

$$T(o) = X_2(o)X_1^{t}(o)\;G_{11x}^{-1}(o) \tag{35}$$

The exponential-form of the solution of Equation (4) is

$$X(t) = e^{At}X(o) = \phi_A\;X(o) \tag{36}$$

where

$$\phi_A(t) = e^{At} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \tag{37}$$

Note that

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \ldots \tag{38}$$

and

$$e^{At} \neq \begin{bmatrix} e^{A_{11}t} & e^{A_{12}t} \\ e^{A_{21}t} & e^{A_{22}t} \end{bmatrix} \tag{39}$$

but for Block diagonal or scalar diagonal systems

$$e^{\begin{pmatrix} a_{11} & o \\ o & a_{22} \end{pmatrix} t} = \begin{bmatrix} e^{a_{11}t} & o \\ o & e^{a_{22}t} \end{bmatrix} \tag{40}$$

Using Equations (26) and (37) in Equation (36)

$$\begin{bmatrix} X_1\,(t) \\ X_2\,(t) \end{bmatrix} = \begin{bmatrix} I \\ T \end{bmatrix} X_1 = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} I \\ T(o) \end{bmatrix} X_1\,(o) \tag{41}$$

or element-wise

$$X_1(t) = [\phi_{11} + \phi_{12} T(o)]X_1(o) \tag{42}$$

$$X_2(t) = [\phi_{21} + \phi_{22} T(o)]X_2(o) \tag{43}$$

By Equation (21)

$$X_1(o) = T^{-1}(o) X_2(o) \tag{44}$$

or Equation (44) in Equation (43)

$$X_2(t) = [\phi_{21} T^{-1}(o) + \phi_{22}]X_2(o) \tag{45}$$

By Equation (22)

$$T(t) = X_2(t)X_1^{*}(t) \tag{46}$$

and by Equation (42) for full-rank factors

$$X_1^{*}(t) = X_1^{*}(o)[\phi_{11} + \phi_{12} T(o)]^{-1} \tag{47}$$

Using Equations (47) and (43) in Equation (46)

$$T(t) = [\phi_{21} + \phi_{22} T(o)][\phi_{11} + \phi_{12} T(o)]^{-1} \tag{48}$$

where, by Equation (34)

$$T(o) = X_2(o)X_1^{t}(o)[X_1(o) X_1^{t}(o)]^{-1} \tag{49}$$

Clearly Equation (48) is a solution to the Ricatti Equation (33) in terms of the block elements ϕij of ϕ.

By Equation (22)

$$\underset{n x n}{T(t)} = X_2 X_1^{*} = Y_2 Y_1^{-1} \tag{50}$$

Using the following Grammians and cross-Grammians:

$$X_2 X_1^{t} = G_{x21i}$$

$$X_1 X_2^{t} = G_{x12i}$$

$$X_1 X_1^{t} = G_{x11i}$$

$$X_2 X_2^{t} = G_{x22i} \tag{51}$$

$$Y_1^t\ Y_2 = G_{y12i}$$

$$Y_1^t\ Y_1 = G_{y11i}$$

$$Y_2\ Y_2^t = G_{y22i}$$

By Equation (24)

$$T = G_{x21}G_{x11}^{-1} = G_{y21}\ G_{y11}^{-1} \tag{52}$$

By Equation (21)

$$X_2 = T(t)\ X_1 \tag{53}$$

and by Equation (15)

$$Y_2 = TY_1 \tag{54}$$

Multiply Equation (53) on left by X_2^*

$$X_2X_2^* = I = TX_1X_2^* \tag{55}$$

or

$$I = TX_1X_2^tG_{x22}^{-1} = TG_{x12}G_{x22}^{-1} \tag{56}$$

or

$$T = G_{x22}G_{x12}^{-1} \tag{57}$$

or by Equations (57) and (52)

$$T = G_{x21}G_{x11}^{-1} = G_{x22}G_{x12}^{-1} \tag{58}$$

and similar analysis for the Y's or

$$T(t) = G_{y21}G_{y11}^{-1} = G_{y22}\ G_{y12}^{-1} \tag{59}$$

One can now obtain the dynamics of T in terms of the dynamics of the Grammian and Cross-Grammian matrices of Equations (58) and (59) to arrive at the results of Equation (33); thus we would like to have the dynamics of the Grammians, Cross-Grammians and their inverses for the X and Y systems. The derivative of Equation (59) is

$$\dot{T} = \dot{G}_{y21}\ G_{y11}^{-1} + G_{y21}\ \dot{G}_{y11}^{-1} \tag{60}$$

By Equations (33) and (59) equated to Equation (60)

$$\dot{T} = -TA_{11} + A_{22}T - T A_{12} T + A_{21}$$

$$= \dot{G}_{y21} G_{y11}^{-1} + G_{y21} \dot{G}_{y11}^{-1}$$

$$= -G_{y21} G_{y11}^{-1} A_{11} + A_{22} G_{y2} G_{y11}^{-1}$$

$$-G_{y21} G_{y11}^{-1} A_{12} G_{y21} G_{y11}^{-1} + A_{21} \tag{61}$$

Multiplying Equation (61) on right by G_{y11}

$$\dot{G}_{y21} = -G_{y21} \dot{G}_{y11}^{-1} G_{y11} + G_{y21} B_{11}$$

$$+ A_{22} G_{y21} - G_{y21}(G_{y11}^{-1} A_{12})G_{y21} \tag{62}$$

$$+ A_{21} G_{y11}$$

where

$$B_{11} = G_{y11}^{-1} A_{11} G_{y11}$$

$$B_{22} = A_{22}$$

$$B_{12} = G_{y11}^{-1} A_{12}$$

$$B_{21} = A_{21} G_{y11} \tag{63}$$

or

$$\dot{G}_{y21} = - G_{y21} \dot{G}_{y11}^{-1} G_{y11} + G_{y21} B_{11}$$

$$+ B_{22} G_{y21} - G_{y21} B_{12} G_{y21}$$

$$+ B_{21} \tag{64}$$

The special cases for

$$\dot{G}_{y11}^{-1} = 0$$

or

$$\dot{G}_{y12}^{-1} = 0$$

will be discussed in later sections.

By Equation (4) we have

$$\dot{\phi} = A\,\phi \tag{65}$$

where

$$\underset{2n\times 2n}{\phi(o)} = I = \begin{bmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ I \end{pmatrix} \end{bmatrix} \tag{66}$$

Partition ϕ as

$$\underset{2n\times 2n}{\phi(t)} = [\underset{2n\times n}{\phi_1(t)},\ \underset{2n\times n}{\phi_2(t)}] \tag{67}$$

Form the inverse in terms of the partioned factors

$$\phi^{-1} = \phi^t\,(\phi\phi^t)^{-1} \tag{68}$$

The transpose of Equation (67) is

$$\phi^t = \begin{bmatrix} \phi_1^t \\ \phi_2^t \end{bmatrix} \tag{69}$$

and the Grammian is

$$\phi^t\phi = \begin{bmatrix} \phi_1^t\,\phi_1 & \phi_1^t\,\phi_2 \\ \phi_2^t\,\phi_1 & \phi_2^t\,\phi_2 \end{bmatrix} \tag{70}$$

or

$$\phi^t\phi = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = G_i \tag{71}$$

Using Equation (71) in Equation (68)

$$\phi^{-1} = \begin{bmatrix} \phi_1^t & G_i^{-1} \\ \phi_2^t & G_i^{-1} \end{bmatrix} = \begin{bmatrix} \underset{n\times 2n}{\phi_1^*} & (1,2) \\ \underset{n\times 2n}{\phi_2^*} & (1,2) \end{bmatrix} \tag{72}$$

By Equations (72) and (67)

$$\phi\phi^{-1} = I = (\phi_1,\ \phi_2)\begin{bmatrix} \phi_1^* & (1,2) \\ \phi_2^* & (1,2) \end{bmatrix} \tag{73}$$

or

$$I_{2n} = \underset{2nx2n}{\phi_1 \phi_1^*(1,2)} + \underset{2nx2n}{\phi_2 \phi_2^* (1,2)} \qquad (74)$$

also

$$\phi^{-1}\phi = \begin{bmatrix} \phi_1^*(1,2) \\ \phi_2^*(1,2) \end{bmatrix} (\phi_1, \phi_2) \qquad (75)$$

$$= \begin{bmatrix} \phi_1^*(1,2)\ \phi_1 & \phi_1^*(1,2)\ \phi_2 \\ \phi_2^*(1,2)\ \phi_1 & \phi_2^*(1,2)\ \phi_2 \end{bmatrix} \qquad (76)$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \qquad (77)$$

By Equation (74)

$$\underset{2nx2n}{\phi_2 \phi_2^*(1,2)} = I - \phi_1 \phi_1^*(1,2) \qquad (78)$$

$$P_{\phi 22}(1,2) = \tilde{P}_{\phi 11}(1,2) \qquad (79)$$

where

$$P_{\phi 11} = \underset{2nx2n}{\phi_1 \phi_1^*(1,2)} \qquad (80)$$

is a rank n projector since by Equation (76)

$$\phi_1^*(1,2)\ \phi_1 = \underset{nxn}{I} \qquad (81)$$

and

$$P_{\phi 22}(1,2) = \underset{2nx2n}{\phi_2 \phi_2^*(1,2)} \qquad (82)$$

By Equation (79) we see that the projectors are orthogonal complements that is

$$P_{\phi 22}(1,2)\ P_{\phi 11}(1,2) = \tilde{P}_{\phi 11}\ P_{\phi 11} = 0 \qquad (83)$$

Consider next the singleton Duals, that is

$$\underset{nx2n}{\phi_1^*} = (\phi_1^t \, \phi_1)^{-1} \, \phi_1^t = G_{\phi 11i}^{-1} \, \phi_1^t \tag{84}$$

$$\underset{nx2n}{\phi_2^*} = (\phi_2^t \, \phi_2)^{-1} \, \phi_2^t = G_{\phi 22i}^{-1} \, \phi_2^t \tag{85}$$

or package-wise

$$\phi_s^* = \begin{bmatrix} \phi_1^* \\ \phi_2^* \end{bmatrix} = \begin{bmatrix} G_{11}^{-1} & 0 \\ 0 & G_{22}^{-1} \end{bmatrix} \begin{bmatrix} \phi_1^t \\ \phi_2^t \end{bmatrix} \tag{86}$$

also

$$(\phi_1, \phi_2) = (\phi_1^{*t}, \phi_2^{*t}) \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix} \tag{87}$$

The matrix product of Equations (87) and (67)

$$\phi_s^* \, \phi = \begin{bmatrix} \phi_1^* \\ \phi_2^* \end{bmatrix} (\phi_1, \phi_2) = \begin{bmatrix} G_{11}^{-1} & 0 \\ 0 & G_{22}^{-1} \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} I & G_{11}^{-1} \, G_{12} \\ G_{22}^{-1} \, G_{21} & I \end{bmatrix} \tag{88}$$

The connection matrix between the two pseudo matrices is

$$\phi_s^* = M \, \phi^* \tag{89}$$

or

$$\phi_s^* \, \phi = M \tag{90}$$

which is Equation (88), hence

$$\begin{bmatrix} \phi_1^* \\ \phi_2^* \end{bmatrix} = \begin{bmatrix} I & G_{11}^{-1} \, G_{12} \\ G_{22}^{-1} \, G_{21} & I \end{bmatrix} \begin{bmatrix} \phi_1^*(1,2) \\ \phi_2^*(1,2) \end{bmatrix} \tag{91}$$

and by Equation (180), section (1) the inverse matrix relation gives

$$\begin{bmatrix} \phi_1^*(1,2) \\ \phi_2^*(1,2) \end{bmatrix} = \begin{bmatrix} I + G_{12}\, G_{22}^{-1}\, G_{21}\, G_{11}^{-1} & -G_{12}\, G_{22}^{-1} \\ -G_{21}\, G_{11}^{-1} & I - G_{21}\, G_{11}^{-1}\, G_{12}\, G_{22}^{-1} \end{bmatrix} \begin{bmatrix} \phi_1^* \\ \phi_2^* \end{bmatrix} \qquad (92)$$

The connection between the two singleton projectors with the projectors of Equations (80) and (81) can be established by Equation (91)

$$\phi_1\phi_1^* = P_{11} = \phi_1[\phi_1^*(1,2) + G_{11}^{-1}\, G_{12}\, \phi_2^*(1,2)]$$

$$P_{11} = \phi_1\, \phi_1^*(1,2) + \phi_1\, G_{11}^{-1}\, G_{12}\, \phi_2^*(1,2)$$

$$P_{11} = P_{11}(1,2) + \phi_1^*\, G_{12}\, \phi_2^*(1,2) \qquad (93)$$

and similarly for P_{22}. Note by anti-analogy with Equation (74),

$$P_{11} + P_{22} \neq I \qquad (94)$$

Block Orthogonal Subsytems of ϕ. Consider special case where the block of vectors in ϕ_1 are perpendicular to all of those vectors in block ϕ_2 (or subspace spanned by ϕ_2) that is by Equation (71) the full-rank Grammian is

$$\phi^t\phi = \begin{bmatrix} \phi_1^t\, \phi_1 & 0 \\ 0 & \phi_2^t\, \phi_2 \end{bmatrix} \qquad (95)$$

or

$$\underset{n\times n}{G_{12}} = \phi_1^t\, \phi_2 = 0 \qquad (96)$$

Using Equation (96) in Equation (93) we see that the projectors are equivalent, that is

$$P_{11} = P_{11}(1,2) \qquad (97)$$

$$P_{22} = P_{22}(1,2) \qquad (98)$$

and

$$P_{11} + P_{22} = I \qquad (99)$$

If we put the additional constraint that

$$\phi_1^t\, \phi_1 = \phi_2^t\, \phi_2 = I \qquad (100)$$

then Equation (86) becomes

$$\begin{bmatrix} \phi_1^* \\ \phi_2^* \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} \begin{bmatrix} \phi_1^t \\ \phi_2^t \end{bmatrix} = \begin{bmatrix} \phi_1^t \\ \phi_2^t \end{bmatrix} \qquad (101)$$

The dynamics of these two systems are derived in later sections.

Consider next the sequence of vectors contained in the subspace of $\phi_1(t)$, that is

$$x(o)(2n\rangle_{\phi 1j} = \underset{2nxn}{\phi_1(o)}\ x\ (o)\ (n\rangle_{\phi 1j} \qquad (102)$$

and package-wise

$$\underset{2nxj}{X_{\phi 1}(o)} = \underset{2nxn}{\phi_1(o)}\ \underset{nxj}{X_{\phi 1}(o)} \qquad (103)$$

or full-rank factors, or

$$\underset{2nxj}{X(o)} = \begin{bmatrix} I \\ O \end{bmatrix} X_{\phi 1}(o) = \begin{bmatrix} X_{\phi 1}(o) \\ O \end{bmatrix} \qquad (104)$$

as a time function

$$\underset{2nxj}{X_{\phi 1}(t)} = \phi\ X_{\phi 1}(o) \qquad (105)$$

or by Equation (104)

$$X_{\phi 1}(t) = [\phi_1, \phi_2] \begin{bmatrix} I \\ O \end{bmatrix} X_{1\phi}(o) \qquad (106)$$

$$\underset{2nxj}{X_{\phi 1}(t)} = \underset{2nxn}{\phi_1(t)}\ \underset{nxj}{X_{1\phi}(o)} \qquad (107)$$

or full-rank factors. If we now partition

$$X_{\phi 1}(t) = \begin{bmatrix} \underset{nxj}{X_1(t)} \\ \underset{nxj}{X_2(t)} \end{bmatrix} = \phi_1\ X_{1\phi 1}(o) \qquad (108)$$

and

$$X_{\phi 1}(o) = \begin{bmatrix} X_{1\phi 1}(o) \\ 0 \end{bmatrix} \tag{109}$$

The connection by Equation (21) is

$$\underset{nxj}{X_2(t)_{\phi 1}} = T_{\phi 1}(t)\ X_{1\phi 1}(t) \tag{110}$$

where

$$X_2(o)_{\phi 1} = T_{\phi 1}(o)\ X_{1\phi}(o) \tag{111}$$

By Equation (109)

$$0 = T_{\phi 1}(o) I \tag{112}$$

or

$$T_{\phi 1}(o) = 0 \tag{113}$$

The matrix Ricatti by Equation (33) is

$$\dot{T}_{\phi 1} = -T_{\phi 1}\ A_{11} + A_{22}\ T_{\phi 1} - T_{\phi 1}\ A_{12}\ T_{\phi 1} + A_{21} \tag{114}$$

with initial condition given by Equation (113). By Equation (48) we also have

$$T_{\phi 1}(t) = \phi_{21}(t)\ \phi_{11}^{-1}(t) \tag{115}$$

which is the solution to Equation (114) for the conditions of Equation (113).

Consider next a sequence of vectors belonging to the ϕ_2 subspace, that is

$$X(2)_{2\phi 2j} = \underset{2nxn}{\phi_2(o)}\ X(o)\ (n)_{2\phi 2j} \tag{116}$$

or for the system initial states

$$\underset{2nxj}{X_{\phi 2}(o)} = \phi_2(o) X_{\phi 2}(o) = \begin{bmatrix} 0 \\ I \end{bmatrix} \underset{nxj}{X_{2\phi 2}(o)} \tag{117}$$

$$X_{\phi 2}(t) = \phi(t)\, X_{\phi 2}(o) \tag{118}$$

and by Equation (118) in Equation (117)

$$X_{\phi 2}(t) = \phi(t) \begin{vmatrix} 0 \\ I \end{vmatrix} X_{2\phi 2}(o) \tag{119}$$

$$X_{\phi 2}(t) = \phi_2(t)\, X_{2\phi 2}(o) \tag{120}$$

If we apply Equation (53) to Equation (117) we obtain

$$\underset{nxj}{X_{2\phi 2}(o)} = T(o) X_{1\phi 2}(o) \tag{121}$$

or

$$I = T(o)\ 0 \tag{122}$$

an impossible condition. In this case we can do the following

$$X_{1\phi 2}(t) = T_{\phi 2}(t)\, X_{2\phi 2}(t) \tag{123}$$

for now we have

$$X_{1\phi 2}(o) = T_{\phi 2}(o)\, X_{2\phi 2}(o) \tag{124}$$

or

$$0 = T_{\phi 2}(o)\ I \tag{125}$$

if

$$T_{\phi 2}(o) = 0 \tag{126}$$

If

$$\begin{bmatrix} \dot{X}_{1\phi 2} \\ \dot{X}_{2\phi 2} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_{1\phi 2} \\ X_{2\phi 2} \end{bmatrix} \tag{127}$$

the time derivative of Equation (123) is

$$\dot{X}_{1\phi 2} = \dot{T}_{\phi 2}\, X_{2\phi 2} + T_{\phi 2}\, \dot{X}_{2\phi 2} \tag{128}$$

and after inserting elements of Equation (127) into Equation (128) one obtains

$$\dot{T}_{\phi 2} = A_{11} T_{\phi 2} - T_{\phi 2} A_{22} - T_{\phi 2} A_{21} T_{\phi 2} + A_{12} \qquad (129)$$

$$T_{\phi 2}(o) = 0$$

Contrast Equation (129) with Equation (114). By Equation (123)

$$X_{1\phi 2}(t) X_{2\phi 2}^{*}(t) = T_{\phi 2}(t) \qquad (130)$$

By Equation (119)

$$\begin{bmatrix} X_{1\phi 2}(t) \\ X_{2\phi 2}(t) \end{bmatrix} = \begin{bmatrix} \phi_{12}(t) X_{2\phi 2}(o) \\ \phi_{22}(t) X_{2\phi 2}(o) \end{bmatrix} \qquad (131)$$

The pseudo-inverse for full-rank factor is

$$X_{2\phi 2}^{*}(t) = X_{2\phi 2}^{*}(o) \phi_{22}^{-1}(t) \qquad (132)$$

and used in Equation (130)

$$T_{\phi 2}(t) = \phi_{12}(t) X_{2\phi 2}(o) X_{2\phi 2}^{*}(o) \phi_{22}^{-1}(t)$$

or

$$T_{\phi 2}(t) = \phi_{12}(t) \phi_{22}^{-1}(t) \qquad (133)$$

Solving Equation (115) and Equation (133) for ϕ_{21} and ϕ_{12} respectively

$$\phi_{21}(t) = T_{\phi 1}(t) \phi_{11}(t)$$

$$\phi_{12}(t) = T_{\phi 2}(t) \phi_{22}(t) \qquad (134)$$

Using Equation (134) in Equation (48)

$$T(t) = [T_{\phi 1}(t) \phi_{11}(t) + \phi_{22} T(o)][\phi_{11} + T_{\phi 2}(t) \phi_{22} T(o)]^{-1} \qquad (135)$$

which relates $T(t)$ to $T_{\phi 1}(t)$ and $T_{\phi 2}(t)$.

By Equations (46), (58), and (59) it is seen that $T(t)$ has factors in j space and in n-space.

$$T(t) = \underset{nxj}{X_2} \underset{jxn}{X_1^{*}} = G_{21} G_{11}^{-1} = G_{22} G_{12}^{-1} \qquad (136)$$

One can also obtain lower-upper triangular factors, Lancos factors, and eigenvalue factors as

$$T = LL^t = L_u D L_u^t = U\Lambda_\ell V^t$$
$$= E\Lambda E^{-1} \tag{137}$$

etc; the Lancos factors are orthogonal factors also known as the singular value decomposition. The derivatives of the above factors yield interesting relations, some of which are obtained in later sections.

By Equations (115) and (133) it is seen that $T_{\phi 1}$ and $T_{\phi 2}$ are only dependent on the submatrices of , hence one would suspect that the matrix Riccati's could be derived via partitioning of ϕ only. By Equation (65)

$$\dot{\phi} = (\dot{\phi}_1, \dot{\phi}_2) = A[\phi_1, \phi_2] \tag{138}$$

and

$$\dot{\phi}_1 = A\,\phi_1 \tag{139}$$

or

$$\begin{bmatrix} \dot{\phi}_{11} \\ \dot{\phi}_{21} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \tag{140}$$

or

$$\phi_{21}(t) = T_{\phi 1}\,\phi_{11}(t) \tag{141}$$

and at time zero

$$\phi_{21}(o) = T_{\phi 1}(o)\,\phi_{11}(o) \tag{142}$$

hence

$$T_{\phi 1}(o) = 0 \tag{143}$$

By taking the derivation of Equation (141) one obtains the Riccati's relation of Equation (115)

<u>Dynamics of Singleton Pseudo-Inverses</u>. The dynamics of the singleton pseudo-inverses matrices (or duals) are obtained by Equation (44)

$$X_2 = TX_1 \tag{144}$$

and for full rank T

$$T^{-1} X_2 = X_1 \tag{145}$$

or

$$T^{-1} = X_1 X_2^* \tag{146}$$

Multiply Equation (146) by X_1^*

$$X_1^* T^{-1} = X_1^* X_1 X_2^* \tag{147}$$

$$X_1^* T^{-1} = P_{11} X_2^* \tag{148}$$

Multiply Equation (144) by P_{11}

$$X_2 P_{11} = T X_1 P_{11} = T X_1 \tag{149}$$

since

$$X_1 P_{11} = X_1 \tag{150}$$

By Equations (144) and (149)

$$X_2 = X_2 P_{11} \tag{151}$$

or

$$X_2^t = P_{11} X_2^t \tag{152}$$

Equation (148) becomes

$$X_1^* T^{-1} = P_{11} X_2^t (X_2 X_2^t)^{-1} \tag{153}$$

Using Equation (152) in Equation (153)

$$X_1^* T^{-1} = X_2^* \tag{154}$$

or transposing

$$X_2^{*t} = T^{-t} X_1^{*t} \tag{155}$$

By the identity relation

$$TT^{-1} = I \tag{156}$$

one obtains

$$\dot{T}\,T^{-1} + T\,\dot{T}^{-1} = 0 \tag{157}$$

or

$$\dot{T}^{-1} = -T^{-1}\,\dot{T}\,T^{-1} \tag{158}$$

By Equation (33)

$$\dot{T}^{-1} = -T^{-1}[-TA_{11} + A_{22}T - TA_{12}T + A_{21}]T^{-1}$$

or

$$\dot{T}^{-1} = A_{11}T^{-1} - T^{-1}A_{22} - T^{-1}A_{21}T^{-1} + A_{12} \tag{159}$$

or transposing

$$\dot{T}^{-t} = T^{-t}A_{11}^{\ t} - A_{22}^{\ t}T^{-t} + A_{12}^{\ t} - T^{-t}A_{21}T^{-t} \tag{160}$$

From the general form of the Riccati Equation of Equation (33) and the dynamics of Equation (3) and the connection of Equation (14) we have the associated dynamics for the Riccati Equation of Equation (160) and the connection matrix of Equation (155)

$$\begin{bmatrix} \dot{X}_1^{\ *t} \\ \dot{X}_2^{\ *t} \end{bmatrix} = \begin{bmatrix} -A_{11}^{\ t} & A_{21}^{\ t} \\ A_{12}^{\ t} & -A_{22}^{\ t} \end{bmatrix} \begin{bmatrix} X_1^{\ *t} \\ X_2^{\ *t} \end{bmatrix} \tag{161}$$

The singleton pseudo-inverses are given by Equation (160), Section ( ) as

$$\begin{bmatrix} X_1^{\ *t} \\ X_2^{\ *t} \end{bmatrix} = \begin{bmatrix} G_{11}^{-1} & 0 \\ 0 & G_{22}^{-1} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{162}$$

or

$$X_s^{\ *t} = \text{dia}(G_{ii}^{-1})X \tag{163}$$

where the diagonal block matrix is

$$\text{dia}(G_{ii}^{-1}) = \begin{bmatrix} G_{11}^{-1} & 0 \\ 0 & G_{22}^{-1} \end{bmatrix} \tag{164}$$

Taking the derivative of (163)

$$\dot{X}_s^{*t} = \frac{d}{dt}\, \text{dia}(G_{ii}^{-1})X + \text{dia}(G_{ii}^{-1})\dot{X} \tag{165}$$

and

$$\dot{X}_s^{*t} = \left[\frac{d}{dt}\, \text{dia}(G_{ii}^{-1})\ \text{dia}\ G_1 + \text{dia}(G_{ii}^{-1})A\ \text{dia}\ G\right] X_s^{*t} \tag{166}$$

or in open form

$$\begin{bmatrix} \dot{X}_1^{*t} \\ \dot{X}_2^{*t} \end{bmatrix}_s = \begin{bmatrix} \dot{G}_{11}^{-1} & 0 \\ 0 & \dot{G}_{22}^{-1} \end{bmatrix} \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix} + \begin{bmatrix} G_{11}^{-1} & 0 \\ 0 & G_{22}^{-1} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix} \begin{bmatrix} X_1^{*t} \\ X_2^{*t} \end{bmatrix} \tag{167}$$

also combining matrices

$$\begin{bmatrix} \dot{X}_1^{*t} \\ \dot{X}_2^{*t} \end{bmatrix} = \begin{bmatrix} \dot{G}_{11}^{-1}G_{11} + G_{11}^{-1}A_{11}G_{11} & G_{11}^{-1}A_{12}G_{22} \\ G_{22}^{-1}A_{21}G_{11} & \dot{G}_{22}^{-1}G_{22} + G_{22}^{-1}A_{22}G_{22} \end{bmatrix} \begin{bmatrix} X_1^{*t} \\ X_2^{*t} \end{bmatrix} \tag{168}$$

Dynamics of X^* Complete Pseudo-Inverse For Non-Full Rank Case. The dynamics of the Generalized inverse for the non-full rank case of Equation (20), that is

$$\underset{2nxj}{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} I \\ T \end{bmatrix} X_1 = F_t\, X_1(t) \tag{169}$$

where

$$F_t(t) = \begin{bmatrix} I \\ T(t) \end{bmatrix}$$

has rank n. The pseudo-inverse of full-rank factors of Equation (169) is

$$X^* = X_1^*\ F_t^* \tag{170}$$

The pseudo-inverse is given by

$$F_t^* = (F_t^t\ F_t)^{-1}\ F_t^t \tag{171}$$

where

$$F_t^{\,t} F_t = (I,\ T^t)\begin{pmatrix} I \\ T \end{pmatrix} = I + T^t\ T \qquad (172)$$

Consider the derivative of Equation (169)

$$\dot{X} = \dot{F}_t X_1 + F_t \dot{X}_1 = AX \qquad (173)$$

By Equation (3)

$$\dot{F}_t\ X_1 = F_t(A_{11},\ A_{12})\ F_t\ X_1 + A\ F_t\ X_1 \qquad (174)$$

Multiply Equation (174) on left by X_1^*

$$\dot{F}_t = -F_t(A_{11},\ A_{12})F_t + A\ F_t \qquad (175)$$

which is a rectangular matrix Riccati differential equation.

Partition the matrix A as

$$A = \begin{bmatrix} \underset{n\times 2n}{A_1} \\ \underset{n\times 2n}{A_2} \end{bmatrix} \qquad (176)$$

and Equation (175) becomes

$$\dot{F}_t = -F_t A_1 F_t + AF_t \qquad (177)$$

One can take the derivative of Equation (170)

$$\dot{X}_1^{\,*} = \dot{X}_1^{\,*} F_t^{\,*} + X_1^{\,*} \dot{F}_t^{\,*} \qquad (178)$$

and attempt to obtain $\dot{F}_t^*$ from

$$F_t^{\,*}\ F_t = I$$

or

$$\dot{F}_t^{\,*} F_t + F_t^{\,*} \dot{F}_t = 0$$

etc; however the effort becomes difficult and will not be pursued further here.

Adjoint Dynamical Systems. Consider Equation (23)

$$T = X_2 X_1^{\,t}(X_1 X_1^{\,t})^{1} = G_{21} G_{11}^{-1} \qquad (179)$$

for the special case of

$$X_2 X_1^t = G_{210} \text{ (a constant)} \tag{180}$$

then

$$T = G_{210} G_{11}^{-1}(t) \tag{181}$$

and

$$X_2(t) = G_{210} G_{11}^{-1}(t) X_1(t) \tag{182}$$

By Equation (162)

$$X_2(t) = G_{210} X_1^{*t}(t) \tag{183}$$

The time derivative of Equation (183) is

$$\dot{X}_2(t) = G_{210} \dot{X}_1^{*t} \tag{184}$$

By Equation (183) one has

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1(t) \\ G_{210} X_1^{*t}(t) \end{bmatrix} \tag{185}$$

and by Equation (184)

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} \dot{X}_1 \\ G_{210} \dot{X}_1^{*t} \end{bmatrix} \tag{186}$$

By Equation (183)

$$X_2^{*t} = G_{210}^{-t} X_1(t) \tag{187}$$

and the derivative is

$$\dot{X}_2^{*t} = G_{210}^{-t} \dot{X}_1(t) \tag{188}$$

By Equation (187) the package of singleton-pseudos is

$$\begin{bmatrix} X_1^{*t} \\ X_2^{*t} \end{bmatrix} = \begin{bmatrix} X_1^{*t} \\ G_{210}^{-t} X_1(t) \end{bmatrix} \tag{189}$$

and at the rate level by Equation (188)

$$\begin{bmatrix} \dot{X}_1^{*t} \\ \dot{X}_2^{*t} \end{bmatrix} = \begin{bmatrix} \dot{X}_1^{*t} \\ G_{210}^{-t}\dot{X}_1 \end{bmatrix} \tag{190}$$

Using Equations (185) and (186) in Equation (3)

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} \dot{X}_1 \\ G_{210}\dot{X}_1^{*t} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_1(t) \\ G_{210}\ X_1^{*t} \end{bmatrix} \tag{191}$$

and by Equation (161)

$$\begin{bmatrix} \dot{X}_1^{*t} \\ \dot{X}_2^{*t} \end{bmatrix} = \begin{bmatrix} -A_{11}^{\ t} & A_{21}^{\ t} \\ A_{12}^{\ t} & -A_{22}^{\ t} \end{bmatrix} \begin{bmatrix} X_1^{*t} \\ X_2^{*t} \end{bmatrix} \tag{192}$$

Using Equations (189) and (190) in Equation (192)

$$\begin{bmatrix} \dot{X}_1^{*t} \\ G_{210}^{-t}\ \dot{X}1 \end{bmatrix} = \begin{bmatrix} -A_{11}^{\ t} & A_{21}^{\ t} \\ A_{12}^{\ t} & -A_{22}^{\ t} \end{bmatrix} \begin{bmatrix} X_1^{*t}(t) \\ G_{210}^{-t}\ X_1(t) \end{bmatrix} \tag{193}$$

By Equation (191)

$$\dot{X}_1 = A_{11}X_1 + A_{12}G_{210}X_1^{*t} \tag{194}$$

By Equation (193)

$$G_{210}^{-t}\ \dot{X}_1 = A_{12}^t\ X_1^{*t} - A_{22}^{\ t}\ G_{210}^{-t}\ X_1 \tag{195}$$

or multiplying Equation (195) by G_{210}^t on left

$$\dot{X}_1 = G_{210}^t\ A_{12}^{\ t}\ X_1^{*t} - G_{210}^{\ t}\ A_{22}^{\ t}\ G_{210}^{-t}\ X_1(t) \tag{196}$$

Equating coefficients of Equation (196) and Equation (194) one obtains

$$A_{12}^{\ t} = G_{210}^{-t}\ A_{12}\ G_{210} \tag{197}$$

and

$$-A_{22}^{\ t} = G_{210}^{-t}\ A_{11}\ G_{210}^{\ t} \tag{198}$$

or transposing

$$-A_{22} = G_{210}\ A_{11}^{\ t}\ G_{210}^{-1} \tag{199}$$

also by Equation (191)

$$G_{210}\ \dot{X}_1^{*t} = A_{21}\ X_1 + A_{22}\ G_{210}\ X_1^{*t} \tag{200}$$

and by Equation (193)

$$\dot{X}_1^{*t} = -A_{11}^t\ X_1^{*t} + A_{21}^t\ G_{210}^{-t}\ X_1 \tag{201}$$

Multiply Equation (200) on left by G_{210}^{-1}

$$\dot{X}_1^{*t} = G_{210}^{-1}\ A_{21}\ X_1 + G_{210}^{-1}\ A_{22}\ G_{210}\ X_1^{*t} \tag{202}$$

Equating coefficients of Equations (201) and (202)

$$A_{21}^t = G_{210}^{-1}\ A_{21}\ G_{210}^t \tag{203}$$

and

$$-A_{11}^t = G_{210}^{-1}\ A_{22}\ G_{210} \tag{204}$$

Using Equations (197), (199), and (203) in Equation (193)

$$\begin{bmatrix} \dot{X}_1^{*t} \\ \dot{X}_2^{*t} \end{bmatrix} = \begin{bmatrix} -A_{11}^t & G_{210}^{-1}\ A_{21}\ G_{210} \\ G_{210}^{-1}\ A_{12}\ G_{210} & -G_{210}\ A_{11}^t\ G_{210}^{-1} \end{bmatrix} \begin{bmatrix} X_1^{*t} \\ G_{210}^{-t}\ X_1 \end{bmatrix} \tag{205}$$

The two systems are now Equations (191) and (205); there is only one independent variable $X_1(t)$, hence the coupled system can be written by use of Equation (198) for A_{22} in Equation (191)

$$\begin{bmatrix} \dot{X}_1 \\ G_{210}\ \dot{X}_1^{*t} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & -G_{210}\ A_{11}^t\ G_{210}^{-1} \end{bmatrix} \begin{bmatrix} X_1 \\ G_{210}\ X_1^{*t} \end{bmatrix} \tag{206}$$

If

$$G_{210} = X_2\ X_1^t = I$$

then

$$T(t) = (X_1 X_1^t)^{-1} = G_{11}^{-1} \tag{208}$$

By Equation (197)

$$A_{12}^t = A_{21} \tag{209}$$

is symmetric and by Equation (203)

$$A_{21}^{t} = A_{21} \tag{210}$$

is symmetric, hence by Equation (206)

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_1^{*t} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & -A_{11}^{t} \end{bmatrix} \begin{bmatrix} X_1 \\ X_1^{*t} \end{bmatrix} \tag{211}$$

where

$$X_1^{*t} = G_{11}^{-1} X_1 \tag{212}$$

Clearly now by the Riccati equation Equation (33) one obtains

$$\dot{T} = -TA_{11} - A_{11}^{t} T - TA_{12} T + A_{21} \tag{213}$$

Likewise any Riccati equation of the form of Equation (213) implies a coupled dynamical system like Equation (211).

there are a number of other interesting constraints one could pursue at this point, some of which are presented below:

What is the nature of things geometrically, etc., when

$$X_2 X_1^{t} = 0 \tag{214}$$

What if the G_{22} metric equals the G_{11} metric, then

$$X_2^{t} = X_1^{t} T^{t}$$

or

$$X_2 X_2^{t} = T X_1 X_1^{t} T^{t}$$

or

$$G_{22} = T G_{11} T^{t} = G_{11} \tag{215}$$

a congruent auto-morph constraint.

Clearly for the case of Equation (212)

$$G_{22} = G_{11}^{-1} \tag{216}$$

What if

$$G_{11}(t) = G_{110} \tag{217}$$

a constant like a rigid base or in particular

$$G_{110} = I \tag{218}$$

An ON base (time varying). For case of Equation (211)

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & -A_{11}^{t} \end{bmatrix} = A_H$$

$$A_{12} = A_{12}^{t} \quad , A_{21} = A_{21}^{t}$$

$A = A_h$ is said to be Hamiltonian via Bucy pp (67)

$$e^{Ah} = \phi_h$$

is sympletic.

Rectangular Matrix Riccati. Consider the partitioning of Equation (3) for the two cases: (1) $p_1 \le p_2$ and (2) $p_1 \ge p_2$.

Case (1): for $p_1 \le p_2$ Equation (14) becomes

$$\underset{p_2 \times j}{X_2} = \underset{p_2 \times p_1}{T} \; \underset{p_1 \times j}{X_1} \tag{219}$$

and for j=1 a single vector, Equation (20) becomes

$$\begin{bmatrix} x(t)\rangle_1 \\ x(t)\rangle_2 \end{bmatrix} = \begin{bmatrix} \underset{p_1 \times p_1}{I} \\ \underset{p_2 \times p_1}{T(t)} \end{bmatrix} x(t)\rangle_1 = \underset{(p_1 \times p_2)p_1}{F_T(t)} \quad x(t)\rangle_1 \tag{220}$$

Equation (41) becomes for the sequence

$$\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}_t = \begin{bmatrix} I \\ T(t) \end{bmatrix} X_1(t) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} I \\ T(o) \end{bmatrix} X_1(o) \tag{221}$$

One can interpret Equation (220) as all solutions to

$$\dot{X} = AX \tag{222}$$

such that

$$x(t)\rangle_T = F_T(t)\; x(t)\rangle_1 \tag{223}$$

or also

$$x(t)\rangle_2 = T(t)\; x(t)\rangle_1 \tag{224}$$

which states that the second p_2-tuple of coordinates are a function of the first p_1-tuple of coordinates.

By Equation (220) we are talking about the solutions to Equation (222) constrained to lie in the time-varying subspace of dimension p_1 or less (or spanned by the p_1 column vectors of $F_T(t)$. The dynamics of $F(t)$ is

$$\dot{F}_T(t) = \begin{bmatrix} 0 \\ \dot{T}(t) \end{bmatrix} \tag{225}$$

or by Equation (175)

$$\dot{F}_t = - F_t\,(A_{11}, A_{12})\, F_t + AF_t \tag{226}$$

For the case $F_t(t)$ is full rank, we have a time-varying base.

By Equation (28)

$$\dot{X}_1(t) = (A_{11}, A_{12}) \begin{bmatrix} I \\ T(t) \end{bmatrix} X_1(t) \tag{227}$$

or

$$\dot{X}_1 = \underset{p_1 \times p_1}{V_1(t)}\; X_1(t) \tag{228}$$

where the velocity matrix is

$$V_1(t) = (A_{11}, A_{12}) \begin{bmatrix} I \\ T(t) \end{bmatrix} \tag{229}$$

an "inner product" or map-down.

One may also ask about

$$\underset{p_1 \times j}{\dot{X}_1(t)} = \underset{p_1 \times j}{X_1(t)} \; \underset{j \times j}{V_0(t)} \tag{230}$$

The solution to Equation (228) for time-varying coefficients is

$$X_1(t) = e^{\int_0^t V_1(t)\, dt} \; X_1(0) \tag{231}$$

hence if $X_1(0)$ is full rank that is p_1 then $X_1(t)$ remains full rank. The rank of $T(t)$ is determined by Equation (219) with $T(t)$ having factors

$$\underset{p_2 \times p_1}{X_2(t) X_1^*(t)} = \underset{p_2 \times p_1}{T(t)} \tag{232}$$

By Equation (29)

$$\dot{X}_2(t) = (A_{21}, A_{22}) \begin{bmatrix} I \\ T \end{bmatrix} X_1(t) \tag{233}$$

or

$$\dot{X}_2(t) = V_{21}(t)\; X_1(t) \tag{234}$$

Equations (228) and (234) show the constraints as

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} V_1(t) \\ V_{21}(t) \end{bmatrix} X_1(t) \tag{235}$$

Note the initial conditions

$$\begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} = \begin{bmatrix} I \\ T_0 \end{bmatrix} X_1(0) \tag{236}$$

are fully determined by the rank of $X_1(0)$ since F_0 is always full rank.

Suppose $j=p_1$ and

$$\underset{p_1 \times p_1}{X_1(0)} = \underset{p_1 \times p}{I} \tag{237}$$

and Equation (238) becomes

$$\underset{p_1 \times p_1}{\dot{X}_1(t)} = V_1(t)\ X_1(t) \tag{238}$$

subject to Equation (237) or a fundamental solution in p_1 space. One can make further studies on the rank of T(t) and the dynamics of T*(t) as well as the dynamics of the two projectors

$$\frac{d}{dt}(TT^*) = ?$$

and

$$\frac{d}{dt}(T^*T) = ?$$

<u>Case (2)</u>: if one wants to partition Equation (3) such that $p_1<p_2$ reverse the roles by obtaining

$$\underset{p_1 \times j}{X_1} = \underset{p_1 \times p_2}{L} \underset{p_2 \times j}{X_2} \tag{239}$$

or

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \underset{p_1 \times p_2}{L} \\ \underset{p_2 \times p_2}{I} \end{bmatrix} \underset{p_2 \times j}{X_2} \tag{240}$$

and obtain corresponding results except one obtains

$$X_1 X_2^* = \underset{p_1 \times p_2}{L} \tag{241}$$

and selects initial conditions or $X_2(t)$ such that $X_2(t)$ is always full rank. Clearly $\dot{L}$ will be different from $\dot{T}$. Another interesting observation is when

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{242}$$

and by Equation (227)

$$\dot{X}_1 = T(t)\, X_1(t) \tag{243}$$

That is the velocity matrix is $T(t)$.

28. HOMOGENEOUS SYSTEM DIAGONALIZATION OR DECOUPLING OF TRIANGULAR SYSTEMS

There are a number of methods for block Diagonalizing a matrix, a few of which will be derived in this section. The papers of Friedland are perhaps best known on bias estimation to the practitioner of the estimation game.

The first case to be considered will be a block triangular system, with the generalization case deferred to a later part of the section.

Consider the system of Eq (3) sec (2) with

$$A_{21} = 0 \tag{1}$$

or

$$\begin{bmatrix} \dot{X}\rangle_1 \\ \dot{X}\rangle_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} X\rangle_1 \\ X\rangle_2 \end{bmatrix} \tag{2}$$

as before for sequences of trajectores, these become matrix equations or

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{3}$$

If one wants to solve the particular uncoupled system

$$\begin{bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \tag{4}$$

and generate the solution X_1 as a linear combination of the uncoupled solutions,

$$X_1 = Y_1 + F_{12}\, Y_2 \tag{5}$$

$$= (I,\ F_{12}) \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \tag{6}$$

and

$$X_2 = Y_2 = (0,\ I)\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \tag{7}$$

Packaging Eq (6) and Eq (7) and calling <u>this case i</u>

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} I & F_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \tag{8}$$

or

$$\underset{2nxj}{X} = \underset{2nx2n}{F} \ \underset{2nxj}{Y} \tag{9}$$

where

$$F = \begin{bmatrix} I & F_{12} \\ 0 & I \end{bmatrix} \tag{10}$$

clearly

$$F^{-1} = \begin{bmatrix} I & -F_{12} \\ 0 & I \end{bmatrix} \tag{11}$$

Taking the derivative of Eq (5)

$$\dot{X}_1 = \dot{Y}_1 + F_{12}\ Y_2 + F_{12}\ \dot{Y}_2 \tag{12}$$

Ey Eq (3) and Eq (4)

$$(A_{11}, A_{12})\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = A_{11}\ Y_1 + (\dot{F}_{12} + F_{12}\ A_{22})\ Y_2 \tag{13}$$

or

$$(A_{11}, A_{12}) \begin{bmatrix} I & F_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = (A_{11}, \dot{F}_{12} + F_{12}\ A_{22}) \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \tag{14}$$

For the full rank Y case multiply Eq (14) on the right by Y^* hence

$$(A_{11}, A_{12}) \begin{bmatrix} I & F_{12} \\ 0 & I \end{bmatrix} = (A_{11}, \dot{F}_{12} + F_{12} A_{22}) \tag{15}$$

or

$$\dot{F}_{12} = A_{11} F_{12} - F_{12} A_{22} + A_{12} \tag{16}$$

One can also arrive at the result of Eq (16) by taking the derivative of Eq (9)

$$\dot{X} = \dot{F} Y + F \dot{Y} = AX \tag{17}$$

and specify what $\dot{Y}$ should be, that is

$$\dot{Y} = BY = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} Y \tag{18}$$

Using Eq (18) in Eq (17)

$$\dot{Y} = (-F^{-1} \dot{F} + F^{-1} A F)Y = BY \tag{19}$$

or

$$\begin{bmatrix} A_{11} & -\dot{F}_{12} + A_{11} F_{12} + A_{12} - F_{12} A_{22}) \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} \tag{20}$$

and equating elements.

Note by Eq (8) the initial conditions for the sequence

$$X_1(o) = Y_1(o) + F_{12}(o)X_2(o) \tag{21}$$

one can select $F_{12}(o)$ or $Y_1(o)$, for example suppose

$$F_{12}(o) = I \tag{22}$$

then solve for $Y_1(o)$ or vice-versa.

For a single system, one has a full choice of $F_{12}(o)$, say for example

$$F_{12}(o) = 0 \tag{23}$$

then by Eq (8) for j=1

$$x(o)\rangle_1 = y(o)\rangle_1 + F_{12}(o)\ x(o)\rangle_2 \tag{24}$$

or

$$x(o)\rangle_1 = y(o)\rangle_1 \tag{25}$$

If we apply the constraint that $F_{12}(t)$ be a constant, that is

$$\dot{F}_{12}(t) = 0 \tag{26}$$

then Eq (16) becomes the <u>algebraic</u> <u>Riccatti</u> equation

$$0 = A_{11}F_{12} - F_{12}A_{22} + A_{12} \tag{27}$$

solutions of which will be discussed in section's using <u>Kronecker</u> <u>Products</u> etc.

Observe that X may be obtained by solving the upper Block triangular system of Eq (3)

or the block diagonal system of Eq (4) which has solutions

$$Y = \begin{bmatrix} e^{A_{11}t} & 0 \\ 0 & e^{A_{22}t} \end{bmatrix} \begin{bmatrix} Y_1(o) \\ Y_2(o) \end{bmatrix} \tag{28}$$

where

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \ldots$$

plus solving the <u>matrix</u> <u>Riccatti</u> Differentenal equations of Eq (16) and obtaining X from <u>Eq (9)</u> namely

$$X = FY \tag{29}$$

One can also solve the algebriac Riccatti Equation of Eq (27) to obtain a solution.

<u>Case ii</u>.

Consider next the solutions for X put together via

$$X_1 = Y_1 + F_{12}\ Y_2 \tag{30}$$

$$X_2 = 0Y_1 + F_{22}\ Y_2$$

or matrix-wise

$$X = \begin{bmatrix} I & F_{12} \\ 0 & F_{22} \end{bmatrix} Y = FY \tag{31}$$

The inverse is easily computed as

$$F^{-1} = \begin{bmatrix} I & -F_{12}\,F_{22}^{-1} \\ 0 & F_{22}^{-1} \end{bmatrix} \tag{32}$$

The matrix B of Eq (19) is

$$B = -F^{-1}\,\dot{F} + F^{-1}\,A\,F \tag{33}$$

The first matrix product on the right has elements

$$-F^{-1}\dot{F} = \begin{bmatrix} 0 & -\dot{F}_{12} + F_{12}\,F_{22}^{-1}\,\dot{F}_{22} \\ 0 & -F_{22}^{-1}\,\dot{F}_{22} \end{bmatrix} \tag{34}$$

and the second term

$$F^{-1}AF = \begin{bmatrix} A_{11}-F_{12}F_{22}^{-1}A_{21} & A_{11}F_{12}+A_{12}F_{22}-F_{12}F_{22}^{-1}(A_{21}F_{12}+A_{22}F_{22}) \\ F_{22}^{-1}A_{21} & F_{22}^{-1}(A_{21}F_{12}+A_{22}F_{22}) \end{bmatrix} \tag{35}$$

The requirement

$$0 = F_{22}^{-1}\,A_{21} = B_{21} \tag{36}$$

can be satisfied for

$$A_{21} = 0$$

hence

$$B = \begin{bmatrix} A_{11} & -\dot{F}_{12}+F_{12}F_{22}^{-1}\dot{F}_{22}+A_{11}F_{12}+A_{12}F_{22}-F_{12}F_{22}^{-1}A_{22}F_{22} \\ 0 & F_{22}^{-1}A_{22}F_{22} \end{bmatrix} \tag{37}$$

$$= \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} \tag{38}$$

If one selects F_{22} to be a constant

$$\dot{F}_{22} = 0 \tag{39}$$

then

$$B = \begin{bmatrix} A_{11} & -\dot{F}_{12} + A_{11} F_{12} + A_{12} F_{22} - F_{12} F_{22}^{-1} A_{22} F_{22} \\ 0 & F_{22}^{-1} A_{22} F_{22} \end{bmatrix} \tag{40}$$

and the Riccatti equation is

$$\dot{F}_{12} = A_{11} F_{12} + A_{12} F_{22} - F_{12} F_{22}^{-1} A_{22} F_{22} \tag{41}$$

observe that by Eq (40) if one selects the matrix F_{22} to be the matrix which diagonalizes A_{22}

$$B_{22} = F_{22}^{-1} A_{22} F_{22} = \text{dia } B_{22} = \Lambda_2 \tag{42}$$

then the second block of Y_2 is uncoupled, also

$$Y_2 = e^{\begin{pmatrix} \lambda_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \lambda_n \end{pmatrix} t} Y_2(o) \tag{43}$$

Case (iii) consider the case of

$$X_1(t) = F_{11} Y_1 = Y_2 \tag{44}$$

$$X_2(t) = Y_2$$

or

$$X = \begin{bmatrix} F_{11} & I \\ 0 & I \end{bmatrix} Y \tag{45}$$

The inverse

$$\begin{bmatrix} F_{11} & I \\ 0 & I \end{bmatrix}^{-1} \quad \begin{bmatrix} F_{11}^{-1} & -F_{11}^{-1} \\ 0 & I \end{bmatrix} \tag{46}$$

Equation (33) for this case becomes

$$B = \begin{bmatrix} F_{11}^{-1}(-\dot{F}_{11} + A_{11} F_{11} - A_{21} F_{11}) & F_{11}^{-1}(A_{11} + A_{12} - A_{21} - A_{22} \\ A_{21} F_{11} & A_{21} + A_{22} \end{bmatrix} \tag{47}$$

and

$$A_{21} = 0 \tag{48}$$

Using (48) in the B_{12} element of Eq (47)

$$A_{11} + A_{12} - A_{22} = 0$$

or

$$A_{12} = A_{22} - A_{11} \tag{49}$$

and the

$$B_{22} = A_{22}$$

and the one-one element

$$B_{11} = F_{11}^{-1} (-\dot{F}_{11} + A_{11} F_{11})$$

or

$$\dot{F}_{11} = A_{11} F_{11} - F_{11} B_{11} \tag{50}$$

Clearly by Eq (49) the transformation of Eq (45) will work only for the special case of

$$A = \begin{bmatrix} A_{11} & A_{22} - A_{11} \\ 0 & A_{22} \end{bmatrix} \tag{51}$$

Return now to <u>case one</u>. By way of summary: by Eq (3) and Eq (33) sec (2)

$$\dot{X} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} X \tag{52}$$

$$X_2 = TX_1 \tag{53}$$

$$\dot{T} = A_{22} T - T A_{11} - T A_{12} T \tag{54}$$

By Eq (9) and Eq (10)

$$X = FY$$

$$F = \begin{bmatrix} I & F_{12} \\ 0 & I \end{bmatrix} \tag{55}$$

and by Eq (4)

$$\dot{Y} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} Y \tag{56}$$

and by Eq (16)

$$\dot{F}_{12} = A_{11} F_{12} - F_{12} A_{22} + A_{12} \tag{57}$$

The Y connection matrix is

$$Y_2 = QY_1 \tag{58}$$

with

$$\dot{Q} = A_{22} Q - QA_{11} \tag{59}$$

The matrix Riccatti relation of Eq (57) implies a dynamical system

$$Z_2 = F_{12} Z_1 \tag{60}$$

where

$$\begin{bmatrix} \dot{Z}_1 \\ \dot{Z}_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = CZ \tag{61}$$

Taking the derivative of Eq (60) and using Eq (61) one finds that

$$C = \begin{bmatrix} A_{22} & 0 \\ A_{12} & A_{11} \end{bmatrix} \tag{62}$$

or

$$\begin{bmatrix} \dot{Z}_1 \\ \dot{Z}_2 \end{bmatrix} = \begin{bmatrix} A_{22} & 0 \\ A_{12} & A_{11} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \tag{63}$$

By inspection we see that

$$C = L_c \, A \, L_c \tag{64}$$

where the linear convolution matrix is

$$L_c = L_c^{-1} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{65}$$

and

$$\begin{bmatrix} A_{22} & 0 \\ A_{12} & A_{11} \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{66}$$

Thus if one defines the transformation

$$Z = L_c \, X \tag{67}$$

then

$$\dot{Z} = L_c \, A \, L_c \, X \tag{68}$$

also by Eq (55)

$$Z = L_c \, F \, Y \tag{69}$$

or

$$Z = F_{zy} \, Y \tag{70}$$

where

$$F_{zy} = L_c F = \begin{bmatrix} 0 & I \\ I & F_{12} \end{bmatrix} \tag{71}$$

By Eq (53)

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} I \\ T \end{bmatrix} X_1 \tag{72}$$

and by Eq (60)

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} I \\ F_{12} \end{bmatrix} Z_1 \tag{73}$$

Using Eq (72) and Eq (73) in Eq (67)

$$\begin{bmatrix} I \\ F_{12} \end{bmatrix} Z_1 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} I \\ T \end{bmatrix} X_1 \tag{74}$$

By Eq (67)

$$Z_1 = X_2 \tag{75}$$

Eq (75) in Eq (74) yields

$$\begin{bmatrix} I \\ F_{12} \end{bmatrix} X_2 \quad \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} I \\ T \end{bmatrix} X_1 \tag{76}$$

Multiply Eq (76) on right by X_1^{*}

$$\begin{bmatrix} I \\ F_{12} \end{bmatrix} T = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} I \\ T \end{bmatrix} \tag{77}$$

or

$$\begin{bmatrix} T \\ F_{12}T \end{bmatrix} = \begin{bmatrix} T \\ I \end{bmatrix} \tag{78}$$

or

$$F_{12}T = I \tag{79}$$

which implies

$$F_{12} \stackrel{t}{=} T^{-1} \tag{80}$$

hence we see that the transformation that diagonalizes Eq (52) is given by Eq (55) and Eq (80) as

$$X = \begin{bmatrix} I & T^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \tag{81}$$

a rather interesting result.

The transformation between the Z and Y systems are now readily obtained from Eq (67) and Eq (55) as

$$Z = L_c FY \tag{82}$$

or

$$Z = \begin{bmatrix} 0 & I \\ I & T^{-1} \end{bmatrix} Y = F_{zy} Y \tag{83}$$

where

$$F_{zy} = L_c F = \begin{bmatrix} 0 & I \\ I & T^{-1} \end{bmatrix} \tag{84}$$

and by Eq (16)

$$\dot{T}^{-1} = A_{11}T^{-1} - T^{-1}A_{22} + A_{12} \tag{85}$$

<u>Solution to the Upper Triangular Dynamical system via The Diagonal System and a matrix Riccatti Dynamical System.</u>

This section presents 4 alternatives to solving the upper triangular linear dynamical system given by Eq (3) as

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{86}$$

subject to the initial conditions.

$$X_1(o) = X_{10}$$

and

$$X_2(o) = X_{20} \tag{87}$$

Method one consists of solving Equation (86) and (87) by any standard numerical methods or continuious analog methods.

<u>Method two</u>. Since the second matrix of Eq (86) is uncoupled from the first matrix, one has

$$\dot{X}_2 = A_{22} X_2 \tag{88}$$

with solution

$$X_2 = e^{A_{22}t} X_2(o) = \phi_{A_{22}} X_{20} \tag{89}$$

Equation (89) can now be used in the first equation of Eq (86) as

$$\dot{X}_1 = A_{11} X_1 + A_{12} X_2 \tag{90}$$

$$\dot{X}_2 = A_{11} X_1 + A_{12} e^{A_{22}t} X_{20} \tag{91}$$

which is a non-homogeneous linear differential equation, the solution of which is deferred until a later section.

Method three. This method used the solution of the uncoupled matrix of Eq (89) with equation (14) sec (2)

$$X_1 = T^{-1} X_2 \tag{92}$$

or

$$X_1 = T^{-1} e^{A_{22}t} X_{20} \tag{93}$$

and by Eq (85)

$$\dot{T}^{-1} = A_{11} T^{-1} - T^{-1} A_{22} + A_{12} \tag{94}$$

with initial condition

$$T^{-1}(o) = X_1(o) X_2^*(o) \tag{95}$$

or

$$T^{-1}(o) = X_1(o) X_2^t(o) (X_2(o) X_2^t(o))^{-1} \tag{96}$$

This scheme of things requires the solution of non-homogeneous "bi-linear" (missing the guadratic term) Riccatti Equation. Clearly Eq (94) is more difficult to solve than Eq (91) a method for solving Eq (94) is given in a later section.

Method Four. This method is a variation of the last method and is the basis to the bias estimation scheme of Friedland in reference (6). Levin also in reference (51) relates two solutions of the same Riccatti differential equation to a new system, the details of which are disscussed in more detail in section (33).

The methods obtains the transformation Eq (55)

$$X = \begin{bmatrix} I & F_{12} \\ 0 & I \end{bmatrix} Y \tag{97}$$

which diagonalizes Eq (86) in the special way of Eq (56)

$$\begin{bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \qquad (98)$$

The solutions to the uncoupled systems are

$$Y_1 = e^{A_{11}t} Y_1(o) \qquad (99)$$

$$Y_2 = e^{A_{22}t} Y_2(o) \qquad (100)$$

where the initial conditions are by Eq (97)

$$Y_2(o) = X_2(o) \qquad (101)$$

and

$$X_1(o) - Y_1(o) = F_{12}(o)\, X_2(o) \qquad (102)$$

or

$$F_{12}(o) = X_1(o)\, X_2^*(o) - Y_1(o)\, X_2^*(o) \qquad (103)$$

By Eq (95)

$$F_{12}(o) = T^{-1}(o) - Y_1(o)\, X_2^*(o) \qquad (104)$$

By Eq (104) we see that there are two free choices of i.c. matrices namely $F_{12}(o)$ and $Y_1(o)$. For example if one selects,

$$Y_1(o) = 0 \qquad (105)$$

then

$$F_{12}(o) = T^{-1}(o) \qquad (106)$$

and the case becomes the same as method three for then

$$F_{12}(t) = T^{-1}(t) \qquad (107)$$

However, for all cases in which

$$Y_1(o) \neq 0 \qquad (108)$$

then

$$F_{12}(t) \neq T^{-1}(t) \qquad (109)$$

As an example suppose

$$Y_1(o) = X_1(o) \tag{110}$$

then by Eq (103)

$$F_{12}(o) = 0 \tag{111}$$

Two solutions to the same matrix Riccatti differential equation, as shown in section (33), are related as

$$F_{12}(t) = T^{-1}(t) + N(t) \tag{112}$$

where by Eq (16)

$$\dot{F}_{12} = A_{11} F_{12} - F_{12} A_{22} + A_{12} \tag{113}$$

and by Eq (97)

$$\dot{T}^{-1} = A_{11} T^{-1} - T^{-1} A_{22} + A_{12} \tag{114}$$

Tasking the derivative of Eq (112)

$$\dot{F}_{12} - \dot{T}^{-1} = \dot{N} \tag{115}$$

Using Eq (113) and Eq (114) in Eq (115)

$$\dot{N} = A_{11}N - NA_{22} \tag{116}$$

which is a homogeneous "bi-linear" matrix differential equation. Since the matrix Riccatti Differential Equation of Eq (116) implies a dynamic system connection as

$$W_2 = N W_1 \tag{117}$$

then by Eq (3), Eq (14) and Eq (33) of sec (2) one has

$$\begin{bmatrix} \dot{W}_1 \\ \dot{W}_2 \end{bmatrix} = \begin{bmatrix} A_{22} & 0 \\ 0 & A_{11} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \tag{118}$$

or for full rank $W_1(t)$

$$N(t) = W_2(t) W_1^*(t) \tag{119}$$

or the matrix factors of N have uncoupled dynamics, Eq (115) and Eq (118) can be compared with Eq (59) and Eq (56). Clearly since matrix factors are not in general unique. Let N(t) have matrix factor defined as

$$U_1(t) = N(t)\ U_2(t) \tag{120}$$

or

$$U_1\ U_2^{*} = N(t) \tag{121}$$

Take the derivative of Eq (120)

$$\dot{U}_1 = \dot{N}\ U_2 + N\ \dot{U}_2 \tag{122}$$

where

$$\begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{123}$$

or

$$(D_{11}N + D_{12})U_2 = \dot{N}U_2 + N[D_{21}N + D_{22}]U_2 \tag{124}$$

or

$$\dot{N} = D_{11}N - N(D_{21}N) - ND_{22} + D_{12} \tag{125}$$

and by Eq (116)

$$A_{11}N - NA_{22} = D_{11}N - ND_{22} - ND_{21}\ N + D_{12} \tag{126}$$

which implies

$$D = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \tag{127}$$

or

$$\begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{128}$$

Thus we so that by taking solution differences and selecting factors of N(t) in the right manner we achieve a diagonal system as was done via method three.

By Eq (128)

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} e^{A_{11}t} & 0 \\ 0 & e^{A_{22}t} \end{bmatrix} \begin{bmatrix} U_1(o) \\ U_2(o) \end{bmatrix} \tag{129}$$

or

$$U_2(t) = e^{A_{11}t}\, U_2(o) = \phi_{A_{22}}\, U_2(o) \tag{130}$$

and the psuedo-univses is

$$U_2^*(t) = U_2^*(o)\, e^{-A_{22}t} = U_2^*(o)\, \phi_{A_{22}}^{-1} \tag{131}$$

hence by Eq (132); (130) and Eq (121)

$$N(t) = e^{A_{11}t}\, U_1(o)U_2^*(o)\, e^{-A_{22}t} \tag{132}$$

$$N(t) = \phi_{A_{11}}(t)\, N(o)\, \phi_{A_{22}}^{-1}(t) \tag{133}$$

and the initial conditions are by Eq (132)

$$N(o) = U_1(o)\, U_2^*(o) \tag{134}$$

and by Eq (112)

$$N(o) = F_{12}(o) - T^{-1}(o) \tag{135}$$

By Eq (103) and Eq (95)

$$N(o) = X_1(o)X_2^*(o) - Y_1(o)X_2(o) - X_1\, X_2^* \tag{136}$$

or

$$N(o) = -Y_1(o)X_2^*(o) \tag{137}$$

By Eq (134)

$$N(o) = U_1(o)\, U_2^*(o) \tag{138}$$

thus if the diagonal U system is initialized with

$$\begin{bmatrix} U_1(o) \\ U_2(o) \end{bmatrix} = \begin{bmatrix} Y_1(o) \\ -Y_2(o) \end{bmatrix} = \begin{bmatrix} X_1(o) \\ -X_2(o) \end{bmatrix} \tag{139}$$

then the U system is time-wise related to the Y system as

$$\begin{bmatrix} U_1(t) \\ \\ U_2(t) \end{bmatrix} = \begin{bmatrix} Y_1(t) \\ \\ -Y_2(t) \end{bmatrix} = \begin{bmatrix} \phi_{A_{11}}(t)Y_1(o) \\ \\ \phi_{A_{22}}(t)Y_2(o) \end{bmatrix} \tag{140}$$

Using Eq (139) in Eq (136)

$$N(t) = -e^{A_{11}t} X_1(o)X_2^*(o) e^{-A_{22}t} \tag{141}$$

and using Eq (141) in Eq (112)

$$F_{12}(t) = T^{-1}(t) - \phi_{A_{11}}(t) X_1(o) X_2^*(o) \phi_{A_{22}}^{-1}(t) \tag{142}$$

By Eq (97)

$$X_1(t) = Y_1(t) + F_{12}(t) X_2(t) \tag{143}$$

using Eq (142) in Eq (143)

$$X_1(t) = -T^{-1}(t) \phi_{A_{22}}(t) X_2(o) \tag{144}$$

which is the same as Eq (93) under method three.

The Diagonalizing Transformation via Kronecker Matrix Product Solutions.

By Equation (19) the dynamic system matrices are related as

$$B = -F \dot{F} + F^{-1} A F \tag{145}$$

or

$$\dot{F} = AF - FB \tag{146}$$

By Eq (33) sec (G)

$$\text{vec } (AF) = (I \otimes A)) \text{ vec } F \tag{147}$$

and by Eq (49) appendix G

$$\text{vec } FB = (B^t \otimes I) \text{ vec } F \tag{148}$$

or

$$\text{vec } \dot{F} = \left\{ (I \otimes A) - (B^t \otimes I) \right\} \text{vec } F \tag{149}$$

which is a linear matrix differential equation in the tensor product space of dimension $4n^2$, that is the 2nx2n matrix $\dot{F}$ becomes a column vector of size $4n^2 \times 1$. One may now ask whether a transformation which is a constant matrix can be found or

$$\dot{F} = 0 \tag{150}$$

and Eq (149) becomes

$$\left\{(I\otimes A) - (B^t\otimes I)\right\} \text{vec } F = 0 \tag{151}$$

The matrix $I\otimes A - B^t\otimes I$ is called the nivellateur in the literative.

As an example of a solution to Eq (157) consider the 2x2 matrices of Eq (146)

$$\begin{bmatrix} a_{11} & a_{12} \\ o & a_{22} \end{bmatrix} \begin{bmatrix} 1 & f_{12} \\ o & 1 \end{bmatrix} = \begin{bmatrix} 1 & f_{12} \\ o & 1 \end{bmatrix} \begin{bmatrix} a_{11} & o \\ o & a_{22} \end{bmatrix} \tag{152}$$

Equation (151) becomes

$$\left\{ \begin{bmatrix} A & o \\ o & A \end{bmatrix} - \begin{bmatrix} a_{11}I & o \\ o & a_{22}I \end{bmatrix} \right\} \begin{bmatrix} 1 \\ o \\ f_{12} \\ 1 \end{bmatrix} = \begin{bmatrix} o \\ o \\ o \\ o \end{bmatrix} \tag{153}$$

or

$$\begin{bmatrix} o & a_{12} & o & o \\ o & a_{22}-a_{11} & o & o \\ o & o & a_{11}-a_{22} & a_{12} \\ o & o & o & o \end{bmatrix} \begin{bmatrix} 1 \\ o \\ f_{12} \\ 1 \end{bmatrix} = \begin{bmatrix} o \\ o \\ o \\ o \end{bmatrix} \tag{154}$$

By the third row of Eq (154)

$$(a_{11} - a_{22})f_{12} = -a_{12} \tag{155}$$

or

$$f_{12} = -(a_{11} - a_{22})^{-1}a_{12} \tag{156}$$

hence we see that a constant transformation matrix

$$F = \begin{bmatrix} 1 & -(a_{11}-a_{22})^{-1}a_{12} \\ o & 1 \end{bmatrix} \tag{157}$$

diagonalizes the systems.

Section 29 TRIANGULARIZATION AND DIAGONALIZATION OR DE-COUPLING OF ARBITRARY NON-HOMOGENEOUS DYNAMICAL SYSTEM. The fully coupled non-homogeneous form is

$$\dot{x}\left(2n\right\rangle = \begin{pmatrix} \dot{x}\rangle_1 \\ \dot{x}\rangle_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x\rangle_1 \\ x\rangle_2 \end{pmatrix} + \begin{pmatrix} f\rangle_1 \\ f\rangle_2 \end{pmatrix} \tag{1}$$

A full-rank base change of the form

$$x\left(2n\right\rangle = Fy\left(2n\right\rangle \tag{2}$$

yields y dynamics by taking the derivative of Equation (2)

$$Ax\rangle + f\rangle = \dot{F}y\rangle + F\dot{y}\rangle \tag{3}$$

or

$$\dot{y}\rangle = (-F^{-1}\dot{F} + F^{-1}AF)y\rangle + F^{-1}y\rangle \tag{4}$$

or

$$\dot{y}\rangle = By\rangle + f\rangle_y \tag{5}$$

where

$$B = -F^{-1}\dot{F} + F^{-1}AF \tag{6}$$

In the previous section the form of F was taken so that F^{-1} could be easily obtained from triangular forms for F. Also it was shown that via the matrix Kronecker products that F^{-1} term was not needed. The Kronecker product solutions via Equation (149) of Section (27) can be difficult. This section shows some easily invertible matrix forms used by triangular matrix products.

This section obtains the composite (product) maps by going from the full system of Equation (1) to an upper traingular system, that is

$$x\left(2n\right\rangle = F_{xy}y\left(2n\right\rangle \tag{7}$$

where the B of Equation (6) is upper triangular

$$B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} \tag{8}$$

followed by the transformation to a new system

$$y\rangle = F_{yz}z\rangle \tag{9}$$

with dynamic matrix C which is diagonal that is

$$\dot{z}\rangle = \begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix} z\rangle + f\rangle_z \tag{10}$$

By Equation (9) in Equation (7)

$$x\langle 2n \rangle = F_{xy}F_{yz}z\rangle = F_{xz}z\rangle \tag{11}$$

or

$$F_{xz} = F_{xy}F_{yz} \tag{12}$$

and

$$F_{xz}^{-1} = F_{yz}^{-1}F_{xy}^{-1} \tag{13}$$

By Equation (6)

$$C = -F_{xz}^{-1}\dot{F}_{xz} + F_{xz}^{-1}AF_{xz} \tag{14}$$

and the inverse matrix F_{xz} is easily obtained via Equation (13) as the product of simple triangular matrix inverses.

Returning now to the first stage via Equation (6) with proper subscripts

$$\begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} = -F_{xy}^{-1}\dot{F}_{xy} + F_{xy}^{-1}AF_{xy} \tag{15}$$

with

$$F_{xy} = \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} \tag{16}$$

and

$$F_{xy}^{-1} = \begin{pmatrix} I & 0 \\ -L & I \end{pmatrix} \tag{17}$$

The first matrix product term on the right of Equation (15) is

$$F_{xy}^{-1}\dot{F}_{xy} = \begin{pmatrix} I & 0 \\ -L & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \dot{L} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \dot{L} & 0 \end{pmatrix} \tag{18}$$

The similarity transformation term is

$$F_{xy}^{-1}AF_{xy} = \begin{pmatrix} A_{11}+A_{12}L & A_{12} \\ -L(A_{11}+A_{12}L)+A_{21}+A_{22}L & -LA_{12}+A_{22} \end{pmatrix} \tag{19}$$

Using Equation (18), Equation (19) in Equation (15)

$$\begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}+A_{12}L & A_{12} \\ -\dot{L}-L(A_{11}+A_{12}L)+A_{21}+A_{22}L & -LA_{12}+A_{22} \end{pmatrix} \tag{20}$$

we find

$$\begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}+A_{12}L & A_{12} \\ 0 & -LA_{12}+A_{22} \end{pmatrix} \tag{21}$$

with

$$\dot{L} = A_{22}L-LA_{11}-LA_{12}L+A_{21} \tag{22}$$

Note that L is same as Equation (33) Section (26) that is the Ricatti equation for T.

By the previous section the upper triangular system was transformed to a block diagonal system via the transformation

$$y\rangle = F_{yz}z\rangle \tag{23}$$

where

$$F_{yz} = \begin{pmatrix} I & M \\ 0 & I \end{pmatrix} \tag{24}$$

The dynamic matrices for this case become

$$C = -F_{yz}^{-1}\dot{F}_{yz} + F_{yz}^{-1}BF_{yz} \tag{25}$$

By analogy with Equation (20) Section (3)

$$\begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & -\dot{M}+B_{11}M-MB_{22}+B_{12} \\ 0 & B_{22} \end{pmatrix} \tag{26}$$

$$\dot{M} = B_{11}M-MB_{22}+B_{12} \tag{27}$$

The B_{ij} are given in terms of the A_{ij} via Equation (21).

The composite transformation is by Equation (24) and Equation (16) in Equation (12)

$$F_{xz} = F_{xy}F_{yz} \tag{28}$$

or

$$F_{xz} = \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} \begin{pmatrix} I & M \\ 0 & I \end{pmatrix} \tag{29}$$

$$F_{xz} = \begin{pmatrix} I & M \\ L & LM+I \end{pmatrix} \tag{30}$$

with inverse given by Equation (13)

$$F_{xz}^{-1} = \begin{pmatrix} I & -M \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -L & I \end{pmatrix} \tag{31}$$

$$F_{xz}^{-1} = \begin{pmatrix} I+ML & -M \\ -L & I \end{pmatrix} \tag{32}$$

where by Equation (20) in Equation (26)

$$C = \begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix} = \begin{bmatrix} A_{11}+A_{12}L & 0 \\ 0 & A_{22}-LA_{12} \end{bmatrix} \tag{33}$$

and by Equation (27) and Equation (20)

$$\dot{M} = (A_{11}+A_{12}L)M-M(A_{22}-LA_{12})+A_{12} \tag{34}$$

In case one wants constant transformation

$$\dot{L} = \dot{M} = 0$$

and Equation (22) and Equation (34) become algegraic matrix Ricatti equations or

$$0 = A_{22}L-LA_{11}-LA_{12}L+A_{21} \tag{35}$$

and

$$0 = (A_{11}+A_{12}L)M-M(A_{22}-LA_{12})+A_{12} \tag{36}$$

Section 30 TRANSFORMATION OF HOMOGENEOUS DYNAMICAL SYSTEM TO BLOCK COMPANION MATRIX FORM. This section considers the constant transformation to map the the dynamical

$$\dot{X} = \begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{1}$$

to a system in block companion matrix form

$$\dot{Y} = \begin{pmatrix} \dot{Y}_1 \\ \dot{Y}_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = BY \tag{2}$$

where

$$X = FY \tag{3}$$

and

$$F^{-1}AF = B \tag{4}$$

If one selects the constant matrix F of the form

$$F = \begin{pmatrix} I & F_{12} \\ 0 & F_{22} \end{pmatrix} \tag{5}$$

it is easily established that

$$F^{-1} = \begin{pmatrix} I & -F_{12}F_{22}^{-1} \\ 0 & F_{22}^{-1} \end{pmatrix} \tag{6}$$

and

$$F^{-1}AF = B \tag{7}$$

or

$$\begin{pmatrix} A_{11}-F_{12}F_{22}^{-1}A_{21} & (A_{11}-F_{12}F_{22}^{-1}A_{21})F_{12} + (A_{12}-F_{12}F_{22}^{-1}A_{22})F_{22} \\ F_{22}^{-1}A_{21} & F_{22}^{-1}(A_{21}F_{12} + A_{22}F_{22}) \end{pmatrix} = \begin{pmatrix} 0 & I \\ B_{21} & B_{22} \end{pmatrix} \tag{8}$$

Equating the (1,1) elements of Equation (8)

$$A_{11}-F_{12}F_{22}^{-1}A_{21} = 0 \tag{9}$$

or

$$F_{12}F_{22}^{-1} = A_{11}A_{21}^{-1} \tag{10}$$

Equating the (1,2) elements and using Equation (9)

$$(A_{12}-F_{12}F_{22}^{-1}A_{22})F_{22} = I \tag{11}$$

Using Equation (10) in Equation (11)

$$F_{22} = (A_{12}-A_{11}A_{21}^{-1}A_{22})^{-1} \tag{12}$$

By Equation (10)

$$F_{12} = A_{11}A_{21}^{-1}F_{22} \tag{13}$$

and by Equation (12)

$$F_{12} = A_{11}A_{21}^{-1}(A_{12}-A_{11}A_{21}^{-1}A_{22})^{-1} \tag{14}$$

or

$$F_{12} = A_{11}[A_{12}A_{21}-A_{11}A_{21}^{-1}A_{22}A_{21}]^{-1} \tag{15}$$

Equating the (2,1) elements of Equation (8)

$$B_{21} = F_{22}^{-1}A_{21} \tag{16}$$

or Equation (12) in Equation (16)

$$B_{21} = (A_{21}-A_{11}A_{21}^{-1}A_{22})A_{21} \tag{17}$$

Equating the (2,2) elements

$$B_{22} = F_{22}^{-1}(A_{21}F_{12} + A_{22}F_{22}) \tag{18}$$

Using Equation (13) in Equation (18)

$$B_{22} = F_{22}^{-1}[A_{21}A_{11}A_{21}^{-1}F_{22} + A_{22}F_{22}] \tag{19}$$

$$B_{22} = F_{22}^{-1}[A_{21}A_{11}A_{21}^{-1} + A_{22}] \tag{20}$$

Scalar Case. When the matrix A of Equation (1) is 2 x 2 in size

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{21}$$

then by Equation (15)

$$f_{12} = \frac{a_{11}}{a_{12}a_{21} - a_{11}a_{22}} \tag{22}$$

and by Equation (12)

$$f_{22} = \frac{a_{21}}{a_{12}a_{21} - a_{11}a_{22}} \tag{23}$$

and by Equation (17)

$$b_{21} = a_{12}a_{21} - a_{11}a_{22} \tag{24}$$

and by Equation (20)

$$b_{22} = a_{11} + a_{22} \tag{25}$$

The matrix relations are

$$F^{-1}AF = \begin{pmatrix} 0 & 1 \\ a_{12}a_{21} - a_{11}a_{22} & a_{11} + a_{22} \end{pmatrix} \tag{26}$$

When A is a Hamiltonian matrix Section (27), or

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \tag{27}$$

Then by Equation (25)

$$b_{22} = a - a = 0 \tag{28}$$

and

$$\begin{pmatrix} \dot{y}_1 \\ y2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ bc-a^2 & 0 \end{pmatrix} \begin{pmatrix} Y_1 \\ y_2 \end{pmatrix} \tag{29}$$

If

$$\begin{pmatrix} y^1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} \tag{30}$$

We see that

$$\ddot{y} = (bc-a^2)y \tag{31}$$

or the system has no damping or velocity dissipative term, that is the system of Equation (31) is an oscillator when $bc-a^2$ has the correct sign (negative feedback).

At the matrix level the condition is by Equation (20)

$$B_{22} = 0 = A_{21}A_{11}A_{21}^{-1} + A_{22} \tag{32}$$

and for the matrix Hamiltonian case

$$A_{22} = A_{11}^T \tag{33}$$

or

$$A_{21}A_{11}A_{21}^{-1} = A_{11}^T \tag{34}$$

Section 31 NON-HOMOGENEOUS DYNAMICAL SYSTEM $\dot{X} = A X + F(t)$

This section considers the dynamical system

$$\dot{x}\rangle = \underset{2n\times 2n}{A}x\rangle + f\rangle \tag{1}$$

and a sequence of j systems

$$\underset{2n\times j}{\dot{X}} = AX + F \tag{2}$$

Two different methods of solution will be presented. The homogeneous system is obtained in Section (25) as

$$\dot{\Phi} = A\Phi \tag{3}$$

where

$$\Phi(to) = I \tag{4}$$

and

$$\Phi(t,to) = e^{A(t-to)} \tag{5}$$

where $\Phi(t,to)$ is the state transition matrix or the fundamental matrix. It always forms a base. Make a base change on $x(t)$

$$x(t)\rangle = \Phi(t,to)x^{\phi(t)}\rangle \tag{6}$$

where the coordinates of the vector $x(t)\rangle$ in the Φ basis is $x^{\phi}(t)\rangle$ that is the linear combination of the columns of Φ

$$x(t)\rangle = \phi(t)\rangle_1 x_1^{\phi} + \ldots + \phi(t)\rangle_{2n}\, x_{2n}^{\phi}(t) \tag{7}$$

Taking the derivative of Eq. (6)

$$\dot{x}\rangle = \dot{\Phi}x^{\phi}\rangle + \Phi\dot{x}^{\phi}\rangle \tag{8}$$

By Eq. (1) and Eq. (3) in Eq. (8)

$$Ax\rangle + f(t)\rangle = A\Phi x^{\phi}(t)\rangle + \Phi\dot{x}^{\phi}\rangle \tag{9}$$

and by Eq. (6) in Eq. (9)

$$Ax\rangle + f\rangle = Ax\rangle + \Phi\dot{x}^{\phi}\rangle \tag{10}$$

or

$$\Phi(t,to)\dot{x}^{\phi}(t)\rangle + f(t)\rangle \tag{11}$$

inverting

$$\dot{x}^{\phi}\rangle = \Phi^{-1}(t,to)\; f(t)\rangle \tag{12}$$

which is the "apparent velocity" as observed by an observer or the Φ basis.

Integrating Eq. (12)

$$x^{\phi}(t)\rangle = x^{\phi}(to)\rangle + \int_{to}^{t} \Phi^{-1}(\tau,to)f(t)\rangle \, d\tau \tag{13}$$

By Eq. (16)

$$x(to)\rangle = Ix^{\phi}(to)\rangle \tag{14}$$

hence Eq. (13) is

$$x^{\phi}(t)\rangle = x(to)\rangle + \int_{to}^{t} \Phi^{-1}(\tau,to)f(\tau)\rangle \, d\tau \tag{15}$$

Using Eq. (15) in Eq. (6)

$$x(t)\rangle = \Phi(t,to)x(to)\rangle + \Phi(t,to) \int_{to}^{t} \Phi^{-1}(\tau,to)f(\tau)\rangle \, d\tau \tag{16}$$

Using transition matrix properties

$$\Phi^{-1}(t,to) = \Phi(to,t) \tag{17}$$

$$\Phi(t,to)\Phi^{-1}(\tau,to) = \Phi(t,to)\Phi(to,\tau) = \Phi(t,\tau) \tag{18}$$

hence Eq. (18) in Eq. (16)

$$x(t)\rangle = \Phi(t,to)x(to)\rangle + \int_{to}^{t} \Phi(t,\tau)f(\tau)\rangle \, d\tau \tag{19}$$

Eq. (17) and (18) are obvious from

$$\Phi(t,to) = e^{A(t-to)} \tag{20}$$

$$\Phi^{-1}(t,to) = e^{-A(t-to)} = e^{A(to-t)} \tag{21}$$

and

$$\Phi(t,\tau) = e^{A(t-to)} e^{A(to-\tau)} = e^{A(t-\tau)} \tag{22}$$

By Eq. (22) in Eq. (19)

$$x(t)\rangle = \Phi(t,to)x(to)\rangle + \int_{to}^{t} e^{A(t-\tau)} f(\tau)\rangle \, d\tau \tag{23}$$

The convolution integral of Eq. (16) was very naturally obtained in the Φ base.

Section 32 VARIANCES RELATED TO $\dot{x}\rangle_j = Ax\rangle_j + U\rangle_j$ Consider the sequence of j continuous time trajectories

$$\underset{nxj}{\dot{X}} = \underset{nxj}{AX} + \underset{nxj}{U} \tag{1}$$

where

$$\underset{mxj}{U1^*} \; j\rangle = \mu_u(t)\rangle \tag{2}$$

Taking the arithmetic mean of Equation (1)

$$\dot{X}1^*\rangle = \dot{\mu}_x\rangle = A\mu_x\rangle + \mu_u\rangle \tag{3}$$

Subtracting Equation (3) from Equation (1)

$$\underset{nxj}{\dot{X}} - \dot{\mu}_x\rangle\langle j \; 1 = \tilde{X} = A\tilde{X} + \tilde{U} \tag{4}$$

where

$$\tilde{X} = X - \mu\rangle_x\langle 1 \tag{5}$$

and

$$\tilde{U} = U - \mu\rangle_u\langle 1 \tag{6}$$

By Equation (4) the dynamics of the package of residual vectors is

$$\tilde{X} = A\tilde{X} + U \tag{7}$$

The variance is given by

$$\underset{\tilde{x}\tilde{x}}{} = \tilde{X}\tilde{X}^T \frac{}{jmax} = \left(\sum_{j=11}^{jmax} \tilde{x} \; {}_j \; {}^j \tilde{x} \right) \frac{1}{jmax} \tag{8}$$

The dynamics of the variance is by Equation (8)

$$\frac{d}{dt} \underset{\tilde{x}\tilde{x}}{} = (\tilde{X}\tilde{X}^T + \tilde{X}\tilde{X}^T) \frac{1}{jmax} \tag{9}$$

Consider an alternate expression of the variance in the Φ base

$$\tilde{X}(t) = \Phi(t,to)\tilde{X}^{\phi}(t) \tag{10}$$

and

$$\underset{\tilde{x}\tilde{x}}{} = \Phi(t,to)\tilde{X}^{\phi}(t)X^{\phi T}(t)\Phi^T(t,to) \tag{11}$$

$$(t)_{\tilde{x}\tilde{x}} = \Phi(t,to) \quad {}^{\phi}(t) \; \Phi^T(t,to) \tag{12}$$

The derivative of Equation (12) is

$$\dot{\underset{\sim}{x}}\dot{\underset{\sim}{x}} = \dot{\Phi}\Sigma^{\phi}\Phi^{T} + \Phi\dot{\Sigma}^{\phi}\Phi^{T} + \Phi^{T}\Sigma^{\phi}\dot{\Phi}^{T} \tag{13}$$

Now by Equations (11) and (12)

$$\phi = \tilde{X}^{\phi}\tilde{X}^{\phi T} \tag{14}$$

and

$$\phi = \tilde{X}^{\phi}\tilde{X}^{\phi T} + \tilde{X}^{\phi}\tilde{X}^{\phi T} \tag{15}$$

The derivative of Equation (10) is

$$\tilde{X} = \dot{\Phi}\tilde{X}^{\phi} + \Phi\tilde{X}^{\phi} \tag{16}$$

Using the relation

$$\dot{\Phi} = A\ \Phi \tag{17}$$

and Equation (8) in Equation (16)

$$A\tilde{X} + U = A\Phi\tilde{X}^{\phi} + \Phi\tilde{X}^{\phi} \tag{18}$$

or

$$\tilde{X}^{\phi}(t) = \Phi^{-1}(t,to)U(t) \tag{19}$$

Integrating Equation (19)

$$\tilde{X}^{\phi}(t) = \tilde{X}^{\phi}(t,o) + \int_{to}^{t}\Phi^{-1}(\tau,to)U(\tau)d\tau \tag{20}$$

where by Equation (10)

$$\tilde{X}^{\phi}(to) = \tilde{X}(to) \tag{21}$$

Using Equation (19) and (20) and their transposes in Equation (15) the first term on the right is

$$\tilde{X}^{\phi}(t)\tilde{X}^{\phi T} = \Phi^{-1}(t,to)U(t)[\tilde{X}(to) + \int_{to}^{t}\Phi^{-1}(\tau,to)U(\tau)d\tau]^{T} \tag{22}$$

$$\tilde{X}^{\phi}\tilde{X}^{\phi T} = \Phi^{-1}(t,to)U(t)[\tilde{X}^{T}(to) + \int_{to}^{t}U^{T}(\tau)\Phi^{-T}(\tau,to)d\tau \tag{23}$$

or

$$\tilde{X}^{\phi}\tilde{X}^{\phi T} = \Phi^{-1}(t,to)\ U(t)\tilde{X}^{T}(to) + \Phi^{-1}(t,to)\int_{to}^{t}U(t)U^{T}(\tau)\Phi^{-T}(\tau,to)d\tau \tag{24}$$

If the noise is serially - uncorrelated and zero-mean, that is

$$E(u(t)\rangle\langle u(t)) = U(t)U^T(\tau) = Q(t,\tau)\delta(\tau-t) \tag{25}$$

If we assume further that

$$E[u(t)\rangle\langle\tilde{x}(to)] = 0 = U(t)\tilde{X}^T(to) \tag{26}$$

that is the initial state-errors are uncorrelated with respect to the process noise.

Using Equation (25) and (26) in Equation (24)

$$\tilde{X}^{\phi}\tilde{X}^{\phi T} = \Phi^{-1}(t,to)\int_{to}^{t} Q(t,\tau)\delta(\tau-t)\Phi^{-T}(\tau,to)dt \tag{27}$$

or

$$\tilde{X}^{\phi}\tilde{X}^{\phi T} = \Phi^{-1}(t,to)Q(t,t)\Phi^{-T}(t,to) \tag{28}$$

Clearly the second term of Equation (16) is the transpose of Equation (28) or

$$\tilde{X}^{\phi}\tilde{X}^{\phi T} = \Phi^{-1}(t,to)Q(t,t)\Phi^{-T}(t,to) \tag{29}$$

Combining Equation (28) and Equation (29) in Equation (16).

$$\phi = \Phi^{-1}(t,to)Q(t,t)\Phi^{-T}(t,to) \tag{30}$$

Using Equation (30) and the relation

$$\dot{\Phi} = A\Phi \tag{31}$$

in Equation (4)

$$\tilde{x}\tilde{x} = A\Phi\Sigma^{\phi}\phi^T + Q(t,t) + \Phi\Sigma^{\phi}\Phi^T A^T \tag{32}$$

By Equation (13) in Equation (32)

$$\tilde{x}\tilde{x} = A\Sigma_{\tilde{x}\tilde{x}} + \Sigma_{\tilde{x}\tilde{x}}A^T + Q(t) \tag{33}$$

which is the matric Riccati differential equation of the variance.

Section 33 DIFFERENCES OF MATRIX RICCATTI SOLUTIONS

This section discusses some interesting relations involving differences of solutions of the matrix Riccatti Differential Equation. Levin in reference (51) drives some of these. Friedland applies some in his paper on Bias Estimation in reference (32). Lainiotis in reference (48) applies differencing methods to obtain what he calls partitioned Riccatti solutions and then, obtaining the generalized Chandrasekhar relations. Consider the partitioned system

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{1}$$

with

$$X_2 = TX_1 \tag{2}$$

By Eq. (33) Section (6)

$$\dot{T} = -TA_{11} + A_{22}T - TA_{12}T + A_{21} \tag{3}$$

and by (159) Section (6)

$$\dot{T}^{-1} = A_{11}T^{-1} - T^{-1}A_{22} + A_{12} - T^{-1}A_{21}T^{-1} \tag{4}$$

Both equations are non-homogeneous and quadratic.

Suppose we have one solution to Eq. (3) $T_i(t)$ with initial condition

$$T_i(to) = T_{io} \tag{5}$$

and want a second solution T (t) with initial condition

$$T_{i+1}(to) = T_{(i+1)o} \tag{6}$$

obtaining

$$T_{i+1}(t) = T_i(t) + N_i(t) \tag{7}$$

The difference is

$$N_i(t) = T_{i+1} - T_i \tag{8}$$

and the derivative is

$$\dot{N}_i = \dot{T}_{i+1} - \dot{T}_i \tag{9}$$

Using Eq. (3) in Eq. (9)

$$\begin{aligned} \dot{N}_i = {} & -T_{i+1}A_{11} + A_{22}T_{i+1} - T_{i+1}A_{12}T_{i+1} + A_{21} \\ & - (-T_iA_{11} + A_{22}T_i - T_iA_{12}T_i + A_{21}) \end{aligned} \tag{10}$$

or

$$\dot{N}_i = -(T_{i+1} - T_i)A_{11} + A_{22}(T_{i+1} - T_i) - T_{i+1}A_{12}T_{i+1} + T_iA_{12}T_i \tag{11}$$

Using Eq. (7) in Eq. (11) quadratic term

$$\dot{N}_i = -N_iA_{11} + A_{22}N_i - (T_i + N_i)A_{12}(T_i + N_i) + T_iA_{12}T_i \tag{12}$$

$$= -N_iA_{11} + A_{22}N_i - T_iA_{12}T_i - T_iA_{12}N_i - N_iA_{12}T_i - N_iA_{12}N_i + T_iA_{12}T_i \tag{13}$$

or

$$\dot{N}_i = -N_i(A_{11} + A_{12}T_i) + (A_{22} - T_iA_{12})N_i - N_iA_{12}N_i \tag{14}$$

which is homogeneous-quadratic in $N_i(t)$. Or

$$\dot{N}_i = -N_iB_{11} + B_{22}N_i - N_iA_{12}N_i \tag{15}$$

which implies a dynamical system by Eq. (33) Section (2)

$$\begin{pmatrix} \dot{y}>_1 \\ \dot{y}>_2 \end{pmatrix} = \begin{bmatrix} B_{11} & -A_{12} \\ 0 & B_{22} \end{bmatrix} \begin{pmatrix} y>_1 \\ y>_2 \end{pmatrix} \tag{16}$$

where

$$B_{11} = A_{11} + A_{12}T_i \tag{17}$$

$$B_{22} = A_{22} - T_iA_{12} \tag{18}$$

where

$$y>_2 = N_i y>_1 \tag{19}$$

or for a sequence or package of trajectories.

$$Y_2 = N_iY_1 \tag{20}$$

with $N_i(t)$ factors

$$N_i(t) = Y_2Y_1^* \tag{21}$$

The inverse Riccati relation is

$$\dot{N}_i^{-1} = -N_i^{-1}\dot{N}_iN_i^{-1} \tag{22}$$

or

$$\dot{N}_i^{-1} = B_{11}N_i^{-1} - N_i^{-1}B_{22} + A_{12} \tag{23}$$

which is non-homogeneous bi-linear.

If now one has two solution of Eq. (23)

$$N_{i+1}^{-1} - N_i^{-1} = M_i(t) \tag{24}$$

then

$$\dot{M}_i(t) = \dot{N}_{i+1}^{-1} - \dot{N}_i^{-1} \tag{25}$$

or

$$\dot{M}_i(t) = B_{11}M_i - M_iB_{22} \tag{26}$$

which is homo-geneous non-quadratic. Eq. (26) implies a dynamical system

$$\begin{pmatrix} \dot{Z}>_1 \\ \dot{Z}>_2 \end{pmatrix} = \begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix} \begin{pmatrix} Z>_1 \\ Z>_2 \end{pmatrix} = \begin{pmatrix} B_{22} & 0 \\ 0 & B_{11} \end{pmatrix} \begin{pmatrix} Z>_1 \\ Z>_2 \end{pmatrix} \tag{27}$$

with connection

$$Z>_2 = M_i Z>_1 \tag{28}$$

$$C_{11} = B_{22} = A_{22} - T_iA_{12} \tag{29}$$

$$C_{22} = B_{11} = A_{11} - T_iA_{12} \tag{30}$$

The dynamics of successive differences and inverses are applied in following the obtain solutions to Riccati equations in terms of a controlled initial condition solution. Some of these variations are

$$\dot{T} = A_{22}T$$

$$\dot{T} = -TA_{11}$$

$$\dot{T} = -TA_{11} + A_{22}T$$

$$\dot{T} = -TA_{12}T$$

$$\dot{T} = A_{22}T - TA_{12}T$$

$$\dot{T} = -TA_{11} - TA_{12}T$$

$$\dot{T} = A_{22}T - TA_{11} - TA_{12}T$$

all of the above are homogeneous and linear, bi-linear, and or quadratic.

The non-homogeneous cases are

$$\dot{T} = A_{22}T + A_{21}$$

$$\dot{T} = A_{11}T + A_{21}$$

$$\dot{T} = -TA_{12}T + A_{21}$$

$$\dot{T} = -TA_{11} + A_{22}T - TA_{12}T + A_{21}$$

Section 34 SOLUTIONS OF MATRIX RICCATI VIA DIAGONALIZATION TECHNIQUES

This section draws upon the relation's of section 2 for block diagonalizing a system to obtain solutions to the matrix Riccati differential equation.

Suppose we start with the given Riccati Eq. (3) section 31

$$\dot{T} = -TA_{11} + A_{22}T - TA_{12}T + A_{21} \tag{1}$$

with i.c. of

$$T(o) = To \tag{2}$$

Eq. (1) implies a coupled homogeneous system given by Eq. (1) of section 31,

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{3}$$

with

$$X_2 = TX_1 \tag{4}$$

or T(t) has the matrix factors.

$$T(t) = X_2 X_1^* \tag{5}$$

By Eq. (30) sec 28

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{bmatrix} I & M(t) \\ L(t) & LM+I \end{bmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \tag{6}$$

where by Eq. (33) of sec 28

$$\begin{pmatrix} \dot{Z}_1 \\ \dot{Z}_2 \end{pmatrix} = \begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \tag{7}$$

$$= \begin{pmatrix} A_{11} + A_{12}L & 0 \\ 0 & A_{22} - LA_{12} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \tag{8}$$

with

$$Z_2 = S(t)Z_1 \tag{9}$$

or

$$S = Z_2 Z_1^* \tag{10}$$

By Eq. (21) sec 31, the solution

$$S(t) = \Phi_{c_{22}}(t) S(o) \Phi^{-1}_{c_{11}}(t) \tag{11}$$

where

$$\Phi_{c_{11}}(t) = e^{\int_{to}^{t} (A_{11} + A_{12} L) dt} \tag{12}$$

$$\Phi_{c_{22}}(t) = e^{\int_{to}^{t} (A_{22} - L A_{12}) dt} \tag{13}$$

and by Eq. (34) sec 28

$$\dot{M} = (A_{11} + A_{12} L) M - M(A_{22} - L A_{12}) + A_{12} \tag{14}$$

and by Eq. (22) sec 28

$$\dot{L} = A_{22} L - L A_{11} - L A_{12} L + A_{21} \tag{15}$$

note that Eq. (14) can be written as

$$\dot{M} = C_{11} M - M C_{22} + A_{12} \tag{16}$$

and has solution given by Eq. (26) sec 32 as

$$M(t) = \Phi_{c_{22}} \left[M(o) + \int_{o}^{t} \Phi^{-1}_{c_{22}}(\tau, o) A_{12} \, \Phi_{c_{11}}(\tau, o) d\tau \right] \Phi^{-1}_{c_{11}}(t, o) \tag{17}$$

The initial conditions by Eq. (6)

$$\begin{pmatrix} X_1(o) \\ X_2(o) \end{pmatrix} = \begin{bmatrix} I & M(o) \\ L(o) & LoMo + I \end{bmatrix} \begin{pmatrix} Z_1(o) \\ Z_2(o) \end{pmatrix} \tag{18}$$

If we select

$$Mo = Lo = 0 \tag{19}$$

then

$$\begin{pmatrix} X_1(o) \\ X_2(o) \end{pmatrix} = \begin{pmatrix} Z_1(o) \\ Z_2(o) \end{pmatrix} \tag{20}$$

and by Eq. (16) we see that $\dot{M}(t)$ is a non-homogeneous system and Eq. (17) is simplified and

$$M(t) = \Phi_{c_{22}}(t) \int_o^t \left[\Phi_{c_{22}}^{-1}(\tau, o) A_{12} \Phi_{c_{11}}(\tau, o) d\tau\right] \Phi_{c_{11}}^{-1} \tag{21}$$

We next seek the connection between the matrix S(t) and the matrix T(t). By Eq. (4) and Eq. (9) in Eq. (6)

$$\begin{pmatrix} I \\ T \end{pmatrix} X_1 = \begin{pmatrix} I & M \\ L & LM+1 \end{pmatrix} \begin{pmatrix} I \\ S \end{pmatrix} Z_1 \tag{22}$$

or

$$X_1 = (I + MS)Z_1 \tag{23}$$

Using Eq. (23) in the second element of Eq. (22)

$$TX_1 = T(I + MS)Z_1 = \left[L + (LM + I)S\right] Z_1 \tag{24}$$

or

$$T(I + MS) = L(I + MS) + S \tag{25}$$

or

$$T = L + S(I + MS)^{-1} \tag{26}$$

If we set

$$N = S(I + MS)^{-1} \tag{27}$$

then

$$T = L + N \tag{28}$$

which is of the form of Eq. (7) of sec 31.

Eq. (25) also be written as

$$T = L + (L - T)MS + S \tag{29}$$

or

$$T = L - NMS + S \tag{30}$$

which implies that

$$N = -NMS + S \tag{31}$$

which agrees with Eq. (27).

By Eq. (30)

$$T = (I - NM)\,S + L \tag{32}$$

we see by Eq. (9)

$$\dot{Z}_2 = \dot{S}Z_1 + S\dot{Z}_1 \tag{33}$$

or by Eq. (8) in Eq. (33)

$$(A_{22} - LA_{12})Z_2 - S(A_{11} + A_{22}L) = \dot{S}Z_1 \tag{34}$$

or

$$(A_{22} - LA_{12})S - S(A_{11} + A_{12}L) = \dot{S} \tag{35}$$

By Eq. (14) sec 7

$$\dot{N} = -N(A_{11} + A_{12}L) + (A_{22} - LA_{12})N - NA_{12}N \tag{36}$$

which is homogeneous quadratic in N.

The initial conditions by Eq. (30)

$$To = Lo - NoMoSo + So \tag{37}$$

and by Eq. (19) in Eq. (37)

$$To = So\ . \tag{38}$$

Section 35 SOLUTION OF MATRIX $\dot{T} = -TA_{11} + A_{22}T$. Consider the system

$$\dot{X} = \begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = AX \tag{1}$$

where

$$X_2 = TX_1 \tag{2}$$

Equation (1) becomes

$$\dot{X} = \begin{pmatrix} \dot{X}_1 \\ \dot{T}X_1 + T\dot{X}_1 \end{pmatrix} \quad \begin{pmatrix} I \\ T \end{pmatrix} \dot{X}_1 + \begin{pmatrix} 0 \\ \dot{T} \end{pmatrix} X_1 \tag{3}$$

Using the orthogonal property

$$(-T, I) \begin{pmatrix} I \\ T \end{pmatrix} = 0 \tag{4}$$

applied to Equation (3)

$$(-T, I) \begin{pmatrix} \dot{X}_1 \\ \dot{X}2 \end{pmatrix} = \dot{T}X_1 \tag{5}$$

Using Equation (1) in Equation (5)

$$(-T, I)\, A \begin{pmatrix} I \\ T \end{pmatrix} X_1 = \dot{T}X_1 \tag{6}$$

or

$$(-T, I)\, A \begin{pmatrix} I \\ T \end{pmatrix} = \dot{T} \tag{7}$$

or

$$\dot{T} = (-T, I) \begin{pmatrix} A_{11} + A_{12}T \\ A21 + A_{22}T \end{pmatrix} \tag{8}$$

$$\dot{T} = -TA_{11} - TA_{12}T + A_{22}T + A_{21} \tag{9}$$

if

$$A_{12} = A_{21} = 0 \tag{10}$$

then

$$\dot{T} = -TA_{11} + A_{22}T \tag{11}$$

with the associated dynamical system

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{12}$$

with

$$\Phi_{A_{11}} = e^{A_{11}(t-to)} \tag{13}$$

and

$$X_1(t) = \Phi_{A_{11}} X_1(0) \tag{14}$$

and

$$\Phi_{A_{22}} = e^{A_{22}(t-to)} \tag{15}$$

and

$$X_2(t) = \Phi_{A_{22}}(t) X_2(0) \tag{16}$$

The generalized inverse is

$$X_1^*(t) = X_1^*(0)\Phi^{-1}_{A_{11}} \tag{17}$$

By Equation (2) T(t) has factor

$$T(t) = X_2 X_1^*(t) \tag{18}$$

or

$$T(t) = \Phi_{A_{22}}(t) X_2(0) X_1^*(0) \Phi^{-1}_{A_{11}}(t,to) \tag{19}$$

or

$$T(t) = \Phi_{A_{22}}(t) T(0) \Phi^{-1}_{A_{11}}(t) \tag{20}$$

Section 36 SOLUTION OF $\dot{T} = -TA_{11} + A_{22} T + A_{21}$

The solution of the bi-linear non-homogeneous matrix differential equation of the form

$$\dot{T} = -TA_{11} + A_{22} T + A_{21} \tag{1}$$

is obtained. Via Eq. (9) of the previous section

$$A_{12} = 0 \tag{2}$$

hence Eq. (1) becomes

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{3}$$

with

$$T(t) = X_2(t)X_1*(t) \tag{4}$$

and

$$T(o) = X_2(to)X_1*(to) \tag{5}$$

Obtain T(t) as a solution of the system

$$\dot{T}_c(t) = -T_cA_{11} + A_{22}T_c + A_{21} \tag{6}$$

with

$$T(t) = T_c(t) + N(t) \tag{7}$$

or

$$N(t) = T(t) - T_c(t) \tag{8}$$

and

$$\dot{N}(t) = -N(t)A_{11} + A_{22}N(t) \tag{9}$$

with initial condition

$$N(to) = T(to) - T_c(to) \tag{10}$$

select

$$T_c(to) = 0 \tag{11}$$

then

$$N(to) = T(to) \tag{12}$$

By Eq. (21) of section 8

$$N(t) = \Phi_{A_{22}}(t)\ T(to)\ \Phi^{-1}_{A_{11}}(t) \tag{13}$$

The system $T_c(t)$ via Eq. (4) has factors

$$T_c(t) = [X_2(t)X_1{*}(t)]_c \tag{14}$$

with

$$X_{2c}(t) = T_c(t)X_{1c}(t) \tag{15}$$

and initial condition

$$X_{2c}(to) = T_c(to)X_{1c}(to) = 0\ . \tag{16}$$

Select

$$X_{1c}(to) = X_1(to) \tag{17}$$

By Eq. (3) the first coordinate is uncoupled hence

$$X_{1c}(t) = e^{A_{11}(t-to)}\ X_1(to) \tag{18}$$

The second matrix coordinate of Eq. (3) is

$$\dot{X}_{2c}(t) = A_{22}X_{2c}(t) + A_{21}X_{1c}(t) \tag{19}$$

or by Eq. (18) in Eq. (19)

$$\dot{X}_{2c}(t) = A_{22}X_{2c}(t) + A_{21}\Phi_{A_{11}}(t,\ to)X_1(to) \tag{20}$$

which is a linear non-homogeneous equation with solution given by Eq. (16) section 34

$$X_{2c}(t) = \Phi_{A_{22}}(t,\ to)X_{2c}(to)+\Phi_{A_{22}}(t,\ to)\int_{to}^{t}\Phi^{-1}_{A_{22}}(\tau,\ to)A_{21}\Phi_{A_{11}}(\tau,\ to)X_1(to)d\tau \tag{21}$$

The initial condition by Eq. (16) is zero, hence Eq. (21) becomes

$$X_{2c}(t) = \Phi_{A_{22}}(t, to)\int_{to}^{t}\Phi_{A_{22}}^{-1}(\tau, to)A_{21}\Phi_{A_{11}}(\tau, to)X_1(to)d\tau \tag{22}$$

By Eq. (18)

$$X_{1c}^*(t) + X_1^*(to)\Phi_{A_{11}}^{-1}(t, to) \tag{23}$$

Using Eq. (22) and Eq. (23) in Eq. (14)

$$T_c(t) = \Phi_{A_{22}}(t, to)\left[\int_{to}^{t}\Phi_{A_{22}}^{-1}(\tau, to)A_{21}\Phi_{A_{11}}(\tau, to)d\tau\right]\Phi_{A_{11}}^{-1}(t, to) \tag{24}$$

Using Eq. (13) and Eq. (24) in Eq. (7)

$$T(t) = \Phi_{A_{22}}(t)T(to)\Phi_{A_{11}}^{-1}(t, to)$$

$$+ \Phi_{A_{22}}(t, to)\int_{to}^{t}\Phi_{A_{22}}^{-1}(\tau, to)A_{21}\Phi_{A_{11}}(\tau, to)d\tau \quad \Phi_{A_{11}}^{-1}(t, to) \tag{25}$$

or

$$T(t) = \Phi_{A_{22}}(t, to)\left[T(to)+\int_{to}^{t}\Phi_{A_{22}}^{-1}(\tau, to)A_{21}\Phi_{A_{11}}(\tau, to)d\tau\right]\Phi_{A_{11}}^{-1}(t, to) \tag{26}$$

which is the solution of Eq. (1).

Note by section 32 Eq. (12)

$$\dot{\Phi}_{A_{11}} = A_{11}\,\Phi_{A_{11}}(t, to) \tag{27}$$

with

$$\Phi_{A_{11}}(to) = I \tag{28}$$

and

$$\dot{\Phi}_{A_{22}} = A_{22}\,\Phi_{A_{22}}(t, to) \tag{29}$$

with

$$\Phi_{A_{22}}(to) = I \tag{30}$$

Section 37 SOLUTION $\dot{T} = -TA_{12}T$

The solution of the homogeneous quadratic differential equation

$$\dot{T} = -TA_{12}T \tag{1}$$

with implied coupled dynamics

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} 0 & A_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{2}$$

where

$$X_2 = TX_1 \tag{3}$$

By Eq. (2)

$$\dot{X}_2(t) = 0 \tag{4}$$

or

$$X_2(t) = X_2(to) \tag{5}$$

a constant.

Using Eq. (5) in Eq. (2)

$$\dot{X}_1 = A_{12}X_2 = A_{12}X_2(to) \tag{6}$$

integrating Eq. (6)

$$X_1(t) = X_1(to) + A_{12}X_2(to)(t-to) \tag{7}$$

Using Eq. (3) in Eq. (7)

$$X_1(t) = [I + A_{12}T(to)(t-to)]X_1(to) \tag{8}$$

The psuedo inverse of Eq. (8) is

$$X_1^*(t) = X_1^*(to)[I + A_{12}T(to)(t-to)]^{-1} \tag{9}$$

using Eq. (9) in Eq. (3)

$$\begin{aligned} T(t) &= X_2X_1^*(t) \\ &= X_2(to)X_1^*(to)[I + A_{12}T(o)(t-to)]^{-1} \end{aligned} \tag{10}$$

$$T(t) = T(to)[I + A_{12}T(o)\ (t-to)]^{-1} \tag{11}$$

The above derivations are straight forward, however, one can obtain a solution via

$$\dot{T}^{-1} = -\ T^{-1}\dot{T}T^{-1} \tag{12}$$

or using Eq. (1) in Eq. (12)

$$\dot{T}^{-1} = A_{12} \tag{13}$$

Integrating Eq. (13)

$$T^{-1}(t) = T^{-1}(to) + A_{12}(t-to) \tag{14}$$

also Eq. (14) can be written as

$$T^{-1}(t) = [I + A_{12}T(to)(t-to)]T^{-1}(to) \tag{15}$$

Inverting Eq. (15) one obtains

$$T(t) = T(to)[I + A_{12}T(to)(t-to)]^{-1} \tag{16}$$

which agrees with Eq. (11).

An interesting case occurs when A_{12} is a rank-one dyad say

$$A_{12} = a\rangle_1 \, {}^2\langle a \tag{17}$$

By Eq. (14)

$$T(t) = [T^{-1}(to) + a\rangle_1 \, {}^2\langle a \, \sigma]^{-1} \tag{18}$$

where

$$\sigma = t-to \tag{19}$$

Using the Householder matrix inversion limma one obtains

$$T(t) = T(to) \quad I - \frac{a\rangle_1 \, {}^2\langle a \, \sigma}{1 + {}_1\langle aa\rangle_2 \sigma} \tag{20}$$

Section 38 <u>SOLUTION $\dot{T} = -TA_{12}T + A_{21}$</u>. This section obtains a solution of the non-homogeneous quadratic equation

$$\dot{T} = - TA_{12}T + A_{21} \tag{1}$$

$$T(to) = To \tag{2}$$

in terms of a controlled matrix solution to Equation (1) with a controlled initial condition which may simplify the computation. Assume we can select the IC such that

$$\dot{T}_c = - T_cA_{12}T_c + A_{12} \tag{3}$$

with

$$T_c(to) = T_{co} \tag{4}$$

Note that the simplified dynamical system is

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{5}$$

with

$$X_2 = T(t)X_1 \tag{6}$$

Since Equation (3) is non-homogeneous it can have dynamics with

$$T_c(to) = 0 \tag{7}$$

Note that if we try the inverse dynamics

$$\dot{T}^{-1} = A_{12} - T^{-1}A_{21}T^{-1} \tag{8}$$

which is the same form as Equation (1) with no simplification.

Let the desired solution be

$$T(t) = T_c(t) + N(t) \tag{9}$$

where

$$N(t) = T(t) - T_c(t) \tag{10}$$

$$N(to) = T(to) \tag{11}$$

The derivative of Equation (10 is

$$\dot{N} = \dot{T} - \dot{T}_c \tag{12}$$

Using Equation (1) and Equation (3) in Equation (12)

$$\dot{N} = - TA_{12}T + T_cA_{12}T_c \tag{13}$$

and by Equation (9) in Equation (13)

$$\dot{N} = - N(A_{12}T_c) + (- T_cA_{12})N - NA_{12}N \tag{14}$$

or

$$\dot{N} = - NB_{11} + B_{22}N - NB_{12}N \tag{15}$$

with

$$N(to) = T(to) \tag{16}$$

and

$$\begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} = \begin{pmatrix} A_{12}T_c & A_{12} \\ 0 & -T_cA_{12} \end{pmatrix} \tag{17}$$

The homogeneous system of Equation (15) is solved in Section (26) by Equation (7) and is

$$N(t) = \Phi_{B_{22}}(t,to)[T(to) + \int_{to}^{t} \Phi_{B_{11}}(\tau,to)A_{12}\Phi_{B_{22}}(\tau,to)d\tau]^{-1}\Phi_{B_{11}}^{-1}(t,to) \tag{18}$$

with

$$\dot{\Phi}_{B_{11}} = A_{12}T_c(t)\Phi_{B_{11}}(t,to) \tag{19}$$

with

$$\Phi_{B_{11}}(to,to) = I \tag{20}$$

and

$$\dot{\Phi}_{B_{22}} = - T_cA_{12}\Phi_{B_{22}}(t,to) \tag{21}$$

with

$$\Phi_{B_{22}}(to,to) = I \tag{22}$$

Using Equation (18) in Equation (9)

$$T(t) = T_c(t) + N(t) \tag{23}$$

or

$$T(t) = T_c(t) + \Phi_{B_{22}}(t,to)[T(to) + \int_{to}^{t} \Phi_{B_{11}}(\tau,to)A_{12}\Phi_{B_{22}}(\tau,to)d\tau]^{-1}\Phi_{B_{11}}^{-1}(t,to) \tag{24}$$

Section 39 SOLUTION OF $\dot{T} = -TA_{11} + A_{22}T - TA_{12}T$

This section obtains a solution of the homogeneous bilinear quadratic equation

$$\dot{T} = -TA_{11} + A_{22}T - TA_{12}T \tag{1}$$

with implied dynamical system

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{2}$$

where

$$X_2 = TX_1 \tag{3}$$

The dynamics of T^{-1} is

$$\dot{T}^{-1} = -T^{-1}\,\dot{T}T^{-1} \tag{4}$$

or

$$\dot{T}^{-1} = A_{11}T^{-1} - T^{-1}\,A_{22} + A_{12} \tag{5}$$

which is non-homogeneous and bi-linear. The solution of Equation (5) by section 33, equation (26) is

$$T^{-1}(t) = \Phi_{A_{11}}(t, to)\,\Big[T(to) + \int_{to}^{t} \Phi_{A_{11}}(\tau, to)A_{12}\Phi_{A_{22}}(\tau, to)d\tau\Big]\Phi_{A_{22}}^{-1}(t, to) \tag{6}$$

or inverting one obtains

$$T(t) = \Phi_{A_{22}}(t, to)\Big[T(to) + \int_{to}^{t} \Phi_{A_{11}}(\tau, to)A_{12}\Phi_{A_{22}}(\tau, to)d\tau^{\;-1}\Big]\Phi_{A_{11}}^{-1}(t, to) \tag{7}$$

Section 40 SOLUTION $\dot{T} = -TA_{11} + A_{22}T - TA_{12}T + A_{21}$

This section like the previous section obtains a solution of a non-homogeneous with bilinear and quadratic terms as a function of the solution to the same equation but with controlled (arbitrary initial conditions). As before let

$$T(t) = T_c(t) + N(t) \tag{1}$$

with

$$T(to) = N(to) \tag{2}$$

and

$$T_c(to) = 0 \tag{3}$$

By Equation (1)

$$N(t) = T(t) - T_c(t) \tag{4}$$

or

$$\dot{N} = \dot{T} - \dot{T}_c(t) \tag{5}$$

Using the equation

$$\dot{T} = -TA_{11} + A_{22}T - TA_{12}T + A_{21} \tag{6}$$

in equation (5) one obtains

$$\dot{N}(t) = -N(A_{11} + A_{12}T_c) + (A_{22} - T_cA_{12})N - NA_{12}N \tag{7}$$

or

$$\dot{N} = -NB_{11} + B_{22}N - NB_{12}N \tag{8}$$

where

$$B_{11} = A_{11} + A_{12}\,T_c \tag{9}$$

$$B_{22} = A_{22} - T_c\,A_{12} \tag{10}$$

$$B_{12} = A_{12} \tag{11}$$

Note that equation (8) is missing the force function B_{21}, or is homogeneous. The solution of equation (8) is obtained in section 38, equation (7) as

$$N(t) = \Phi_{B_{22}}(t, to)\left[T(to) + \int_{to}^{t} \Phi_{B_{11}}(\tau, to) A_{12} \Phi_{B_{22}}(\tau, to) d\tau\right]^{-1} \Phi_{B_{11}}^{-1}(t, to) \quad (12)$$

where

$$\dot{\Phi}_{B_{11}}(t, to) = (A_{11} + A_{12}\, T_c)\, \Phi_{B_{11}}(t, to) \quad (13)$$

$$\Phi_{B_{11}}(t, to) = I \quad (14)$$

$$\dot{\Phi}_{B_{22}}(t, to) = (A_{22} - T_c\, A_{12})\, \Phi_{B_{22}}(t, to) \quad (15)$$

with

$$\Phi_{B_{22}}(t, to) = I \quad (16)$$

Using equation (12) in equation (11)

$$T(t) = T_c(t) + N(t) \quad (17)$$

where

$$T_c(to) = 0 \quad (18)$$

and $N(t)$ is given by equation (12) through equation (16).

It should be noted that in ref (47) Lainiotis uses this solution technique for the special case of the Hamiltonian system, that is

$$A_h = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & -A_{11}^T \end{pmatrix} \quad (19)$$

with

$$A_{12}^T = A_{12} \quad (20)$$

and

$$A_{21}^T = A_{21} \quad (21)$$

Section 41 THE CONTINUOUS KALMAN FILTER

Consider a sequence of $j \geq n$ dynamical systems driven by a deterministic forcing vector functions $f(t)$ and a stochastic forcing vector $u(t)_j$ where $u(t)$ is different for each j, that is the package

$$[u\rangle_1,\ u\rangle_2\ .\ .\ u\rangle_j] = \underset{l\times j}{U} \tag{1}$$

for $u(t)$ an ℓ dimensional vector, with mean

$$U1^*\rangle = \mu\rangle_u \tag{2}$$

has full rank, that is

$$\rho U = \ell \tag{3}$$

The deterministic forcing vector $f(t)$ for the package has rank one

$$[f\rangle,\ f\rangle,\ .\ .\ .\ f\rangle] = f\rangle\langle 1 \tag{4}$$

and mean

$$(f\rangle\langle 1)1^*\rangle = f\rangle \tag{5}$$

Having thus characterized the nature of the forcing vectors, the differential equation is assumed to be of the form

$$\dot{x}\rangle_g = Ax\rangle_g + f\rangle + By\rangle_g \tag{6}$$

or for the package

$$\dot{X} = AX + f\rangle\langle 1 + BU \tag{7}$$

Assume further that we have the linear observational equation relating the state vector to the measurement vector $z\rangle$ of dimension m through the known transformation matrix $H(t)$ plus additive measurement noise.

$$Z\rangle_j = \underset{m\times n}{H}x\rangle_j + u\rangle_j \tag{8}$$

or for the package

$$\underset{m\times j}{Z} = HX + \underset{m\times j}{V} \tag{9}$$

with measurement vector noise mean of

$$V1^*\rangle = \mu\rangle_v \tag{10}$$

Equation (6) is called the process dynamics or plant and equation (8) the measurement vector. Equation (6) is to be thought of as the "true process dynamics" and will be referred to as the "truth model". In actual practice one usually linearize the "true" non-linear process dynamics; in which case Equation (6) would represent the linearized equations and must be treated with all due respect for linear regions in which they are valid.

The process noise and the measurement noise are further characterized by their variances. Each of the stochastic noise vectors can be described as

$$\tilde{u}\rangle_j = u\rangle_j - \mu\rangle_u \tag{11}$$

a package-wise

$$\tilde{U} = U - \mu\rangle_u\langle 1 \tag{12}$$

The rank-one dyad of Eq. (12) by Eq. (2) can be written as

$$\mu\rangle_u\langle 1 = U1^*\rangle\langle 1 = UP_{11} = \hat{U} \tag{13}$$

where in "population space" or the larger j-space

$$\underset{j \times j}{P_{11}} = 1^*(j\rangle\langle j) = \frac{1\rangle\langle 1}{\langle 11\rangle} \tag{14}$$

is the rank-one projector.

Using Eq. (13) in Eq. (12)

$$\tilde{U} = U - UP_{11} = U(I - P_{11}) \tag{15}$$

or

$$\tilde{U} = U\tilde{P}_{11} \tag{16}$$

where the orthogonal complement projector is

$$\tilde{P}_{11} = I - 1^* \quad 1 \tag{17}$$

which is rank j-1.

A similar relation holds for the measurement vector

$$v_j = \mu_v + \tilde{v}_j \tag{18}$$

with zero mean $\tilde{v}$, that is

$$\tilde{V}1^* = V\tilde{P}_{11}1^* = 0 \tag{19}$$

The variance of the process noise is

$$E[u_j - \mu_u)(\,_ju - \mu_u)] = E\,\tilde{u}_j\,{}^j\tilde{u} \tag{20}$$

or matrix-wise

$$Q(t,t) = \lim_{j\to\infty} \tilde{U}\tilde{U}^T \frac{1}{jmax} = \sum_{j=1}^{jmax} (\tilde{u}\rangle_j {}^j\langle\tilde{u}) \frac{1}{jmax} \tag{21}$$

The limit as j max goes to countably infinite will be be assumed in all of the relation and not written down.

The serially uncorrelated assumption is used, that is

$$E(\tilde{u}\rangle_j {}^j\langle\tilde{u}) = Q(t,\tau)\delta(t,\tau) \tag{22}$$

and similarly for the measurement noise

$$E(\tilde{u}\rangle_j {}^j\langle\tilde{u}) = R(t,\tau)\delta) \tag{23}$$

The correlation between the two vectors is assumed zero, that is

$$\tilde{U}\tilde{V}^T = E(\tilde{u}\rangle\langle\tilde{v} = 0 \tag{24}$$

The mean of Eq. (7) is

$$\dot{\mu}_x\rangle = A\mu_x\rangle + f(t)\rangle + B\mu\rangle_u \tag{25}$$

and the mean of Eq. (9) is

$$\mu\rangle_z = H\mu\rangle_x + \mu\rangle_v \tag{26}$$

The dynamics of the error vectors is

$$\tilde{x}\rangle_j = \dot{x}\rangle_j - \dot{\mu}\rangle_x \tag{27}$$

a package-wise

$$\tilde{X} = A\tilde{X} + B\tilde{U} \tag{28}$$

The observation error is

$$z\rangle_j - \mu\rangle_z = \tilde{z}\rangle_j \tag{29}$$

or package-wise

$$\tilde{Z} = H\tilde{X} + \tilde{V} \tag{30}$$

$$\tilde{Z} = (H,I) \begin{pmatrix} \tilde{X} \\ \tilde{V} \end{pmatrix}$$

at time τ the transpose of Eq. (30) is

$$\tilde{Z}^T = [\tilde{X}^T,\ \tilde{V}^T] \begin{bmatrix} H^T \\ I \end{bmatrix} \tag{31}$$

or

$$\tilde{Z}\tilde{Z}^T = (H,I) \begin{bmatrix} \tilde{X}\tilde{X}^T & \tilde{X}\tilde{V}^T \\ \tilde{V}\tilde{X}^T & \tilde{V}\tilde{V}^T \end{bmatrix} \begin{pmatrix} H^T \\ I \end{pmatrix} \quad (32)$$

assume the measurement noise is independent of the state or

$$E[\tilde{x}\rangle\langle\tilde{v}] = 0 = \tilde{X}\tilde{V}^T \quad (33)$$

Dividing Eq. (32) by j max one obtains

$$E[\tilde{z}\rangle\langle\tilde{z}] = \Phi_{\tilde{z}\tilde{z}}(t,\tau) \quad (34)$$

or Eq. (33) in Eq. (32)

$$\Phi_{\tilde{z}\tilde{z}}(t,\tau) = (H,I) \begin{bmatrix} \Phi_{\tilde{x}\tilde{x}}(t,\tau) & 0 \\ 0 & R(t,\tau)\delta(t,\tau) \end{bmatrix} \begin{pmatrix} H^T \\ I \end{pmatrix} \quad (35)$$

or

$$\Phi_{\tilde{z}\tilde{z}} = H\Phi_{\tilde{x}\tilde{x}}H^T + R(t,\tau)\delta(t,\tau) \quad (36)$$

Consider next a man-made estimate of the state vector with dynamics having the structure

$$\dot{\hat{x}}(t,t)\rangle = A\,\hat{x}(t,t)\rangle + f(t)\rangle + C(t)\rangle \quad (37)$$

where the control vector $c(t)\rangle$ (controlling $\hat{x}(t,t)\rangle$ to follow $x(t)\rangle$ in some sense to be defined later) has the structure

$$c(t)\rangle = W(t)\tilde{z}(t,t)\rangle \quad (38)$$

where the error vector $\tilde{z}(t,t)$ is given by

$$\tilde{z}(t,t)\rangle = z(t)\rangle - \hat{z}(t,t)\rangle$$

$$\hat{Z}(t,t)\rangle_j = H\hat{x}(t,t)\rangle + \tilde{U}\rangle \quad (39)$$

and $\hat{z}(t,t)$ is the estimate of the measurement vector

$$\hat{z}(t,t)\rangle = H\,\hat{x}(t,t)\rangle + \mu\rangle_u \quad (40)$$

Using Eq. (38) in Eq. (37)

$$\dot{\hat{x}}(t,t)\rangle_j = A\hat{x}(t,t)\rangle_j + f(t)\rangle + B\mu(t)\rangle_u + W(t)\,\tilde{z}(t,t)\rangle_j \quad (41)$$

One can think of Eq. (41) as a Curve follower or positioning-serve problem with feedback matrix W(t) and when $\tilde{z}(t,t)$ is zero, the system of Eq. (41) is following Eq. (6)

The error in estimate of the state vector is

$$\tilde{x}(t,t)\rangle = x(t)\rangle - \hat{x}(t,t)\rangle \quad (42)$$

or package-wise

$$\tilde{X}(t,t) = X(t) - \hat{X}(t,t) \tag{43}$$

The variance of the state estimate is given by

$$E(\tilde{x}\rangle\langle\tilde{x}(t,t)) = \frac{\tilde{X}(t,t)\tilde{X}^T(t,t)}{j\ max} = P(t,t) \tag{44}$$

The minimization criterion is to find a W(t) which minimizes the trace of P(t,t) which is

$$E(\underset{j}{\langle}\tilde{x}\tilde{x}\rangle_j) = tr\left(\frac{\tilde{X}\tilde{X}^T}{j\ max}\right) = tr\ P(t,t) \tag{45}$$

and then to find the dynamics $\dot{P}(t)$ for the optimal W(t).

The error dynamics Eq. (6) minus Eq. (4) is

$$\dot{\tilde{x}}(t,t)\rangle_j = A\tilde{x}(t,t)\rangle + B\tilde{u}(t)\rangle_j - W(t)\ \tilde{z}(t,t)\rangle_j \tag{46}$$

Using Eq. (39) in Eq. (46)

$$\dot{\tilde{x}}(t,t)\rangle_j = (A-WH)\ \tilde{x}(t,t)\rangle_j + B\tilde{u}\rangle_j - W\tilde{u}\rangle_j \tag{47}$$

or package-wise

$$\dot{\tilde{X}}(t,t) = A_E\tilde{X}(t,t) + B\tilde{U}(t,t) - W\tilde{V}(t) \tag{48}$$

with

$$A_E = A - WH \tag{49}$$

The homogeneous system has dynamics

$$\dot{\tilde{X}}_h = A_E\tilde{X}_h \tag{50}$$

and the fundamental matrix dynamics

$$\dot{\Phi}_E = A_E\Phi_E \tag{51}$$

or

$$\Phi_E(t,to) = e^{\int_{to}^{t}(A-WH)dt} \tag{52}$$

with

$$\Phi_E(to,to) = I \tag{53}$$

The solution matrix to Eq. (48) in fundamental base Φ_E is

$$\tilde{X}(t,to) = \Phi_E(t,to)\tilde{X}^{\phi}(t,to) \tag{54}$$

The time derivative yields.

$$\dot{\tilde{X}}(t,to) = \dot{\Phi}_E\tilde{X}^{\phi} + \Phi_E\dot{\tilde{X}}^{\phi} \tag{55}$$

or via Eq. (48) and Eq. (52) in Eq. (55)

$$A_E\tilde{X} + B\tilde{U} - W\tilde{V} = A\Phi_E\tilde{X}^{\phi} + \Phi_E\dot{\tilde{X}}^{\phi} \tag{56}$$

or

$$B\tilde{U} - W\tilde{V} = \Phi_E\dot{\tilde{X}}^{\phi} \tag{57}$$

or the apparent velocity is

$$\dot{\tilde{X}}^{\phi} = \Phi_E^{-1}(t,to)[B\tilde{U}-W\tilde{V}] \tag{58}$$

The solution in the Φ_E bases is

$$\tilde{X}^{\phi}(t,to) = \tilde{X}^{\phi}(to,to) + \int_{to}^{t} \Phi_E^{-1}(\tau,to)[B\tilde{U}-W\tilde{V}]d\tau \tag{59}$$

The variance by Eq. (44) and Eq. (54) is

$$P(t) = \Phi_E(t,to)P^{\phi}(t)\Phi_E(t,to) \tag{60}$$

where the variance observed in the Φ_E base is

$$P^{\phi}(t,to) = \tilde{X}^{\phi}(t,to)\tilde{X}^{\phi T}(t,to) \; \frac{1}{j\ max} \tag{61}$$

The dynamics of Eq. (60) is

$$\dot{P} = \dot{\Phi}_E(t,to)P^{\phi}\Phi_E^T + \Phi_E\dot{P}^{\phi}\Phi_E^T + \Phi_E P^{\phi}\dot{\Phi}_E^T \tag{62}$$

Using Eq. (51) and its transpose in Eq. (62)

$$\dot{P} = A_E\Phi_E P^{\phi}\Phi_E^T + \Phi_E P^{\phi}\Phi_E^T A_E^T + \Phi_E\dot{P}^{\phi}\Phi_E^T \tag{63}$$

and by Eq. (60) in Eq. (63)

$$\dot{P} = A_E P + PA_E^T + Q_E\dot{P}^{\phi}\Phi_E^T \tag{64}$$

The final term of Eq. (64) must next be evaluated by Eq. (61)

$$\dot{P}^{\phi} = (\dot{\tilde{X}}^{\phi}\tilde{X}^{\phi T} + \tilde{X}^{\phi}\dot{\tilde{X}}^{\phi T}) \; \frac{1}{j\ max} \tag{65}$$

Consider the last term of Eq. (65) by Eq. (59) and the transpose of Eq. (58)

$$\tilde{X}^{\phi}\dot{\tilde{X}}^{\phi T} = \left\{\tilde{X}^{\phi}(to,to) + \int_{to}^{t} \Phi_E^{-1}(\tau,to)[B\tilde{U}(\tau) - W\tilde{V}(\tau)]d\tau\right\} \tag{66}$$

$$x[\tilde{U}(t)B^T - \tilde{V}^T(t)W^T(t)]\Phi_E^{-1}(t,to)$$

note by Eq. (54) and (53)

$$\tilde{X}^{\phi}(to,to) = \tilde{X}(to,to) \tag{67}$$

and the assumption is made

$$E[\tilde{x}(to,to)\rangle\langle\tilde{u}(t)] = E[\tilde{x}(to,to)\rangle\langle\tilde{v}(t)] = 0 \tag{68}$$

or

$$\tilde{X}(to,to)\tilde{U}^{T}(t) = 0 \tag{69}$$

and

$$\tilde{X}(to,to)\tilde{V}^{T}(t) = 0 \tag{70}$$

thus Eq. (66) becomes

$$\tilde{X}^{\phi}(t)\tilde{X}^{\phi T} = \int_{to}^{t} \Phi^{-1}(\tau,to)[B\tilde{U}(\tau) - W\tilde{V}(\tau)]\ [\tilde{U}^{T}(t)B^{T} - \tilde{V}^{T}W^{T}]\Phi^{-T}(t,to)d\tau \tag{71}$$

The parenthesis terms become

$$[B\tilde{U}(\tau) - W\tilde{V}(\tau)]\ [\tilde{U}^{T}(t)B^{T} - \tilde{V}^{T}(t)W^{T} = B\tilde{U}(\tau)\tilde{U}^{T}(t)B^{T} + W\tilde{V}(\tau)\tilde{V}^{T}(t)W \tag{72}$$

where the relation of Eq. (24) has been used. Using Eq. (22) and Eq. (23) in and Eq. (72) in Eq. (71)

$$\tilde{X}^{\phi}\tilde{X}^{\phi T} = \int_{to}^{t} \Phi_{E}^{-1}(\tau,to)[BQ(\tau,t)\delta(\tau-t)B^{T} + WR(\tau,t)\delta(\tau,t)W^{T}]\Phi_{E}^{-1}d\tau \tag{73}$$

applying the $\delta(\tau,t)$ function as defined in appendix (a) Eq. (17) we find

$$\tilde{X}^{\phi}\tilde{X}^{\phi T} = \tfrac{1}{2}\Phi_{E}^{-1}(t,to)[BQ(t)B^{T} + WR(t)W^{T}]\Phi_{E}^{-1}(t,to) \tag{74}$$

The transpose of Eq. (74) is

$$\tilde{X}^{\phi}\tilde{X}^{\phi T} = \tfrac{1}{2}\Phi^{-1}(t,to)[BQB^{T} + WRW^{T}]\Phi^{-T}(t,to) \tag{75}$$

Using Eq. (74) and Eq. (75) in Eq. (65)

$$\dot{P}^{\phi} = \Phi_{E}^{-1}(t,to)[BQB^{T} + WRW^{T}]\Phi_{E}^{-T}(t,to) \tag{76}$$

which is the varience dynamics observed by an observer in the Φ_E space. Using Eq. (76) in Eq. (64)

$$\boxed{\dot{P} = A_{E}P + PA_{E}^{T} + BQB^{T} + WRW^{T}} \tag{77}$$

Eq. (77) is valid for any W(t) selected. An optimal W(t) has not as yet been found. Eq. (77) is a matrix Riccatti and implies a dynamical system

$$Y_2 = PY_1 \tag{78}$$

or

$$\tilde{X}\tilde{X}^T = P = Y_2 Y_1^* \tag{79}$$

or

$$P = Y_2 Y_1^T (Y_1 Y_1^T)^{-1} \tag{80}$$

Since P(t) is symmetric so is

$$G_{21} = Y_2 Y_1^T = Y_1 Y_2^T \tag{81}$$

By Eq. (211) Section (25)

$$\begin{pmatrix} \dot{Y}_1 \\ \dot{Y}_2 \end{pmatrix} = \begin{pmatrix} -A_E^T & 0 \\ BQB^T + WRW^T & A_E \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \tag{82}$$

which is a Hamiltonian system with solution given by Eq. (12) Section (36) as

$$P(t) = \Phi_{E22}(t,t0) \quad P(to) + \int_{to}^{t} \Phi_E^{-1}(\tau,to)[BQB^T + WRW^T]\Phi_{E11}(\tau,to)]\Phi_{E11}^{-1}(t,to) \tag{83}$$

where

$$\dot{\Phi}_{E11} = -A_E^T \Phi_{E11} \tag{84}$$

and

$$\dot{\Phi}_{E22} = A_E \Phi_{E22} \tag{85}$$

note that the term of Eq. (62)

$$\dot{P}_p = \Phi_E \dot{P}^{\Phi} \Phi_E^T \tag{86}$$

by Eq. (76) becomes

$$\dot{P}_p = BQB^T + WRW^T \tag{87}$$

The homogeneous part of Eq. (64) is

$$\dot{P}_h = A_E P_h + P_h A_E^T \tag{88}$$

with solution given by Eq. (20) Section (30) as

$$P_h(t) = \Phi_E(t,to)P(to,to)\Phi_E^T(t,to) \tag{89}$$

Thus Eq. (77) can be viewed as

$$\dot{P} = \dot{P}_h + \dot{P}_p \tag{90}$$

Consider now Eq. (77) which we seek to minimize, that is

$$\dot{P} = A_E P + PA_E^T + BQB^T + WRW^T \tag{91}$$

where by Eq. (49)

$$A_e = A - WH \tag{92}$$

By definition of

$$P = E\left\{\tilde{x}(t,t)\right\rangle\left\langle\tilde{x}(t,t)\right\} = \lim_{jmax \to 0} \left[\tilde{X}\tilde{X}^T \frac{1}{jmax}\right] \tag{93}$$

when trace of $\dot{P}$ is minimized with respect to the matrix W, then trace P is also a minimum. The gradient matrice's of appendix (F) are used to find optimal W, via

$$\frac{\partial \dot{P}}{\partial W} = \frac{\partial}{\partial W^T}\left(A_E P + P A_E^T + WRW^T\right) \tag{94}$$

since

$$\frac{\partial}{\partial W^T}\left(BQB^T\right) = 0 \tag{95}$$

By appendix (F) summary

$$\frac{\partial}{\partial W^T}\left[(A-WH)P\right] = -HP \tag{96}$$

Transposing Eq. (96)

$$\frac{\partial (pA_E^T)}{\partial W^T} = -HP \tag{97}$$

The quadratic term of Eq. (94) has gradient given by the transpose of Eq. (51) appendix (F) as

$$\frac{\partial (WRW^T)}{\partial W^T} = 2RW^T \tag{98}$$

Using Eq. (96), Eq. (97) and Eq. (98) in Eq. (24)

$$\frac{\partial \dot{P}}{\partial W^T} = -HP - HP + 2RW^T = 0 \tag{99}$$

or

$$HP = RW^T \tag{100}$$

or

$$W^T = R^{-1}HP \tag{101}$$

and transposing

$$W = PH^TR^{-1} \tag{102}$$

which is the optimal gain.

Using Eq. (102) in Eq. (91)

$$\dot{P} = (A-PH^TR^{-1}H)P + P(A^T-H^TR^{-1}HP) + BQB^T + PH^TR^{-1}RR^{-1}HP \tag{103}$$

or

$$\dot{P} = AP + PA^T - PH^TR^{-1}HP + BQB^T \tag{104}$$

Equation (104) is a matrix Riccatti and implies a dynamical system

$$X_2(t) = P(t)X_1(t) \tag{105}$$

where by Eq. (213) Section (25)

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} -A^T & H^TR^{-1}H \\ BQB^T & A \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{106}$$

which is a Hamiltonian system. If X_1 and X_2 are nxn matrices

$$P(t) = X_2X_1^{-1} \tag{107}$$

Expressing the inverse as

$$X_1^{-1} = X_1^T(X_1X_1^T)^{-1} \tag{108}$$

or

$$P(t) = X_2X_1^T(X_1X_1^T)^{-1} \tag{109}$$

The product

$$X_2X_1^T = I \qquad 110$$

and

$$P(t) = (X_1X_1^T)^{-1} \tag{111}$$

The initial condition for Eq. (105) for any $X_1(t)$ implies

$$X_2(to) = P(to)X_1(to) \tag{112}$$

Suppose

$$X_1(to) = I \tag{113}$$

then

$$X_2(to) = P(to) \tag{114}$$

By Eq. (110), for any $X_1(t)$

$$X_2(t) = X_1^{-T}(t) \tag{115}$$

thus Eq. (106) becomes

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_1^{-T} \end{pmatrix} = \begin{pmatrix} -A^T & H^T R^{-1} H \\ BQB^T & A \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_1^{-T}(t) \end{pmatrix} \tag{116}$$

with

$$X_2(to) = P(to) X_1(to) \tag{117}$$

Via Eq. (116) and Eq. (82) one may ask what is T for

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = T \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{118}$$

Summarizing we have the plant stochastic dynamics given by Eq. (6)

$$\dot{x}(t)\rangle = Ax\rangle + f(t)\rangle + B(t)u(t)\rangle \tag{119}$$

with noisy measurement given by Eq. (8)

$$z(t)\rangle = H(t)\, x(t)\rangle + u(t)\rangle \tag{120}$$

The optimal estimate of the states has dynamics given by Eq. (37), Eq. (38) and Eq. (102)

$$\dot{\hat{x}}(t,t)\rangle = A\hat{x}(t,t)\rangle + f(t)\rangle + P(t)H^T R^{-1}\tilde{z}(t,t)\rangle \tag{121}$$

where by Eq. 39

$$\tilde{z}(t,t)\rangle = z(t)\rangle - \hat{z}(t,t)\rangle \tag{122}$$

and

$$\hat{z}(t,t)\rangle = H(t)\, \hat{x}(t,t)\rangle \tag{123}$$

and by Eq. (104)

$$\dot{P} = AP + PA^T - PH^T R^{-1} H + BQB^T \tag{124}$$

One must initilize Eq. (121) and Eq. (124) with an $\hat{x}(o,o)\rangle$ and P(0).

Section 42 CHANDRASEKHAR TYPE ALGORITHMS FOR SOLUTION OF OPTIMAL KALMAN GAIN MATRIX

Recently a number of papers have appeared which solve the matrix Riccatti differential equation which describes the dynamics of the variance of the state estimation via a system of equations called the Chandrasekhar equations. Kailath and Lainiotis in references 42 and 48 have attempted to obtain algorithms which are stable and require less computations than the straight forward solution of the non-linear Riccatti equation. There is also a close relation between the bias estimation equation's of Friedland and the Chandrasekhar type equations to be discussed in a later section. Both methods diagonalize or decouple equations.

The variance Dynamics by Eq. (124) section 41 is

$$\dot{P} = AP + PA^T - PH^TR^{-1}HP + BQB^T \tag{1}$$

and by Eq. (77) section 40 for any W

$$\dot{P} = A_EP + PA_E^T + BQB^T + WRW^T \tag{2}$$

where by Eq. (49) section 41

$$A_E = A - WH \tag{3}$$

and by Eq. (102) section 41 for the optimal W

$$W = PH^TR^{-1} \tag{4}$$

By Eq. (104) section 41 for the optimal W

$$\dot{P} = AP + PA^T - PH^TR^{-1}HP + BQB^T \tag{5}$$

or

$$\dot{P} = AP + PA^T - WRW^T + BQB^T \tag{6}$$

Eq. (6) should be compared with Eq. (2).

By Eq. (105) and Eq. (106) section 41

$$X_2 = PX_1 \tag{7}$$

with

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} -A^T & H^TR^{-1}H \\ BQB^T & A \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{8}$$

Kailath derives the Chandrasekhar type algorithms for the time invariant case in ref 42 and Lainiotis in ref 48 derives the time varying case. This section will merely obtain the constant matrix system case for tutorial purposes.

The derivative of Eq. (6) is

$$\ddot{P} = A\dot{P} + \dot{P}A^T - \dot{W}RW^T - WR\dot{W}^T \tag{9}$$

where the assumption on Eq. (4) is

$$\dot{W} = \dot{P}H^TR^{-1} \tag{10}$$

The transpose of Eq. (10) is

$$\dot{W}^T = R^{-1}\ H\ \dot{P} \tag{11}$$

and using Eq. (10) and Eq. (11) in Eq. (9)

$$\ddot{P} = A\dot{P} + \dot{P}A^T - \dot{P}H^TW^T - WH\dot{P} \tag{12}$$

or

$$\ddot{P} = (A-WH)\dot{P} + \dot{P}(A^T-H^TW^T) \tag{13}$$

which is homogeneous and non-quadratic, with solution given by Eq. (20) sec 34

$$\dot{P} = \Phi_{AE_{22}}(t)\ \dot{P}(o)\ \Phi^{-1}_{AE_{11}}(t) \tag{14}$$

The Eq. (13) implies

$$V_2 = \dot{P}V_1 \tag{15}$$

with

$$\begin{pmatrix} \dot{V}_1 \\ \dot{V}_2 \end{pmatrix} = \begin{pmatrix} -(A-WH)^T & 0 \\ 0 & (A-WH) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{16}$$

and the fundamental matrix of Eq. (14)

$$\Phi_{AE_{11}}(t) = e^{-\int_{to}^{t}(A-WH)^T dt} \tag{17}$$

and

$$\Phi_{AE_{22}}(t) = e^{\int_{to}^{t} (A-WH)dt} \tag{18}$$

Note that Eq. (16) implies adjoint

$$V_2 = V_1{*}^T \tag{19}$$

hence by Eq. (15)

$$\dot{P} = V_1{*}^T V_1{*} \tag{20}$$

with

$$V_1{*} = V_1^T(V_1V_1^T)^{-1} \tag{21}$$

and

$$V_1{*}^T = (V_1V_1^T)^{-1}V_1 \tag{22}$$

or

$$\dot{P} = (V_1V_1^T)^{-1} \tag{23}$$

also Eq. (14) can be written as

$$\dot{P}(t) = \Phi_{AE_{22}}(t, to)\ \dot{P}(o)\ \Phi^T_{AE_{22}}(t, to) \tag{24}$$

The constant matrix $\dot{P}(0)$ can be expressed as a lower triangular-diagonal-upper triangular (LDU) decomposition

$$\dot{P}(o) = Lo\ S_\sigma\ Lo^T \tag{25}$$

where S_σ is the signature matrix of $\dot{P}(0)$. One can find a theorem on page 91 of Perlis which states that a real symmetric matrix P(0) of rank α is congruent to a matrix S_σ

$$S_\sigma = \begin{pmatrix} I_\beta & 0 & 0 \\ 0 & -I_{\alpha-\beta} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{26}$$

Eliminating the 0's of Eq. (26)

$$\underset{n\times n}{\dot{P}(o)} = \underset{n\times\alpha}{Lo}\ \underset{\alpha\times\alpha}{S_\sigma}\ \underset{\alpha\times n}{Lo^T} \tag{27}$$

Partition Lo in

$$\underset{n\times\alpha}{Lo} = (\underset{n\times\beta}{L1},\ \underset{n\times(\alpha-\beta)}{L2}) \tag{28}$$

and

$$\dot{P}(o) = (L_1, L_2) \begin{pmatrix} I_\beta & 0 \\ 0 & -I_{(\alpha-\beta)} \end{pmatrix} \begin{pmatrix} L_1{}^T \\ L_2{}^T \end{pmatrix} \quad (29)$$

or

$$\dot{P}(o) = L_1 L_1{}^T - L_2 L_2{}^T \quad (30)$$

Using Eq. (25) in Eq. (24)

$$\dot{P}(t) = \Phi_{AE_{22}} L_o S L_o^T \, \Phi_{AE_{22}}^T \quad (31)$$

$$= Y(t, o) \, SY^T(t, o) \quad (32)$$

where

$$Y(t, o) = \Phi_{AE_{22}}(t, o) L_o \quad (33)$$

and

$$Y^T(t, o) = L_o^T \, \Phi_{AE_{22}}^T \quad (34)$$

Eq. (33) can be written as

$$Y(t, o) = \left[Y_1, \; Y_2 \right] = \left[\Phi_{AE_{22}} L_1, \; \Phi_{AE_{22}} L_2 \right] \quad (35)$$

Using Eq. (35) in Eq. (32)

$$\dot{P}(t) = (Y_1, Y_2) \begin{pmatrix} I_\beta & 0 \\ 0 & -I_{(\alpha-\beta)} \end{pmatrix} \begin{pmatrix} Y_1{}^T \\ Y_2{}^T \end{pmatrix} \quad (36)$$

or

$$\underset{n x n}{\dot{P}(t)} = \underset{(n x \beta)}{Y_1} \; \underset{(\beta x n}{Y_1{}^T} - \underset{n(\alpha-\beta)n}{Y_2 Y_2{}^T} \quad (37)$$

By Eq. (32) in Eq. (10)

$$\dot{W} = Y(t, o) SY^T(t, o) H^T R^{-1} \quad (38)$$

and by Eq. (33)

$$\dot{Y}(t, o) = (A-WH) Y(t, o) \quad (39)$$

with

$$Y(o, o) = L_o \tag{40}$$

Equations (38) through Eq. (40) and the equations given by Kailath and for some systems said to be easier to solve.

APPENDIX A
THE BINOMIAL MATRICES, RUTISHAUSER MATRICES AND THEIR INVERSES

There are many areas of stochastic and deterministic dynamical (continuous and discrete) system theory where one is lead to an n-tuple of elements raised to powers. One of the most fundamental is the expansion of the exponentials

$$e^{ut} = 1 + (ut)^1 + \frac{(ut)^2}{2!} + \cdots \frac{(ut)^n}{n!} + \cdots \tag{1}$$

where u may be a real scalar, a complex scalar, a real or complex square matrix. The vector Taylor series expansion of a vector along a scalar (time) generated trajectory, the characteristic equation as a power-series in the eigenvalues, or its matrix analog via the Caley-Hamilton condition, or the fundamental solution of a differential equation. The Fourier series frequency domain analysis of differential equations, or of covariance matrices yielding power spectral matrices through the Laplace, Fourier, etc., transforms. The complex and matrix case will not be strongly developed in this paper.

Consider sequence of reals raised to powers

$$\underline{u} = (u^0, u^1, u^2, u^3, \cdots u^j \cdots d^{d-1}) \tag{2}$$

and the change of variable

$$u = z(x \pm y) = zw \tag{3}$$

where

$$w = (x \pm y) \tag{4}$$

then the j^{th} power term of the vector of equation (2) is

$$u^j = [z(x \pm y)]^j = z^j [x \pm y]^j \tag{5}$$

Bankier in his paper "Generalizations of Pascals Triangle" reference (6) gives

$$(x+y)^j = \sum_{i=0}^{j} \binom{j}{i} x^i y^{j-i} \tag{6}$$

also due to commutativity of x and y

$$(x+y)^j = \sum_{i=0}^{j} \binom{j}{i} x^{j-i} y^i \tag{7}$$

where the binomial coefficients are given by Morrison, page 26 of reference (61), as

$$\binom{j}{i} = \frac{j!}{i!(j-i)!} \tag{8}$$

where j is a non-negative integer and i is any integer; also

$$\binom{j}{i} = \binom{j}{j-i} = \frac{j!}{(j-i)!i!} \tag{9}$$

If $i>j$, then $j-i<0$ and the binomial coefficient is taken to be zero.

Returning now to the transformation of equation (3) applied to the powers in equation (1) we obtain first by the two factor products

$$\langle u = (w^0, w, w^2, \cdots w^j, \cdots w^{d-1}) \begin{bmatrix} z^0 & & & & & & \\ & z^1 & & & & & \\ & & z^2 & & & & \\ & & & \ddots & & & \\ & & & & z^j & & \\ & & & & & \ddots & \\ & & & & & & z^{d-1} \end{bmatrix} \tag{10}$$

$$\langle u = \langle w D(z^j) \tag{11}$$

where the row vector $\langle w$ is to the powers and the diagonal matrix $D(z^j)$ is given in equation (10). Thus we need consider the $\langle w$ vector and for finite-finess say a 6×6 matrix obtained from the following elements

$$w_1^0 = 1 = \langle x e\rangle_1 \tag{12}$$

where

$$\langle x = (x^0, x^1, x^2 \cdots x^5) \tag{13}$$

and

$$_{1}\langle e = (1,0,0,0,0,0) \tag{14}$$

$$w^1 = (x \pm y) = (1,x,x^2,x^3,x^4,x^5)\begin{bmatrix} \pm y \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

$$w^2 = (x \pm y)^2 = x^2 \pm 2 \times y + y^2 \tag{16}$$

or

$$w^2 = \langle x \begin{bmatrix} y^2 \\ \pm 2y \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

etc. Packaging the powers

$$\langle w = \langle x \begin{bmatrix} 1 & \pm y & y^2 & \pm y^3 & y^4 & \pm y^5 \\ 0 & 1 & \pm 2y & 3y^2 & \pm 4y^3 & 5y^4 \\ 0 & 0 & 1 & \pm 3y & 6y^2 & \pm 10y^3 \\ 0 & 0 & 0 & 1 & \pm 4y & 10y^2 \\ 0 & 0 & 0 & 0 & 1 & \pm 5y \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

$$\langle w = \langle x \; T_u(\pm y) \tag{19}$$

Equation (18) can also be written as

$$\langle w = (1, \pm y, y^2, \pm y^3, y^4, \pm y^5) \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 & x^5 \\ 0 & 1 & 2x & 3x^2 & 4x^3 & 5x^4 \\ 0 & 0 & 1 & 3x & 6x^2 & 10x^3 \\ 0 & 0 & 0 & 1 & 4x & 10x^2 \\ 0 & 0 & 0 & 0 & 1 & 5x \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

By equation (18) for positive y unity we have generating equation (7)

$$w^j = (x+1)^j = \sum_{i=0}^{j} \binom{j}{i} x^{j-i} = \sum_{i=0}^{j} \binom{j}{i} x^i \tag{21}$$

or matrix-wise

$$\langle w = \langle x \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & 1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \langle xB = \langle xT_u(+1) \tag{22}$$

which are the well-used binomial coefficients. We will call the B matrix the Binomial matrix. By equation (18) for the negative y case we have

$$w_b^j = (x-1)^j = \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} x^i = \sum_{i=0}^{j} \binom{j}{i} (-1)^i x^{j-i} \tag{23}$$

and packaging the powers

$$\langle w_b = \langle x \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 0 & 1 & -3 & 6 & -10 \\ 0 & 0 & 0 & 1 & -4 & 10 \\ 0 & 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \langle x T_u(-1) \qquad (24)$$

Just as we observed in Section (4) equation (164) for the triangular Taylor matrix the forward and backward expansions were inverses of each other, we observe here

$$T_u(+1)\ T_u(-1) = I = BB(-1)$$

$$I = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & 1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 0 & 1 & -3 & 6 & -10 \\ 0 & 0 & 0 & 1 & -4 & 10 \\ 0 & 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (25)$$

One may be thought of as a forward propogation by unity and the other as a backward (or reverse propogation by unity.

Consider the translation on the real axis of equation (21)

$$w = x + 1 \qquad (26)$$

which power-wise generates the transformation

$$\langle w = \langle xB \qquad (27)$$

and the inverse translation

$$w' = w - 1 \qquad (28)$$

which generates the transformation

$$\langle w' = \langle wB(-1) \qquad (29)$$

Substituting equation (20) in equation (28)

$$w' = (x+1) - 1 = x \qquad (30)$$

or using equation (27) in equation (29)

$$\langle w' = \langle xBB(-1) = \langle x \qquad (31)$$

hence in general

$$BB(-1) = I \qquad (32)$$

or

$$B(-1) = B^{-1} \qquad (33)$$

which generates the inverse of the matrix of binomial coefficients.

Morrison on page 27 states that the binomial coefficients $\binom{j}{i}$ may be conveniently displayed using the well-known Pascal's triangle in which each number is the sum of the two above it.

```
                    1
                  1   1
                1   2   1
              1   3   3   1
            1   4   6   4   1
          1   5  10  10   5   1
        1   6  15  20  15   6   1
      1   7  21  35  35  21   7   1
    1   8  28  56  70  56  28   8   1
  1   9  36  84 126 126  84  36   9   1
1  10  45 120 210 252 210 120  45  10   1
```

FIGURE (1)
FAMOUS PASCAL TRIANGLE DISPLAY

We find on page 12 of Miller's book, reference [59] an improvement of the display by a table array shown below for the 7×7 case with slight modifications

i \ j	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6
2	0	0	1	3	6	10	15
3	0	0	0	1	4	10	20
4	0	0	0	0	1	5	10
5	0	0	0	0	0	1	6
6	0	0	0	0	0	0	1

TABLE I

BINOMIAL COEFFICIENTS $\binom{j}{i}$

Miller also points out that it is easily seen that the sum of any two integers in consecutive rows is the same as the integer to the right of the second one in the next column. Note that these properties stated by Morrison and Miller in their mnemonic representations are quite trivial with respect to their analytical matrix representations and related matrix properties.

If we designate the binomial matrix in index notation where i is the row and j the column indexed from i,j=0,1,2,···d-1 we have

$$B = [b_{ij}] = [\binom{j}{i}] = \left[\frac{j!}{i!(j-i)!}\right] \tag{34}$$

or

$$\begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots \\ b_{10} & & & \\ \vdots & & & \end{bmatrix} =$$

$$\begin{bmatrix}
\binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \binom{g}{0} & \cdots & \binom{d-1}{0} \\
 & \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \binom{j}{1} & & \binom{d-1}{2} \\
 & & \binom{2}{2} & \binom{3}{2} & \vdots & & \binom{d-1}{2} \\
 & & & \binom{3}{3} & & & \\
 & & & & \binom{j}{i} & & \vdots \\
 & & & & \vdots & & \\
 & & & & \binom{j}{j} & & \\
 & & & & & & \binom{d-1}{d-1}
\end{bmatrix} \tag{35}$$

The 10 10 binomial matrix can be written as factorial ratios as

$$
B = \begin{bmatrix}
\frac{0!}{0!(0)!} & \frac{1!}{0!1!} & \frac{2!}{0!2!} & \frac{3!}{0!3!} & \frac{4!}{0!(4-0)!} & \frac{5!}{0!(5-0)!} & \cdots & \frac{9!}{0!(9-0)!} \\
 & \frac{1!}{1!0!} & \frac{2!}{1!1!} & \frac{3!}{1!2!} & \frac{4!}{1!(4-1)!} & \frac{5!}{1!(5-1)!} & & \frac{9!}{1!(9-1)!} \\
 & & \frac{2!}{2!0!} & \frac{3!}{2!(3-2)!} & \frac{4!}{2!(4-2)!} & \frac{5!}{2!(5-2)!} & & \\
 & & & \frac{3!}{3!(3-3)!} & \frac{4!}{3!(4-3)!} & \frac{5!}{3!(5-3)!} & & \vdots \\
 & & & & \frac{4!}{4!(4-4)!} & \frac{5!}{4!(5-4)!} & & \\
 & & & & & \frac{5!}{5!(5-5)!} & & \frac{9!}{5!(9-5)!} \\
 & & & & & & & \vdots \\
 & & & & & & & \frac{9!}{9!(0)!}
\end{bmatrix} \tag{36}
$$

and the triangle of Figure (1) the 11x11 case is

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
 & & 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 \\
 & & & 1 & 4 & 10 & 20 & 35 & 56 & 84 & 120 \\
 & & & & 1 & 5 & 15 & 35 & 70 & 126 & 210 \\
 & & & & & 1 & 6 & 21 & 56 & 126 & 252 \\
 & & & & & & 1 & 7 & 28 & 84 & 210 \\
 & & & & & & & 1 & 8 & 36 & 120 \\
 & & & & & & & & 1 & 9 & 45 \\
 & & & & & & & & & 1 & 10 \\
 & & & & & & & & & & 1
\end{bmatrix} = B_{11 \times 11} \tag{37}
$$

The binomial coefficients $\binom{j}{i}$ are also known in terms of the number of combinations of j things taken i at a time. A few other areas of their occurrence will be pointed out. In section (6) we find a modified version of the matrix by taking successive discrete differences.

Suppose we have the product of two time functions and take the derivative with respect to time

$$z(t) = xy$$

$$\dot{z} = \dot{x}y + z\dot{y}$$

$$\ddot{z} = \ddot{x}y + \dot{x}\dot{y}2 + x\ddot{y}$$

$$\ddot{z} = \dddot{x}y + 3\ddot{x}\dot{y} + 3\dot{x}\ddot{y} + x\dddot{y} \qquad (38)$$

etc, or package wise

$$(z(t),\dot{z}(t),\ddot{z}(t)\dddot{z}) = (x,\dot{x},\ddot{x},\dddot{x}) \begin{bmatrix} y & \dot{y} & \ddot{y} & \dddot{y} \\ 0 & y & 2\dot{y} & 3\ddot{y} \\ 0 & 0 & y & 3\dot{y} \\ 0 & 0 & 0 & y \end{bmatrix} \qquad (39)$$

There is another interesting nested representation of the matrix of Equation (18) we note that for the $d^x d$ case

$$w^0 = 1 = \langle d) {}^x E_{11} y (d \rangle \qquad (40)$$

where

$$E_{11} = e(\overset{}{\underset{1}{d}}\rangle\ \langle\overset{1}{d})e = \begin{bmatrix} 1 & 0 & 0 & \cdot \\ 0 & & & \\ \cdot & & & 0 \end{bmatrix} \qquad (41)$$

$$w^1 = x \pm y = (1,x)\begin{bmatrix} 0 & \pm 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ y \end{pmatrix} = \langle x \underset{d^x 2}{E} \underset{2^x 2}{B_c} \underset{2^x d}{E^T} y \rangle \qquad (42)$$

$$w^2 = (x \pm y)^2 = (1, x, x^2)\begin{bmatrix} & & 1 \\ & \pm 2 & \\ 1 & & \end{bmatrix}\begin{pmatrix} 1 \\ y \\ y^2 \end{pmatrix} = \left\langle x \underset{d \times 3}{E} \underset{(3 \times 3)}{B_c} \underset{3 \times d}{E^T} y \right\rangle \quad (43)$$

$$w^3 = (x \pm y)^3 = (1, x, x^2 x^3) \begin{bmatrix} & & & 1 \\ & & \pm 3 & \\ & 3 & & \\ \pm 1 & & & \end{bmatrix}\begin{pmatrix} 1 \\ y \\ y^2 \\ y^3 \end{pmatrix} = \left\langle (d) x \underset{d \times 4 \times 4}{E} B_c \underset{4 \times d}{E^T} y (d) \right\rangle \quad (44)$$

where the "select" matrices are

$$\underset{d \times 2}{E} = \begin{bmatrix} I_{2 \times 2} \\ 0_{(d-2) \times (2)} \end{bmatrix} \quad (45)$$

etc, the linear-convolved double-signed binomial coefficient matrices are designed B_c. Packaging the equations we have

$$\langle w = \langle x \, [F_o, F_1, F_2, F_3 \cdots F_j, \cdots F_{d-1}] \, y \rangle \quad (46)$$

where

$$\underset{d \times d}{F_j} = E \quad B_c \quad E^T = E_{j+1} \, B_{cj+1} \, E^T \quad (47)$$

The pseudo inverse of Equation (47) is easily obtained

$$F_j^* = E_{j+1} \, B^{-1}_{c(j+1)} \, E^T_{j+1} \quad (48)$$

where for example the inverse of the 5×5 element is

$$\begin{bmatrix} & & & & 1 \\ & & & \pm 4 & \\ & & 6 & & \\ & \pm 4 & & & \\ 1 & & & & \end{bmatrix}^{-1} = \begin{bmatrix} & & & & 1 \\ & & & \pm\frac{1}{4} & \\ & & \frac{1}{6} & & \\ & \pm\frac{1}{4} & & & \\ 1 & & & & \end{bmatrix} \quad (49)$$

Packaging the terms

$$\langle w = \langle x\ [F_0 y\rangle,\ F_1 y\rangle,\ F_2 y\rangle, \cdots F_j y\rangle \cdots\ F_{d-1} y\rangle] \tag{50}$$

or

$$\langle w = \langle x\ T \tag{51}$$

where

$$T = [F_0 y\rangle, F_1 y\rangle \cdots F_{d-1} y\rangle] \tag{52}$$

and

$$T^T = \begin{bmatrix} \langle y F_0^T \\ \langle y F_1^T \\ \vdots \end{bmatrix} \tag{53}$$

The inverse computation requires

$$(T^T T)^{-1}$$

or consideration of

$$T^T T = \begin{bmatrix} \langle y F_0^T F_0 y\rangle & \langle y F_0^T F_1 y\rangle & \cdots & \langle y F_0^T F_{d-1} y\rangle \\ \langle y F_1^T F_0 y\rangle & & & \\ \vdots & & & \\ \langle y F_{d-1}^T F_0 y\rangle & & & \end{bmatrix} \quad (54$$

For the positive y unity case, that is (x+1)=w we have

$$w^1 = (1,x)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{55}$$

$$w^2 = (1,x,x^2)\begin{bmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$w^3 = (1,x,x^2,x^3)\begin{bmatrix} 0 & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}$$

etc. For example consider the product

$$\underset{d \times d}{F_4^T F_3} = E_5 \; B_{c5} \; E_5^T \; E_4 \; B_{c4} \; E_4^T \tag{56}$$

The product

$$\underset{5 \times d}{E_5^T} \; \underset{d \times 4}{E_4} = \begin{bmatrix} \underset{4 \times 4}{I} \\ \langle 4) 0 \end{bmatrix} \tag{57}$$

and using Equation (53) in Equation (52)

$$\underset{d \times d}{F_4^T F_3} = \underset{d \times 5}{E_5} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 \\ 0 & 3.6 & 0 & 0 \\ 0 & 0 & 3.4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} E_4^T \tag{58}$$

It is felt (though time is not available in this report) that the matrices of Equation (58) can be related to some of the coefficients for example in the Legendre and Gram polynomials. Returning now to Equation (18) in Equation (11)

$$\langle u = \langle x \; T_u(\pm y) D(z^j) \tag{59}$$

where the upper triangular matrix $T_u(\pm y)$ of Equation (51) is spelled out in Equation (18). Equation (59) can also be written by Equation (18) in Equation (11)

$$\langle u = \langle(\pm y)\ T_u(x) D(z^j) \tag{60}$$

Returning to Equation (24) the $T_u(-1)$ matrix was generated by the backward propogation

$$w_b^j = (x-1)^j \tag{61}$$

since if we substitute in Equation (7)

$$y = -1$$

we obtain

$$w_b^j = (x-1)^j = \sum_{i=0}^{j} \binom{j}{i} x^{j-i}\ (-1)^i \tag{62}$$

Let us now look at the matrix generated by

$$w_b^j(-1) = (1-x)^j = [(-1)(x-1)]^j = (-1)^j (x-1)^j \tag{63}$$

for $j=0,1,2\cdots d-1$; we have

$$\left.\begin{aligned}
(1-x)^0 &= 1 = (-1)^0(x-1)^0 \\
(1-x)^1 &= (-1)(x-1) \\
&\vdots \\
(1-x)^j &= (-1)^j(x-1)^j \\
(1-x)^{d-1} &= (-1)^{d-1}(x-1)^{d-1}
\end{aligned}\right\} \tag{64}$$

or

$$[(1-x)^0, (1-x), (1-x)^2, \cdots (1-x)^j, \cdots (1-x)^{d-1}] \equiv \langle w_b(-1) \qquad (65)$$

$$= \langle w_b(-1) = \langle w_b \begin{bmatrix} +1 & & & & & \\ & -1 & & & & \\ & & +1 & & & \\ & & & \ddots & & \\ & & & & (-1)^j & \\ & & & & & (-1)^{d-1} \end{bmatrix} \qquad (66)$$

$$\langle w_b(-1) = \langle w_b \; I(-1) \qquad (67)$$

where I(-1) deisgnates the alternating sign matrix of ones. By Equation (63)

$$w_b^j(-1) = (-1)^j \; w_b^j \qquad (68)$$

By Equation (50) and Equation (62) in Equation (68)

$$w_b^j(-1) = (-1)^j \sum_{i=0}^{j} \binom{j}{i} s^{j-i}(-1)^i \qquad (69)$$

Using Equation (24) in Equation (65) we find

$$\langle w_b(-1) = \langle x \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 0 & 1 & -3 & 6 & -10 \\ 0 & 0 & 0 & 1 & -4 & 10 \\ 0 & 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} +1 & & & & & \\ & -1 & & & 0 & \\ & & +1 & & & \\ & & & -1 & & \\ & 0 & & & +1 & \\ & & & & & -1 \end{bmatrix} \qquad (70)$$

$$= \langle x \; T_u(-1) I(-1) \qquad (71)$$

or

$$\langle w_b(-1) = \langle x \begin{bmatrix} +1 & +1 & +1 & +1 & +1 & +1 \\ 0 & -1 & -2 & -3 & -4 & -5 \\ 0 & 0 & +1 & +3 & +6 & +10 \\ 0 & 0 & 0 & -1 & -4 & -10 \\ 0 & 0 & 0 & 0 & +1 & +5 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{72}$$

By Equation (25) we see that Equation (22) is

$$\langle w = \langle xB = \langle xT_u(+1) \tag{73}$$

$$\langle w_b = \langle xB^{-1} = \langle xT_u(-1) \tag{74}$$

$$\langle w_b(-1) = \langle xB^{-1}I(-1) = \langle xT_u(-1)I(-1) \tag{75}$$

Westlake on page (140) calls a matrix the Rutishauser

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{76}$$

and we see that the R matrix is the matrix of Equation (72) and Equation (70), that is

$$B^{-1}I(-1) = R \tag{77}$$

From Equation (77) it is obvious what the inverse of the Rutishauser matrix is

$$R^{-1} = I^{-1}(-1)B = I(-1)B \tag{78}$$

where

$$I^{-1}(-1) = I(-1) \tag{79}$$

observe by Equation (80) for the 4x4 case

$$R^{-1} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{80}$$

or

$$R^{-1} = I(-1)B = R \tag{81}$$

hence

$$R^2 = I \tag{82}$$

or the Rutishauser matrix is idempotent index 2. Todd on page (240) of reference [86] defines the condition number of a non-singular matrix A

$$c(A) = \left| \frac{\lambda}{\mu} \right| \tag{83}$$

where λ is a root of largest modulus of A and μ a root of least modulus. Todd states that the eigenvalues of the Rutishauser matrix are either 1 or -1, so that

$$c(R) = 1 \qquad (84) \tag{84}$$

Consider the Rutishauser matrix transpose

$$R^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 1 & -3 & 3 & -1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \tag{85}$$

and

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & -1 & -2 & -3 & -4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{86}$$

and its inverse is

$$R^{-1} = (R^T R)^{-1} R^T \tag{87}$$

$$R^{-1} = R^T (RR^T)^{-1} \tag{88}$$

The Grammian of Equation (87) is

$$R^T R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 60 \end{bmatrix} \tag{89}$$

By Equation (35) we see the generalized R matrix is

$$R = \begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \cdots & \binom{j}{0} \cdots & \binom{d-1}{0} \\ & -\binom{1}{1} & -\binom{2}{1} & -\binom{2}{1} & \cdots & -\binom{j}{0} & -\binom{d-1}{0} \\ & & & & & (-1)^i \binom{j}{i} & (-1)^i \binom{d-1}{i} \\ & & & & & \vdots & \vdots \\ & & & & & (-1)^j \binom{j}{j} & \vdots \\ & & & & & \vdots & \\ \cdot & & \cdot & & \cdot & \vdots & (-1)^{d-1} \binom{d-1}{d-1} \end{bmatrix} \tag{90}$$

Hence the Grammian matrix of Equation (89) involves the products of the binomial coefficients. Todd on page (240) of his book gives the matrix product elements as

$$R^TR = \left[\binom{i+j}{i}\right], \quad 0 \le i, \; j \le d-1 \tag{91}$$

where by Equation (8)

$$\binom{i+j}{i} = \frac{(i+j)!}{i!j!} \tag{92}$$

The inverse of R^TR is

$$(R^TR)^{-1} = R^{-1}R^{-T} \tag{93}$$

by Equation (81)

$$R = R^{-1} \tag{94}$$

and transposing

$$R^T = R^{-T} \tag{95}$$

hence Equation (94) and Equation (95) in Equation (93)

$$(R^TR)^{-1} = RR^T \tag{96}$$

and for the 5x5 case

$$RR^T = \begin{bmatrix} 5 & -10 & 10 & -5 & 1 \\ -10 & 30 & -35 & 19 & -4 \\ 10 & -35 & 10 & -27 & 6 \\ -5 & 19 & -27 & 17 & -4 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \tag{97}$$

Since Euqation (97) is also a product of the binomial terms, one can obtain the general expression for the i-jth element which corresponds to Equation (91).

We also have

$$RR^T = RR^T I = RR^T(RR^{-1}) \tag{98}$$

$$RR^T = R(R^T R)R^{-1} \tag{99}$$

and by Equation (96) in Equation (99)

$$RR^T = R(RR^T)^{-1} R^{-1} \tag{100}$$

which implies that the eigenvalues of RR^T and $(RR^T)^{-1}$ coincide.

By Equation (63) we see that any function sequence of the form

$$g^j(t) = \left[1-f_i(t)\right]^j = \sum_{i=0}^{j} \binom{j}{i} (-1)^i f_i(t) \tag{101}$$

or packagewise

$$\langle g(t) = \langle f(t)R \tag{102}$$

generates the Rutishauser matrix. Westlake on page (140) gives the left eigenvector corresponding to the three unity eigenvalues for R as

$$\lambda_i = 1, \ i = 1,2,3$$

as

$$\langle 5)u_i = \left(6x_3-4x_4, 3x_3-2x_4, x_3, x_4, x_5 \right) \tag{103}$$

or factor-wise

$$\langle 5)u_i = (x_3, x_4, x_5) \begin{bmatrix} 6 & 3 & 1 & 0 & 0 \\ -4 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{104}$$

and the two eigenvectors corresponding to

$$\lambda_j = -1 \quad j=4,5$$

as

$$\langle 5)_j u = (0, 2x_4 - x_5, 2x_4 - x_5, x_4, x_5) \tag{105}$$

and factor-wise

$$\langle 5)_j u = (x_4, x_5) \begin{bmatrix} 0 & 2 & 2 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{bmatrix} \tag{106}$$

or packagewise.

$$\begin{bmatrix} {}^1\langle u \\ {}^2\langle u \\ {}^3\langle u \\ {}^4\langle u \\ {}^5\langle u \end{bmatrix} \begin{bmatrix} \underset{3x3}{I} & \underset{3x2}{0} \\ \underset{2x3}{0} & \underset{2x2}{-I} \end{bmatrix} = U\Lambda = UR \tag{107}$$

where

$$\Lambda = \begin{pmatrix} \underset{3x3}{I} & 0 \\ 0 & -\underset{2x2}{I} \end{pmatrix} \tag{108}$$

also the right eigenvectors are given by Westlake as for $\lambda_i = 1$ $i = 1,2,3$

$$v(5\rangle_i = \begin{bmatrix} y_1 \\ y_2 \\ y_5 - y_1 \\ -2y_5 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_5 \end{bmatrix} \tag{109}$$

and for $\lambda_j = -1 \quad j=4,5$

$$v(5\rangle_j = \begin{bmatrix} y_1 \\ \frac{1}{2}(y_4-4y_1) \\ \frac{1}{2}(-3y_4) \\ y_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & \frac{1}{2} \\ 0 & -\frac{3}{2} \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_4 \end{bmatrix} \tag{110}$$

and packagewise

$$\begin{bmatrix} \underset{3\times3}{I} & \underset{3\times2}{0} \\ \underset{2\times3}{0} & \underset{2\times2}{I} \end{bmatrix} V = RV \tag{111}$$

Returning now to Equation (4)

$$w = x \pm y \tag{112}$$

we shall recast three cases corresponding to

$$w = x + y \tag{113}$$

$$w = x - y \tag{114}$$

for $y \neq 0$, normalize Equation (113)

$$\frac{w}{y} = w_y = \frac{x}{y} + 1 = x_y + 1 \tag{115}$$

and Equation (114)

$$\frac{w_b}{y} = w_{by} = \frac{x}{y} - 1 = x_y - 1 \tag{116}$$

and for $x \neq 0$, normalize Equation (114)

$$\frac{w}{x} = w_x = 1 - \frac{y}{x} = 1 - y_x \quad (117)$$

The powers of Equation (115) generate

$$\langle w_y = \langle x_y B \quad (118)$$

and the powers of Equation (116) generate

$$\langle w_{by} = \langle x_y B(-1) \quad (119)$$

and the powers of Equation (117) generate

$$\langle w_x = \langle y_x R = \langle y_x B(-1) I(-1) \quad (120)$$

we also have

$$\langle x_y = \langle x D^{-1}(y^n) \quad (121)$$

$$\langle w_y = \langle w D^{-1}(y^n) \quad (122)$$

$$\langle w_{by} = \langle w_b D^{-1}(y^n) \quad (123)$$

$$\langle w_x = \langle w D^{-1}(x^n) \quad (124)$$

Using Equations (121) and (122) in Equation (118)

$$\langle w = \langle x D^{-1}(y^n) B D(y^n) \quad (125)$$

corresponding to

$$w = x + y \quad (126)$$

and similarly

$$\langle w_b = \langle x D^{-1}(y^n) B(-1) D(y^n) \quad (127)$$

corresponding to

$$w_b = x - y \tag{128}$$

and

$$\langle w_b(-1) = \langle xD^{-1}(y^n)RD(y^n) \tag{129}$$

corresponding to

$$w_b(-1) = x - y \tag{130}$$

where

$$D(y^n) = \begin{bmatrix} 1 & & & & & & & \\ & y & & & & & & \\ & & y^2 & & & & & \\ & & & y^3 & & & & \\ & & & & \ddots & & & \\ & & & & & y^n & & \\ & & & & & & \ddots & \\ & & & & & & & y^{d-1} \end{bmatrix} \tag{131}$$

Reconsider Equation (6) for x=y=1, we have

$$(1+1)^j = 2^j = \sum_{i=0}^{j} \binom{j}{i} \tag{132}$$

or as an innerproduct

$$2^j = \left[\binom{j}{0},\binom{j}{1},\binom{j}{2},\cdots,\binom{j}{i},\cdots,\binom{j}{j}\right] 1(j+1\rangle \tag{133}$$

The binomial matrix of Equation (34) has the property using Equation (133)

$$\langle d)1 \underset{d\times d}{B} = \left[1,2,2^2,2^3,2^4,\cdots,2^j,\cdots,2^{d-1}\right] \tag{134}$$

Now reconsider Equation (7) for

$$\left.\begin{array}{l} x = 1 \\ y = 1 \end{array}\right\} \tag{135}$$

then

$$0 = \sum_{i=0}^{j} (-1)^{i} \binom{j}{i} \tag{136}$$

and as an inner-product

$$0\rangle = \left[\binom{j}{0}\binom{j}{1}\binom{j}{2} \cdots \binom{j}{i} \cdots \binom{j}{j}\right] I(-1)\ 1\rangle \tag{137}$$

The Rutishauser matrix of Equation (76) thus has the property

$$\langle d)\, I\ R = \langle 1\ B^{-1} I(-1) = \langle d)\ 0 \tag{138}$$

<u>y-Binomial Matrix and its Inverse.</u> For lack of better terminology the <u>y-binomial matrix will be defined</u> as the matrix of Equation (18) for plus sign case, that is the real-axis translation

$$w = x + y \tag{139}$$

induces the monomial-base transformation

$$\langle w = \langle x B(y) \tag{140}$$

where B(y) is the binomial matrix carrying along the variable y; by Equation (35)

$$\underset{d\times d}{B(y)} = \begin{bmatrix} \binom{0}{0} & \binom{1}{0}y & \binom{2}{0}y^2 & \binom{3}{0}y^3 & \cdots & \binom{d-1}{0}y^{d-1} \\ 0 & \binom{1}{1}1 & \binom{2}{1}y & \binom{3}{1}y^2 & & \binom{d-1}{1}y^{d-2} \\ 0 & 0 & \binom{2}{2}1 & \binom{3}{2}y & & \vdots \\ \cdot & \cdot & 0 & \binom{3}{3}1 & & \\ \cdot & \cdot & \cdot & 0 & & \binom{d-1}{d-2}y \\ \cdot & \cdot & \cdot & \vdots & & \binom{d-1}{d+1}1 \end{bmatrix} \tag{141}$$

or for the 6×6 case

$$B(y) = \begin{bmatrix} 1 & y & y^2 & y^3 & y^4 & y^5 \\ 0 & 1 & 2y & 3y^2 & 4y^3 & 5y^4 \\ 0 & 0 & 1 & 3y & 6y^2 & 10y^3 \\ 0 & 0 & 0 & 1 & 4y & 10y^2 \\ 0 & 0 & 0 & 0 & 1 & 5y \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{142}$$

If we make the additional variable change on the real axis

$$w' = w - y \tag{143}$$

at the package-of-power level by Equation (18) for negative sign

$$\langle w' - \langle wT_u(-y) = \langle wB(-y) \tag{144}$$

adding Equation (139) to Equation (143)

$$w' - x + y - y = x \tag{145}$$

and package-wise

$$\langle w' = \langle xI \tag{146}$$

Substituting Equation (140) in Equation (144)

$$\langle w' = \langle xB(y)\ B(-y) - \langle xI \tag{147}$$

or

$$B(y)\ B(-y) = I \tag{148}$$

or

$$\boxed{B(-y) = B^{-1}(y)} \tag{149}$$

where for example

$$B^{-1}(y) = \begin{bmatrix} 1 & -y & y^2 & -y^3 & y^4 & -y^5 \\ 0 & 1 & -2y & 3y^2 & -4y^3 & 5y^4 \\ 0 & 0 & 1 & -3y & 6y^2 & -10y^3 \\ 0 & 0 & 0 & 1 & -4y & 10y^2 \\ 0 & 0 & 0 & 0 & 1 & -5y \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{150}$$

If we make the translation by Equation (139)

$$r = w + q = x + (y+q) \tag{151}$$

or package wise

$$\langle r - \langle wB(q) = \langle xB(y+q) \tag{152}$$

using Equation (140) in Equation (152)

$$\langle r = \langle xB(y)\, B(q) = \langle xB(y+q) \tag{153}$$

hence we have

$$\boxed{B(y+q) = B(y)\, B(q)} \tag{154}$$

If

$$q = (n-1)\, y \tag{155}$$

where n is an integer, 0,1,2···

$$\boxed{B^{n}(y) = B(ny)} \tag{156}$$

and

$$\boxed{B^{-n}(y) = B(-ny)} \tag{157}$$

Note by Equation (125)

$$B(y) = D^{-1}(y)\, B(1)\, D(y) \tag{158}$$

or for the 6×6 case

$$B(y) = \begin{bmatrix} 1 & & & & & \\ & \frac{1}{y} & & & & \\ & & \frac{1}{y^{3}} & & & \\ & & & \frac{1}{y^{4}} & & \\ & & & & \frac{1}{y^{6}} & \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 6 & 10 \\ & & & 1 & 4 & 10 \\ & & & & 1 & 5 \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & y & & & & \\ & & y^{2} & & & \\ & & & y^{3} & & \\ & & & & y^{4} & \\ & & & & & y^{5} \end{bmatrix} \tag{159}$$

adding Equation (139) to Equation (143)

$$w' = x + y - y = x \tag{145}$$

and package-wise

$$\langle w' = \langle xI \tag{146}$$

Substituting Equation (140) in Equation (144)

$$\langle w' = \langle xB(y)\ B(-y) = \langle xI \tag{147}$$

or

$$B(y)\ B(-y) = I \tag{148}$$

or

$$\boxed{B(-y) = B^{-1}(y)} \tag{149}$$

where for example

$$B^{-1}(y) = \begin{bmatrix} 1 & -y & y^2 & -y^3 & y^4 & -y^5 \\ 0 & 1 & -2y & 3y^2 & -4y^3 & 5y^4 \\ 0 & 0 & 1 & -3y & 6y^2 & -10y^3 \\ 0 & 0 & 0 & 0 & 1 & -5y \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If we make the translation by Equation (139)

$$r = w + q = x + (y+q) \tag{151}$$

or package-wise

$$\langle r = \langle wB(q) = \langle xB(y+q) \tag{152}$$

using Equation (140) in Equation (152)

Powers of B(y) can be expressed as powers of B(1), that is

$$\boxed{B^n(y) = D^{-1}(y)\, B^n(1)\, D(y)} \qquad (160)$$

Appendix (B) BACKWARD AND FORWARD FRACTORIAL FUNCTION MATRICES, STERLING AND ASSOCIATED MATRICES AND RELATIONS TO RUTISHAUSER AND BINOMIAL MATRICES

Consider the $(i - j)^{th}$ element of the Binomial matrix of Eq. (35) sec (A)

$$\binom{j}{i} = \frac{j!}{i!(j-i)!} \tag{1}$$

and the quotient of Eq. (1)

$$\frac{j!}{(j-i)!} = \frac{j(j-1)(j-2)\ \dots\ (j-i+1)(j-i)\ \dots\ 2\ ..1}{(j-i)(j-i-1)\ \dots\ 2\ .1} \tag{2}$$

and after cancelling common terms

$$\frac{j!}{(j-i)!} = j(j-1)(j-2)\ \dots\ (j-i+1) \tag{3}$$

Equation (3) is a special case of the backward-factorial function of order i written as

$$j^{(i)} = j(j-1)(j-2)\ \dots\ (j-i+1) \tag{4}$$

when j is less than i, then Eq. (4) is equal to zero as can be seen by entering the values in Eq. (4).

Brand in his paper "Binomial Expansion in Factorial Powers" states that in the calculus of finite differences the factorial powers x(n) plays a role analogous to that of xn in the differential calculus. He states that there is not only a wide diversity of notations for the factorial powers but also a disagreement as to its definition when the index is negative. Brand says that moreover the notation (ax + b)(n) is sometimes used in a way which may be a fertile source of error and that the way out of this confusion is to use a single definition applicable to all real values of the index. He states further that the alleged factorial powers (ax + b)(n) are readily converted to genuine factorial powers.

When n is a positive integer the backward factorial function (x a real variable) of order n is defined as the product

$$x^{(n)} = x(x-1)(x-2)(x-3)\ \dots\ (x-n+1) \tag{5}$$

Thus $x^{(n)}$ is the product of n terms starting from x and counting down by unity, for example if n = 3

$$x^{(3)} = x(x-1)(x-2) \tag{6}$$

Also the forward factorial function of order n is defined as

$$x^{[n]} = x(x+1)(x+2)(x+3)\ \dots\ (x+n-1) \tag{7}$$

and for the example n = 3

$$x^{[3]} = x(x+1)(x+2) \tag{8}$$

Note by Eq. (4) that

$$j^{(j)} = j(j-1)(j-2) \ldots 3\cdot2\cdot1 = j! \tag{9}$$

Returning to the definition of Eq. (5), consider

$$x^{(m)} = x(x-1)(x-2) \ldots (x-m+1) \tag{10}$$

and

$$(x-m)^{(\cdot)} = (x-m)(x-m-1) \ldots (x-m-n+1) \tag{11}$$

and the product of Eq. (10) and Eq. (11) yeilds

$$x^{(m)}\ (x-m)^{(n)} = x^{(m+n)} = x^{(n)}\ (x-n)^{(m)} \tag{12}$$

Miller, ref (59) says that the definition of Eq. (5) when n is a negative integer stems from the same type of reasoning used in defining negative exponents in elementary algebra, that there it is easily proved that for n and m positive integers

$$x^n x^m = x^{n+m} \tag{13}$$

and if this rule of exponents is to hold for non-positive integers one is inevitably led to the definitions

$$x^0 = 1$$

and

$$x^{-n} = 1/x^n$$

Brand considers Eq. (12) to obtain a definition for all real values of m and n when m = 0, Eq. (12) becomes

$$x^{(0)}x^{(n)} = x^{(n)} \tag{14}$$

hence if $x \neq 0$

$$x^{(0)} \equiv 1 \tag{15}$$

When $m = -n$ Eq. (12) yields

$$x^{(0)} = x^{(-n)}\ (x+n)^{(n)} \tag{16}$$

or

$$x^{(-n)} = \frac{1}{(x+n)^{(n)}} = \frac{1}{(x+n)(x+n-1)\ \ldots\ (x+2)(x+1)} \tag{17}$$

One can also show very easily that

$$x^{[m+n]} = x^{[m]}\,(x+m)^{[n]} = x^{[n]}\,(x-n)^{[m]} \tag{18}$$

Clearly when the variable x is complex for example the La Place transform variables or z in the continuous or discrete frequency domain, many of these relations carry over, but will not be pursued in this report.

In the above we also have

$$x^{[0]} = 1 \tag{19}$$

and in accordance with Brand

$$0^{(0)} \equiv 1 \tag{20}$$

and

$$0^{(-n)} = \frac{1}{n!} \tag{21}$$

Returning now the Eq. (3) and Eq. (4) we have

$$j^{(i)} = \frac{j!}{(j-i)!} \tag{22}$$

and using Eq. (22) in Eq. (1)

$$j^{(i)} = i!\binom{j}{i} \tag{23}$$

also

$$\binom{j}{i} = \begin{cases} \dfrac{j^{(i)}}{i!} & i > 0 \\ 1 & i = 0 \\ 0 & i < 0 \end{cases} \tag{24}$$

Forming the column vector for i = 0, 1, 2, ... d-1 we have

$$\begin{bmatrix} j^{(0)} \\ j^{(1)} \\ j^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ j^{(i)} \\ \cdot \\ \cdot \\ \cdot \\ j^{(d-1)} \end{bmatrix} = \begin{bmatrix} 0! & & & & & & & \\ & 1! & & & & & & \\ & & 2! & & & & & \\ & & & \ddots & & & & \\ & & & & i! & & & \\ & & & & & \ddots & & \\ & & & & & & & (d-1)! \end{bmatrix} \begin{bmatrix} \binom{j}{0} \\ \binom{j}{1} \\ \binom{j}{2} \\ \cdot \\ \cdot \\ \cdot \\ \binom{j}{i} \\ \cdot \\ \cdot \\ \cdot \\ \binom{j}{d-1} \end{bmatrix} \tag{25}$$

or

$$j^{(\ )} \left(d\right\rangle_j = F\ b \left(d\right\rangle_j \tag{26}$$

Solving for the vector of binomial coefficients

$$\begin{bmatrix} \binom{j}{0} \\ \binom{j}{1} \\ \binom{j}{2} \\ \cdot \\ \cdot \\ \cdot \\ \binom{j}{d-1} \end{bmatrix} = \begin{bmatrix} 1/0! & & & & \\ & 1/1! & & & \\ & & 1/2! & & \\ & & & \ddots & \\ & & & & \frac{1}{(d-1)!} \end{bmatrix} \begin{bmatrix} j^{(0)} \\ j^{(1)} \\ j^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ j^{(d-1)} \end{bmatrix} \tag{27}$$

The vector of Eq. (26) is seen to be the j^{th} column of the Binomial matrix of Eq. (35) sec (A), hence packaging Eq. (27)

$$B = F^{-1}\ S^{(\ )} \tag{28}$$

where the matrix of backward factorial functions $S^{(\ )}$ is given by (definition herein used is restricted to j = x integers)

$$\underset{d\mathrm{x}d}{S^{(\ )}} = \begin{bmatrix} 0^{(0)} & 1^{(0)} & 2^{(0)} & \cdots & i^{(0)} & \cdots & j^{(0)} & \cdots & (d-1)^{(0)} \\ 0 & 1^{(1)} & 2^{(1)} & & i^{(1)} & & j^{(1)} & & (d-1)^{(1)} \\ & 0 & 2^{(2)} & & i^{(2)} & & j^{(2)} & & \cdot \\ & & 0 & & i^{(3)} & & \cdot & & \cdot \\ & & & & \cdot & & \cdot & & \cdot \\ & & & & \cdot & & \cdot & & \cdot \\ & & & & \cdot & & \cdot & & \cdot \\ & & & & i^{(i)} & & \cdot & & \cdot \\ & & & & 0 & & j^{(j)} & & \cdot \\ & & & & & & & & (d-1)^{(d-1)} \end{bmatrix} \tag{29}$$

or

$$S^{(\ )} = \left[j^{(i)} \right] \tag{30}$$

$$i, j = 0, 1, 2 \ldots (d-1)$$

For example, the 5x5 case is

$$S^{(\ )}_{5\times5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 6 & 12 \\ 0 & 0 & 0 & 6 & 24 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix} \tag{31}$$

The forward factorial function matrix is defined as

$$S^{[\ ]} = j^{[j]} \tag{32}$$

or

$$S^{[\ ]} = \begin{bmatrix} 0^{[0]} & 1^{[0]} & 2^{[0]} & \cdots & i^{[0]} & \cdots & (d-1)^{[0]} \\ & 1^{[1]} & 2^{[1]} & & i^{[1]} & & (d-1)^{[1]} \\ & & 2^{[2]} & & i^{[2]} & & \cdot \\ & & & & \cdot & & \cdot \\ & & & & \cdot & & \cdot \\ & & & & \cdot & & \cdot \\ & & & & i^{[i]} & & \cdot \\ & & & & & & (d-1)^{[d-1]} \end{bmatrix} \tag{33}$$

and for the 5x5 case

$$S^{[\ ]}_{5\times5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 6 & 12 & 20 \\ 0 & 0 & 0 & 60 & 120 \\ 0 & 0 & 0 & 0 & 840 \end{bmatrix} \tag{34}$$

By Eq. (28)

$$S^{(\ )} = F\ B \tag{35}$$

hence we can now easily invert the backward factorial function matrix

$$\left[S^{(\ )}\right]^{-1} = B^{-1}\ F^{-1} \tag{36}$$

where the inverse binomial matrix is given by Eq. (77) of appendix (A) in terms of the Rutishauser matrix

$$B^{-1} = RI(-1) \tag{37}$$

hence

$$\left[S^{(\ )}\right]^{-1} = RI(-1)\ F^{-1} \tag{38}$$

Morrison ref (6) gives the generalization of Eq. (24) as

$$\binom{x}{i} = \begin{cases} \frac{x^{(i)}}{i!} , & i > 0 \\ 1 , & i = 0 \\ 0 , & i < 0 \end{cases} \quad (39)$$

for x any real number and i any interger.

We next seek a relation between powers of x and the backward factorial functions of Eq. (5), which yields for the 5x5 case

$$\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \\ x^{(3)} \\ x^{(4)} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 1 & 0 \\ 0 & -6 & 11 & -6 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \quad (40)$$

or

$$x^{(\ )}\rangle = S_b \, x\rangle \quad (41)$$

The matrix S_b is called the Stirling Matrix of The First Kind and in this report will be further classified as the Backward Stirling Matrix.

If we partition Eq. (40) as

$$\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ x^{(i)} \end{pmatrix} \begin{bmatrix} {}^{0}\angle s_b \\ {}^{1}\angle s_b \\ \cdot \\ \cdot \\ \cdot \\ {}^{i}\angle s_b \end{bmatrix} x\rangle \quad (42)$$

hence the row vectors of Eq. (40) are the elements of ${}^{i}\angle s_b$. The eleven by eleven case is given in Morrison ref (61). Note by Eq. (42)

$$x^{(i)} = {}^{i}\angle s_b \; x\rangle \quad . \quad (43)$$

If we now let x = j (the integers) we have the connection

$$\begin{pmatrix} j^{(0)} \\ j^{(1)} \\ j^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ j^{(i)} \\ \cdot \\ \cdot \\ \cdot \\ j^{(j)} \end{pmatrix} = \underset{(j+1)(j+1)}{S_b} \; j\rangle \tag{44}$$

where

$j = 0, 1, 2, \ldots d-1$

For j = 0, Eq. (44) becomes

$$j^{(\;)}\rangle_0 = 0^{(\;)}\rangle = \begin{pmatrix} 0^{(0)} \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = S_b \begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = e\rangle_1 \tag{45}$$

For j = 1

$$j^{(\;)}\rangle_1 = 1^{(\;)}\rangle = \begin{pmatrix} 1^{(0)} \\ 1^{(1)} \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = S_b \; 1\rangle \tag{46}$$

For j = 2

$$j^{(\;)}\rangle_2 = 2^{(\;)}\rangle = S_b \; 2\rangle \tag{47}$$

for j = i

$$j^{(\;)}\rangle_i = i^{(\;)}\rangle = S_b \; i\rangle \tag{48}$$

and for j = d-1

$$j^{(\ )}\Big\rangle_{d-1} = (d-1)^{(\ )}\Big\rangle = S_b\ (d-1)\Big\rangle \tag{49}$$

or package-wise

$$\begin{bmatrix} 0^{(0)} & 1^{(0)} & 2^{(0)} & \cdots & (d-1)^{(0)} \\ 0 & 1^{(1)} & 2^{(1)} & & (d-1)^{(1)} \\ & 0 & 2^{(2)} & & \cdot \\ & & 0 & & \cdot \\ & & & & \cdot \\ & & & & (d-1)^{(d-1)} \end{bmatrix} = S_b C_E^T \tag{50}$$

where by Eq. (51) sec (5)

$$\underset{d\times d}{C_E^T} = \left[e\Big\rangle_1 \ , \ C^T \right] = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & & i & & d-1 \\ 0 & 1 & 2^2 & 3^2 & & i^2 & & (d-1)^2 \\ 1 & 1 & 2^3 & 3^3 & & \cdot & & \cdot \\ 0 & \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & & \cdot \\ 0 & 1 & 2^{d-1} & 3^{d-1} & & i^{d-1} & & (d-1)^{d-1} \end{bmatrix} \tag{51}$$

or by Eq. (29)

$$S^{(\ )} = S_b C_E^T \tag{52}$$

Using Eq. (38) in Eq. (52)

$$\mathcal{F}B = S_b C_E^T \tag{53}$$

or

$$S_b = \mathcal{F}\ B C_E^{-T} \tag{54}$$

and the inverse of the S_b matrix is

$$S_b^{-1} = C_E^T\ B^{-1}\ \mathcal{F}^{-1} \tag{55}$$

and by Eq. (37)

$$\begin{aligned} S_b^{-1} &= C_E^T\ RI(-1)\ \mathcal{F}^{-1} \\ &= C_E^T\ RF^{-1}I(-1) \end{aligned} \tag{56}$$

The inverse of the Backward Stirling Matrix also called Stirling Matrix of the First Kind is called the Stirling Matrix of the Second Kind and its matrix factors are given by Eq. (56). Morrison on page 25 of ref (61) gives the 11x11 table to construct the matrix, (he does not obtain the matrix factors of Eq. (56), they are derived here for the first time to my knowledge). The first five are

$$S_b^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 7 & 6 & 1 \end{bmatrix} \tag{57}$$

we see also by Eq. (54) the inverse of the C_E matrix can be obtained as

$$C_E^{-T} = B^{-1} \, F^{-1} \, S_b \; . \tag{58}$$

In a similar manner by Eq. (7) we can formulate the forward factorial vector in terms of the monomial base for the 5x5 case as

$$\begin{pmatrix} x^{[0]} \\ x^{[1]} \\ x^{[2]} \\ x^{[3]} \\ x^{[4]} \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 6 & 11 & 6 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \tag{59}$$

or

$$x^{[\;]}\rangle = S_f \, x\rangle \tag{60}$$

If now we let $x = j$ we have the connection

$$j^{[\;]}\rangle = S_f \, j\rangle \tag{61}$$

and the vector of integer powers is

$$\angle j = (1, j, j^2, j^3, \ldots j^i, \ldots j^{d-1}) \tag{62}$$

for

$$j = 0, 1, 2, \ldots d-1$$

The matrix of Eq. (59) and Eq. (60) is called the Forward Sterling Matrix in this report (it is the transpose of the matrix of Morrison p. 85 which he calls the associate Sterling matrix of the First Kind.

The 5x5 case is

$$
\underset{5x5}{S_f} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 6 & 11 & 6 & 1 \end{bmatrix} \tag{63}
$$

If we package Eq. (61) into a row of column vectors for j = 0, 1, 2 ... we obtain the left hand side matrix by Eq. (33) as

$$
S^{[\ ]} = S_f \, C_E^T \tag{64}
$$

in analogy with Eq. (50), or

$$
S_f = S^{[\ ]} \, C_E^{-T} \tag{65}
$$

By Eq. (52) in Eq. (64)

$$
S^{[\ ]} = S_f \, S_b^{-1} \, S^{(\ )} \tag{66}
$$

which connects the forward and backward factorial function matrices.

The inverse of the forward Sterling matrix of Eq. (65) is

$$
S_f^{-1} = C_E^T \, (S^{[\ ]})^{-1} \tag{67}
$$

The inverse of the forward Sterling matrix is called by Morrison the associate Sterling matrix of the Second Kind (he uses the transpose matrix); the 5x5 elements are

$$
\underset{5x5}{S_f^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & -1 & 7 & -6 & 1 \end{bmatrix} \tag{68}
$$

Many other relations between the Binomial, Rutishauser and these matrices can be obtained but are not pursued further here. One obvious one is by Eq. (38)

$$
R = \left[S^{(\ )}\right]^{-1} \ddagger \, I(-1) \tag{69}
$$

and by Eq. (81) appendix (A)

$$
R^2 = I = \left[S^{(\ )}\right]^{-1} F \, I(-1) \left[S^{(\ )}\right]^{-1} F \, I(-1) \tag{70}
$$

or

$$
S^{(\ )} = \ddagger \, I(-1) \left[S^{(\ )}\right]^{-1} \ddagger \, I(-1) \tag{71}
$$

and

$$\left[S^{(\ )} \right]^{-1} = F^{-1}\, I(-1)\, S^{(\ )}\, I(-1)\, F^{-1} \tag{72}$$

which provides the factors of the inverse of the backward factorial function matrix.

If we have a polynomial in the monomial base

$$f(x) = \langle x a \rangle \tag{73}$$

by Eq. (41)

$$f(x) = \langle x^{(\ )} S_b^{-T} a \rangle \tag{74}$$

or

$$f(x) = \langle x^{(\ )} b \rangle \tag{75}$$

where

$$b\rangle = S_b^{-T} a\rangle \tag{76}$$

Polynomials in the factorial function base are called Newton Polynomials and $\langle x^{(\ )}$ will be referred to as the Backward Newton Base, connected to the monomial base via Eq. (4).

APPENDIX C

FORMULAS FOR THE SUM OF THE d^{th} POWERS OF THE FIRST N INTEGERS

L. S. Levy in Reference (52) shows that for each positive integer d there is exactly one polynomial $S_d(N)$ in N such that

$$S_d(N) = 1 + 2^d + 3^D + \cdots N^d \qquad (1)$$

whenever N is a positive integer.

Equation (1) can be written as an inner product

$$S_d(N) = \langle c^d \; 1\rangle \qquad (2)$$

where the row-vector $\langle c^d$ is

$$\langle c^d = (1,\ 2^d,\ 3^d,\ 4^d,\ \cdots N^d) \qquad (3)$$

and the column vector $1\rangle$ is an N dimensional vector of ones

$$1\rangle = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad (4)$$

The following formulas are given in Reference (52) for d=1,2···7. Powers beyond this can be obtained from the papers.

$$S_1(N) = \frac{N}{2} + \frac{N^2}{2} = \frac{N}{2}\,(N+1) \qquad (5)$$

$$S_2(N) = \frac{N}{6} + \frac{N^2}{2} + \frac{N^3}{3} = \frac{N}{6}\,(N+1)\;(2N+1) \qquad (6)$$

$$S_3(N) = \frac{N^2}{4} + \frac{N^3}{2} + \frac{N^4}{4} = \frac{N^2}{4}\,(N+1)^2 \qquad (7)$$

$$S_4(N) = -\frac{N}{30} + \frac{N^3}{3} + \frac{N^4}{4} + \frac{N^5}{5} = \frac{N}{30}(N+1)(2N+1)(3N^2+3N-1) \quad (8)$$

$$S_5(N) = -\frac{N^2}{12} + \frac{5N^4}{12} + \frac{N^5}{2} + \frac{N^6}{6} = \frac{N^2}{12}(N+1)^2(2N^2+2N-1) \quad (9)$$

$$S_6(N) = \frac{N}{42} - \frac{N^3}{6} + \frac{N^5}{2} + \frac{N^6}{2} + \frac{N^7}{7} = \frac{1}{42}N(N+1)(2N+1)(3N^4+6N^3-3N+1) \quad (10)$$

$$S_7(N) = \frac{N^2}{12} - \frac{7N^4}{24} + \frac{7N^6}{12} + \frac{N^7}{2} + \frac{N^8}{8}$$

$$= \frac{N^2}{24}(N+1)^2(3N^4+6N^3-N^2-4N+2) \quad (11)$$

The span width in points

$$W_p = 2M + 1 = N + 1 \quad (12)$$

$$2M = N$$

$$W_p - 1 = N \quad (13)$$

and

$$\frac{W_p-1}{2} = M \quad (14)$$

The summations of Equations (5-11) in terms of W_p

$$S_1(N) = \langle lc \rangle = S_1(N) = \frac{N}{2}(N+1) = \frac{1}{2}(W_p-1)(W_p) \quad (15)$$

$$S_2(N) = \langle lc^2 \rangle = \frac{N}{6}(N+1)(2N+1) \quad (16)$$

$$S_2(N) = \frac{(W_p-1)}{6}(W_p)(2W_p-1) \quad (17)$$

$$S_3(N) = \langle 1c^3 \rangle = \frac{N^2}{4}(N+1)^2 = \frac{(W_p-1)^2}{4} W_p^2 \quad (18)$$

$$S_4(N) = \langle 1c^4 \rangle = \frac{N}{30} (N+1)\ (2N+1)\ (3N^2+3N-1)$$

$$= \frac{W_p}{30} (W_p-1)\ (2W_p-1)\ (3W_p^2-3W_p-1) \quad (19)$$

The summations of Equations (5-11) in terms of M

$$S_1(M) = \langle 1c \rangle = \frac{M}{2} (M+1) \quad (20)$$

$$= \frac{W_p-1}{4}\left[\frac{W_p}{2} - \frac{1}{2} + 1\right]$$

$$= \frac{W_p-1}{4}\left[\frac{W_p+1}{2}\right]$$

$$= \frac{1}{8}\left[W_p^2-1\right]$$

$$S_2(M) = \langle 1c^2 \rangle = \frac{M}{6} (M+1)\ (2M+1)$$

$$= \frac{(W_p-1)}{6(2)}\left[\frac{W_p+1}{2}\right] W_p \quad (21)$$

$$\langle 1c^2(M) \rangle = \frac{1}{24} W_p\ (W_p^2-1)$$

$$S_3(M) = \langle 1c^3(m) \rangle = \frac{M^2}{4} (M+1)^2 \quad (22)$$

$$= \frac{(W_p-1)}{16} \frac{(W_p+1)^2}{4} = \frac{(W_p-1)(W_p+1)^2}{64}$$

$$S_4(M) = \frac{M}{30} (M+1) (2M+1) (3M^2+3M-1) \qquad (23)$$

$$S_4(M) = \frac{W_p}{(30)(16)} (W_p+1) (W_p-1) (3W_p^2-7) \qquad (24)$$

Appendix (D) INTEGRATION BY PARTS OF $\int t^n e^{at} dt$

This appendix derives some classical integrals by integration by parts. Consider the integral where n = o, 1, 2 . . . d-1,

$$\int t^n e^{at} dt = \int u\, dv \tag{1}$$

where

$$u = t^n \tag{2}$$

or

$$du = nt^{n-1} dt \tag{3}$$

and

$$dv = e^{at} dt \tag{4}$$

or

$$v = \int e^{at} dt = \frac{e^{at}}{a} \tag{5}$$

The integration by parts comes from

$$d(uv) = du\, v + u\, dv \tag{6}$$

or

$$d(uv) = uv = \int du\, v + \int u\, dv \tag{7}$$

and

$$\int udv = uv - \int vdu \tag{8}$$

Hence using Eq. (2) through Eq. (4) in Eq. (8)

$$\int t^n e^{at} dt = \frac{t^n e^{at}}{a} - \frac{n}{a} \int t^{n-1} e^{at} dt \tag{9}$$

In the same manner one can show

$$\int t^n e^{-at} dt = -\frac{t^n}{a} e^{-at} + \frac{n}{a} \int t^{n-1} e^{-at} dt \tag{10}$$

If we apply the end points of integration the definite integrals Eq. (10) becomes

$$\int_{t1}^{t2} t^n e^{-at} dt = -\frac{t^n}{a} e^{-at} \Big|_{t1}^{t2} + n \int_{t1}^{t2} t^{n-1} e^{-at} dt \tag{11}$$

As the first case considers the interval (o, ∞), then Eq. (11) becomes for n = 0

$$\int e^{-at}dt = -\frac{e^{-at}}{a}\Big|_0^\infty = \frac{1}{a} \tag{12}$$

for n = 1

$$\int_0^\infty t^{-at}dt = -t\frac{e^{-at}}{a}\Big|_0^\infty + \frac{1}{a}\int_0^\infty e^{-at}dt \tag{13}$$

$$= \frac{-1}{a}\left[\frac{-\infty}{e^{a\infty}}\right] + \frac{1}{a}\int_0^\infty e^{-at}dt \tag{14}$$

The occurance of ∞/∞ necessitates the use of L. Hospitals' rule taken from Schaum's Series "Differential and Integral Calculus" page 114 reference (78)

L. Hospitals' Rule: If b = + ∞, -∞, or ∞, if f (x) and g(x) are differentiable and g(x) = 0 for all ẋ or some interval 1 x 1 > M, if

$$\lim_{x\to b} f(x) = \infty$$

and

$$\lim_{x\to b} g(x) = \infty$$

then when

$$\lim_{x\to b}\left(\frac{f^1(x)}{g^1(x)}\right) \text{ exists}$$

or is infinite

$$\lim_{x\to b}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to b}\left(\frac{f^1(x)}{g^1(x)}\right) \tag{15}$$

as an example

$$\lim_{x\to\infty}\left(\frac{x^2}{e^x}\right) = \lim_{x\to\infty}\left(\frac{2x}{e^x}\right) = \lim_{x\to 0}\left(\frac{2}{e^x}\right) = 0 \tag{16}$$

applying Eq. (15) to Eq. (13) we see

$$+\, t\, e^{-at}\Big|_0^\infty = \lim_{t\to\infty} t_{at} = 0 \tag{17}$$

and likewise by repeated applications; the n^{th} degree term Eq. (11) is

$$t^n e^{-at} \Big|_0^\infty = \lim_{t\to\infty} t^n e^{-at} = 0, \tag{18}$$

which is a repitition of the example of Eq. (16).

Using Eq. (17) in Eq. (13)

$$\int_0^\infty t e^{-at} dt = \frac{1}{a^2} \tag{19}$$

Continuing in this recursive manner one can show that

$$\int_0^\infty t^n e^{-at} = \frac{n!}{a^{n+1}} \tag{20}$$

Clearly if one replaces the interval of integration by $(0, -\infty)$ the term are bounded for

$$\int_0^{-\infty} e^{-at} dt = -\frac{e^{-at}}{a} \Big|_0^{-\infty} = \infty \tag{21}$$

However the positive exponential power of Eq. (9) is finite in the interval $(0, -\infty)$, for $n = 0$ we have

$$\int_0^{-\infty} e^{at} dt = \frac{e^{at}}{a} \Big|_0^{-\infty} = \frac{-1}{a} \tag{22}$$

and for the n^{th} term

$$\int_0^{-\infty} t^n e^{at} dt = \frac{n!(-1)^{n+1}}{a^{n+1}} \tag{23}$$

Contrast this alternating sign result with Eq. (20).

Appendix (E) BASE AND METRIC RELATIONS

Consider two bases connected via Eq. (1) where T is known

$$\langle \bar{t} = \langle \bar{\tau} T \tag{1}$$

and suppose one performs a G-S process as the $\langle \bar{\tau}$ vectors such that B_σ is known

$$\langle \bar{\sigma} = \langle \bar{\tau} B_\sigma \tag{2}$$

where

$$\bar{\sigma}\rangle \textcircled{i} \langle \bar{\sigma} = I \tag{3}$$

and

$$M_{tt} = \bar{t}\rangle \textcircled{i} \langle \bar{t} = M_{tt} = T^T M_{\tau\tau} T \tag{4}$$

We desire to find the G-S connection matrix B_s where

$$\langle \bar{s} = \langle \bar{t} B_s \tag{5}$$

under the constraint

$$\bar{s}\rangle \textcircled{i} \langle \bar{s} = I = \bar{\sigma}\rangle \textcircled{i} \langle \bar{\sigma} \tag{6}$$

By Eq. (2) and Eq. (5) in Eq. (6)

$$B_s^T \bar{t}\rangle \textcircled{i} \langle \bar{t} B_s = B_\sigma^T \bar{\tau}\rangle \textcircled{i} \langle \bar{\tau} B_\sigma \tag{7}$$

or

$$B_s^T M_{tt} B_s = B_\sigma^T M_{\tau\tau} B_\sigma \tag{8}$$

a "quadratic algebraic equation" in B_s the unknown.

Using Eq. (4) in Eq. (8)

$$B_s^T T^T M_{\tau\tau} T B_s = B_\sigma^T M_{\tau\tau} B_\sigma \tag{9}$$

a solution is given by

$$B_s = T^{-1} B_\sigma \tag{10}$$

where

$$B_s^T = B_\sigma^T T^{-T} \tag{11}$$

and substituting Eq. (10) and Eq. (11) in Eq. (9)

$$B_\sigma^T T^{-T} T^T M_{\tau\tau} T T^{-2} B_\sigma = B_\sigma^T M_{\tau\tau} B_\sigma \tag{12}$$

However Eq. (10) is not a unique solution for example another solution

$$B_s = L T^{-1} B_\sigma \tag{13}$$

and

$$B_s^T = B_\sigma^T T^{-T} L^T \tag{14}$$

and using Eq. (13) and (14) in Eq. (9)

$$B_\sigma^T T^{-T} L^T T^T M_{\tau\tau} T L T^{-2} B_\sigma = B_\sigma^T M_{\tau\tau} B_\sigma \tag{15}$$

for all L and T such that

$$TLT^{-1} = I \tag{16}$$

Note that Eq. (9) is a special case of the steady-state matrix Riccati differential Equation and many studies and papers have been published along this line but will not be pursued further here.

Note that we have the further relation using Eq. (1) in Eq. (5)

$$\langle \bar{s} = \tau T B_s \tag{17}$$

and

$$I = \bar{s}\rangle \textcircled{i} \langle\bar{s} = B_s^T T^T \bar{\tau}\rangle \textcircled{i} \langle\bar{\tau} T B_s \tag{18}$$

The metric factors were derived in Eq. (348) of Section (1) as

$$M_{\tau\tau} = \tilde{\tau}\rangle \textcircled{i} \langle\bar{\tau} = (B_\sigma B_\sigma^T)^{-1} \tag{19}$$

or Eq. (19) in Eq. (8)

$$I = B_s^T T^T (B_\sigma B_\sigma^T)^{-1} T B_s \tag{20}$$

and

$$M_{tt} = B_s^{-T} B_s^{-1} = (B_s B_s^T)^{-1} = T^T (B_\sigma B_\sigma^T)^{-1} T \tag{21}$$

and inverting

$$M_{tt}^{-1} = B_s B_s^T = T^{-1} B_\sigma B_\sigma^T \tag{22}$$

Equating factors we find

$$B_s = T^{-1} B_\sigma \tag{23}$$

but we know from standard matrix theorems that triangular factors (for full rank matrices) are only unique when they are in unit triangular factors with a diagonal matrix.

As an example of a third solution

$$B_s = T^{-1} B_\sigma T \tag{24}$$

$$B_s^T = T^T B_\sigma^T T^{-1} \tag{25}$$

and one obtains

$$T^T(B_\sigma^T M_{\tau\tau} B_\sigma)T = B_\sigma^T M_{\tau\tau} B_\sigma \tag{26}$$

hence for all T that realize the congruent automorph relation of Eq. (26) one has Eq. (24) as a solution to Eq. (9).

APPENDIX F

GRADIENTS OF TRACES OF MATRIX FUNCTIONS

Some well known trace properties of matricies are

$$\text{tr } M = \text{tr } M^T \tag{1}$$

and the cyclic property of products. For example

$$\text{tr } \underset{p\times m}{R}\ \underset{m\times \ell}{S}\ \underset{\ell\times p}{T} = \text{tr } \underset{\ell\times p}{T}\ \underset{p\times m}{R}\ \underset{m\times \ell}{S} = \text{tr } \underset{m\times \ell}{S}\ \underset{\ell\times p}{T}\ \underset{p\times m}{R} \tag{2}$$

A. LINEAR CASES. Consider the $\ell\times\ell$ matrix function of the rectangular matrix X of size $p\times m$, that is

$$\underset{\ell\times\ell}{L_1} = \underset{\ell\times p}{A}\ \underset{p\times m}{X}\ \underset{m\times \ell}{B} \tag{3}$$

with trace

$$\text{tr } L_1 = \ell = \text{tr } AXB \quad \text{(Note this } \ell \text{ is not the matrix size.)} \tag{4}$$

the $m\times m$ matrix L_2

$$\underset{m\times m}{L_2} = \underset{m\times \ell}{B}\ \underset{\ell\times p}{A}\ \underset{p\times m}{X} \tag{5}$$

with trace

$$\text{tr } L_2 = \text{tr } L_1 = \ell \tag{6}$$

and the $p\times p$ matrix

$$\underset{p\times p}{L_3} = \underset{p\times m}{X}\ \underset{m\times \ell}{B}\ \underset{\ell\times p}{A} \tag{7}$$

and by Equation (2)

$$\ell = \text{tr } AXB = \text{tr } BAX = \text{tr } XBA \tag{8}$$

The differential of the trace is

$$d\ell = d \text{ tr } (AXB) = d \text{ tr } (BAX) = d \text{ tr } (XBA) \tag{9}$$

or

$$d\ell = \text{tr } \underset{\ell\times p}{A}\ \underset{p\times m}{dX}\ \underset{m\times \ell}{B} = \text{tr } \underset{m\times m}{BAdX}$$

$$= \text{tr } \underset{p\times p}{(dXBA)} \tag{10}$$

The differential of the m×m matrix can be written as

$$\underset{m\times m}{dL_2} = \underset{m\times p}{BA}\ \underset{p\times m}{dX} = \underset{m\times p}{\frac{\partial \ell}{\partial X^T}}\ \underset{p\times m}{dX} \tag{11}$$

and the differential of the p×p matrix can be written as

$$\underset{p\times p}{dL_3} = \underset{p\times m}{dX}\ \underset{m\times p}{BA} = \underset{p\times m}{dX}\ \underset{m\times p}{\frac{\partial \ell}{X^T}} \tag{12}$$

The differential of the traces are

$$d\ell = \mathrm{tr}\left[\underset{p\times m}{\frac{\partial \ell}{\partial X^T}}\ \underset{m\times p}{dX}\right] = \mathrm{tr}\left[\underset{p\times m}{dX}\ \underset{m\times p}{\frac{\partial \ell}{\partial X^T}}\right] \tag{13}$$

and by Equation (10) and Equation (13)

$$\frac{\partial \ell}{\partial X^T} = \underset{m\times \ell}{B}\ \underset{\ell\times p}{A} \tag{14}$$

or

$$\frac{\partial}{\partial X^T}\,\mathrm{tr}\ \underset{\ell\times\ell}{(AXB)} = \frac{\partial}{\partial X^T}\,\mathrm{tr}\ \underset{m\times m}{(BAX)} = \frac{\partial}{\partial X^T}\ \underset{p\times p}{(XBA)} = \underset{m\times \ell}{B}\ \underset{\ell\times p}{A} \tag{15}$$

For the special case when the L_i are scalers, i.e.: A and B are vectors

$$\frac{\partial}{\partial X^T}\,\mathrm{tr}\ \langle p)\ a\ \underset{p\times m}{X}\ b\ (m\rangle = b(m\rangle\langle p)a \tag{16}$$

$$\frac{\partial\,\mathrm{tr}}{\partial X^T}\left[b(m\rangle\langle p)a\ \underset{p\times m}{X}\right] = b(m\rangle\langle p)a \tag{17}$$

and

$$\frac{\partial}{\partial X^T} \operatorname{tr} \left[\underset{p\times m}{X} \, b(m\rangle\langle p)a \right] = b(m\rangle\langle p)a \tag{18}$$

By Equation (3) for m=ℓ and B=Im

$$\frac{\partial \operatorname{tr}}{\underset{m\times p}{\partial X^T}} \left(\underset{m\times p}{A} \; \underset{p\times m}{X} \right) = \underset{m\times p}{A} \tag{19}$$

By Equation (3) for ℓ=p and $A=I_p$

$$\frac{\partial \operatorname{tr}}{\underset{m\times p}{\partial X^T}} \left(\underset{p\times m}{X} \; \underset{m\times p}{B} \right) = \underset{m\times p}{B} \tag{20}$$

In the above equation if m=1

$$\frac{\partial \operatorname{tr}}{\underset{1\times p}{\partial X^T}} \left[x(p\rangle\langle p)b \right] = \langle p)b \tag{21}$$

Since the trace of the dyadic product is the inner product

$$\operatorname{tr}\, (x\rangle\langle b) = \langle bx\rangle = \ell \tag{22}$$

one has

$$\ell \langle \frac{\partial}{\partial x} = \langle \frac{\partial \ell}{\partial x} = \langle b \; x\rangle\langle \frac{\partial}{\partial x} = \langle b \tag{23}$$

where the matrix

$$x\rangle\langle \frac{\partial}{\partial x} = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^p \end{bmatrix} \left[\frac{\partial}{\partial x_1}, \cdots \frac{\partial}{\partial x_p} \right] = I \tag{24}$$

when the coordinates are all independent.

One can verify all of the previous results via the tedious process of partitioning. For example consider Equation (20)

$$\underset{p\times p}{L} = \underset{p\times m}{X} \; \underset{m\times p}{B} \tag{25}$$

and partition X into its column vectors and B into its row vectors to obtain

$$XB = \left[x(p\rangle_1, \cdots x(p\rangle_m \right] \begin{bmatrix} {}^1\langle p)b \\ \vdots \\ {}^m\langle p)b \end{bmatrix} = \sum_{i=1}^{m} x(p\rangle_i \, {}^i\langle p)b \tag{26}$$

the trace of Equation (26) is

$$\mathrm{tr}\; XB = \mathrm{tr}\left(x\rangle_1 {}^1\langle b \right) + \mathrm{tr}\left(x\rangle_2 {}^2\langle b \right) + \cdots + \mathrm{tr}\left(x\rangle_m {}^m\langle b \right) \tag{27}$$

or

$$\ell = \mathrm{tr}\; XB = {}_1\langle bx\rangle_1 + {}_2\langle bx\rangle_2 + \cdots + {}_m\langle bx\rangle_m \tag{28}$$

$$= \ell_1 + \ell_2 + \cdots + \ell_m \tag{29}$$

The differential of ℓ is

$$d\ell = d\ell_1 + d\ell_2 + \cdots + d\ell_m \tag{30}$$

where each of the ℓ_i is a function of the vector $x\rangle_i$ or by Equation (13)

$$d\ell_i = {}_i\left\langle \frac{\partial \ell_i}{\partial x} \, dx \right\rangle_i \tag{31}$$

Repackaging Equation (31)

$$d\ell = {}_1\left\langle \frac{\partial \ell_i}{\partial x} \, dx(p\right\rangle_1 + \cdots + {}_m\left\langle \frac{\partial \ell_m}{\partial x} \, dx(p\right\rangle_m \tag{32}$$

By Equation (31) in Equation (28)

$$ {}_i\left\langle \frac{\partial \ell}{\partial x} \right. = {}_i\left\langle \frac{\partial \ell_i}{\partial x} \right. \qquad (33)$$

Consider the product of the matrix of gradient vectors with the matrix of dX vectors that is

$$\begin{bmatrix} {}_1\langle p)\frac{\partial \ell}{\partial x} \\ {}_2\langle p)\frac{\partial \ell}{\partial x} \\ \vdots \\ {}_m\langle p)\frac{\partial \ell}{\partial x} \end{bmatrix} \begin{bmatrix} dx(p\rangle_1, \cdots dx(p\rangle_m \end{bmatrix} \qquad (34)$$

$$= \begin{bmatrix} {}_1\left\langle \frac{\partial \ell}{\partial x}\, dx \right\rangle_1 & \cdots & {}_1\left\langle \frac{\partial \ell}{\partial x}\, dx \right\rangle_m \\ \vdots & & \vdots \\ {}_m\left\langle \frac{\partial \ell}{\partial x}\, dx \right\rangle_1 & \cdots & {}_m\left\langle p)\frac{\partial \ell}{\partial x}\, dx \right\rangle_m \end{bmatrix} \qquad (35)$$

$$= \underset{m\times p}{\frac{\partial \ell}{\partial X^T}} \underset{p\times m}{dX} \qquad (36)$$

Clearly

$$\operatorname{tr}\left[\frac{\partial \ell}{\partial X^T}\, dX\right] = d\ell \qquad (37)$$

The other results can be shown by similar partitioning and can be found in detail in reference [4].

B. <u>QUADRATIC CASES.</u> Consider the $\ell\times\ell$ matrix product

$$\underset{\ell\times\ell}{Q_1} = \underset{\ell\times p}{A}\ \underset{p\times m}{X}\ \underset{m\times m}{C}\ \underset{m\times p}{X^T}\ \underset{p\times\ell}{B} \tag{38}$$

or

$$\underset{\ell\times\ell}{Q_1} = \left[\underset{\ell\times m}{AX}\right]\left[\underset{m\times\ell}{C\ X^T\ B}\right] \tag{39}$$

Form the new matrix of size m×m from Equation (39) cyclically permuted via parentheses

$$\underset{m\times m}{Q_2} = \left[\underset{m\times\ell}{C\ X^T\ B}\right]\ \underset{\ell\times m}{AX} \tag{40}$$

Form a third matrix of size p×p by cyclically permuting the A matrix of Equation (39)

$$\underset{p\times p}{Q_3} = \underset{p\times m}{X}\ \underset{m\times m}{C}\ \underset{m\times p}{X^T}\ \underset{p\times\ell}{B}\ \underset{\ell\times p}{A} \tag{41}$$

The differential of Equation (40) is

$$\underset{m\times m}{dQ_2} = C\ dX^TBAX + CX^TBAdX \tag{42}$$

The first matrix on the right of Equation (42) under the trace transposition and permutation rules becomes

$$\text{tr}\ (CdX^TBAX) = \text{tr}\ C^TX^TA^TB^TdX \tag{43}$$

Using Equation (43) in the trace of Equation (42) yields

$$\text{tr}\ \underset{m\times m}{dQ_2} = \text{tr}\ (C^TX^TA^TB^T+CX^TBA)\ dX \tag{44}$$

The gradient factors of dQ_2 can be taken to be

$$\underset{m\times m}{dQ_2} = \frac{\partial tr dQ_2}{\underset{m\times p}{\partial X^T}} \underset{p\times m}{dX} \tag{45}$$

and by Equation (45) and Equation (44)

$$\frac{\partial tr}{\partial X^T} (XCX^T BA) = C^T X^T A^T B^T + CX^T BA \tag{46}$$

By Equation (38), Equation (40) and Equation (43) the traces are all equal, that is

$$tr\ Q_1 = tr\ Q_2 = tr\ Q_3 \tag{47}$$

and the

$$d\ tr\ Q_i = tr\ dQ_i \qquad i = 1, 2, 3 \tag{48}$$

hence

$$\frac{\partial tr}{\underset{m\times p}{\partial X^T}} \left[\underset{\ell\times p}{A} \ \underset{p\times m}{X} \ \underset{m\times m}{C} \ \underset{m\times p}{X^T} \ \underset{p\times \ell}{B} \right] = C^T X^T A^T B^T + CX^T BA \tag{49}$$

and

$$\frac{\partial tr}{\partial X^T} \left[CX^T BAX \right] = C^T X^T A^T B^T + CX^T BA \tag{50}$$

If C is square and equal to I, we obtain by Equation (46), Equation (49) and Equation (50)

$$\frac{\partial tr}{\partial X^T} \left[\underset{p\times p}{XX^T} \ BA \right] = \frac{\partial tr}{\partial X^T} \left[AXX^T B \right] = \frac{\partial tr}{\partial X^T} \left[X^T BAX \right] = X^T (A^T B^T + BA) \tag{51}$$

If

$$BA = I \tag{52}$$

then Equation (51) becomes

$$\frac{\partial tr}{\partial X^T}\left[X^T X\right] = \frac{\partial tr}{\partial X^T}\left[X X^T\right] = 2 \underset{m\times p}{X^T} \tag{53}$$

For the special case when Equation (49) is a scalar, one obtains

$$\frac{\partial}{\partial X^T}\left[\langle a X C X^T b\rangle\right] = C^T X^T a\rangle\langle b + C X^T b\rangle\langle a \tag{54}$$

and for C=I

$$\frac{\partial}{\partial X^T}\left[\langle a X X^T b\rangle = \underset{m\times p}{X^T}\left[a\rangle\langle b+b\rangle\langle a\right] \tag{55}$$

For p=1 by Equation (51)

$$\frac{\partial}{\partial X^T}\langle x B A x\rangle = \langle x\left(a\rangle\langle b+b\rangle\langle a\right) \tag{56}$$

For C=I in Equation (54)

$$\frac{\partial tr}{\partial X^T}\left[X^T b\rangle\langle a X\right] = X^T\left(a\rangle\langle b+b\rangle\langle a\right) \tag{57}$$

As a final quadratic case consider the (full rank) grammian-matrix

$$X^T X = G \tag{58}$$

and its inverse

$$(X^T X)^{-1} = G^{-1} \tag{59}$$

where

$$G^{-1} G = I \tag{60}$$

The differential of Equation (60) is

$$dG^{-1}G + G^{-1}dG = 0 \tag{61}$$

or

$$dG^{-1} = -G^{-1}dGG^{-1} \tag{62}$$

$$= -G^{-1}[dX^TX+X^TdX]\ G^{-1} \tag{63}$$

or

$$dG^{-1} = -G^{-1}dX^TXG^{-1} - G^{-1}X^TdXG^{-1} \tag{64}$$

The generalized inverse, for the full-rank case, is defined as

$$\underset{m\times p}{X^*} = (X^TX)^{-1}\ X^T = G^{-1}X^T \tag{65}$$

and its transpose is

$$\underset{p\times m}{X^{*T}} = \underset{p\times m}{X}\ \underset{m\times m}{G^{-1}} \tag{66}$$

Using Equation (65) and Equation (66) in Equation (64)

$$dG^{-1} = -\ G^{-1}\ dX^TX^{*T} - X^*dXG^{-1} \tag{67}$$

The trace of Equation (67) is

$$\mathrm{tr}\ dG^{-1} = -\ \mathrm{tr}\ (dX^TX^{*T}G^{-1}) - \mathrm{tr}\ (G^{-1}X^*dX)$$

Using the trace transpose property on the first right hand side term of Equation (68)

$$\mathrm{tr}\ dG^{-1} = -\ \mathrm{tr}\ (G^{-1}X^*dX+G^{-1}X^*dX) \tag{69}$$

$$= -\ 2\ \mathrm{tr}(G^{-1}X^*dX)$$

or

$$\underset{m\times p}{\frac{\partial tr}{\partial X^T}} \underset{m\times m}{(X^TX)^{-1}} = -2\,(X^TX)^{-1}X^* \tag{71}$$

Using Equation (65) in Equation (71)

$$\frac{\partial tr}{\partial X^T} (X^TX)^{-1} = -2\,\underset{m\times m}{(X^TX)^{-2}}\;\underset{m\times p}{X^T} \tag{72}$$

C. "CUBIC" CASE. The following cases do not involve cubics but the matrix X and X^T appear three times in the generalized inverse relation (full rank)

$$\underset{m\times p}{X^*} = \underset{m\times m}{(X^TX)^{-1}}\;\underset{m\times p}{X^T} \tag{73}$$

Consider the linear form in X* that is

$$\underset{m\times m}{Q} = \underset{m\times p}{X^*}\;\underset{p\times m}{B} \tag{74}$$

and

$$dQ = dX^*B \tag{75}$$

$$dQ = d\,[(X^TX)^{-1}X^T]\,B$$

$$= d(X^TX)^{-1}\,X^TB + (X^TX)^{-1}\,dX^TB \tag{76}$$

by Equation (63) in Equation (76)

$$dQ = -\,G^{-1}\,[dX^TX+X^TdX]\,G^{-1}X^TB + G^{-1}dX^TB \tag{77}$$

or by Equation (73) in Equation (77)

$$dQ = - G^{-1}[dX^T X + X^T dX]X^*B + G^{-1}dX^T B \quad (78)$$

Using the relation for the projector

$$\underset{p\times m}{X}\ \underset{m\times p}{X^*} = \underset{p\times p}{P} = P^T = P^2 \quad (79)$$

in Equation (78)

$$dQ = - G^{-1}\ dX^T\ PB - X^*dXX^*B + G^{-1}\ dX^T B \quad (80)$$

or

$$dQ = G^{-1}\ dX^T\ (I-P)\ B - X^*\ dXX^*B \quad (81)$$

The orthogonal compliment projector is

$$\tilde{P} = I - P = \tilde{P}^T = \tilde{P}^2 \quad (82)$$

Using Equation (82) and Equation (81)

$$dQ = G^{-1}\ dX^T\tilde{P}B - X^*dXX^*B \quad (83)$$

The trace relations yield

$$\mathrm{tr}\ dQ = + \mathrm{tr}(dX^T\tilde{P}BG^{-1}) - \mathrm{tr}\ (X^*BX^*dX) \quad (84)$$

and transpose-wise

$$\mathrm{tr}\ dQ = \mathrm{tr}\ (G^{-1}B^T\tilde{P}dX) - \mathrm{tr}\ X^*BX^*dX \quad (85)$$

or

$$\mathrm{tr}\ dQ = \mathrm{tr}\ [(G^{-1}B^T\tilde{P} - X^*BX^*)dX] \quad (86)$$

Expressing dQ as

$$\underset{m\times m}{dQ} = \underset{m\times p}{\frac{\partial \mathrm{tr} dQ}{\partial X^T}}\ \underset{p\times m}{dX} \quad (87)$$

one obtains

$$\boxed{\frac{\partial tr}{\partial X^T_m}(X^*B)_m = G^{-1} B^T\tilde{P} - X^* BX^*} \tag{88}$$

D. POWERS OF $(X^*B)^n$. The following cases are of higher degree than third. Consider n=2 then the Q of Equation (74) has square

$$Q^2 = X^*BX^*B \tag{89}$$

and

$$dQ^2 = dX^*BX^*B + X^*BdX^*B \tag{90}$$

By Equation (75)

$$dQ = dX^*B \tag{91}$$

hence Equation (91) in Equation (90)

$$dQ^2 = dQX^*B + X^*BdQ \tag{92}$$

By Equation (89) let

$$q_2 = tr\ Q^2 \tag{93}$$

then

$$dQ^2 = \frac{\partial q_2}{\partial X^T} dX = dQQ + QdQ \tag{94}$$

The trace of Equation (92) is

$$tr\ dQ^2 = tr\ 2X^*BdQ \tag{95}$$

By Equation (94) and Equation (95)

$$\tfrac{1}{2}\ \mathrm{tr}\ dQ^2 = \mathrm{tr}\ Q\ dQ = \mathrm{tr}\ X^*BdQ \tag{96}$$

Using dQ given by Equation (83) in Equation (96)

$$\begin{aligned} \mathrm{tr}\ X^*BdQ &= \mathrm{tr}\ \{X^*B\ \ (G^{-1}dX^T\tilde{P}B - X^*dXX^*B)\} \\ &= \mathrm{tr}\ X^*BG^{-1}dX^T\tilde{P}B - \mathrm{tr}\ X^*BX^*dXX^*B \end{aligned} \tag{97}$$

$$= \mathrm{tr}\ dX^T\ (\tilde{P}BX^*BG^{-1}) - \mathrm{tr}\ X^*BX^*BX^*dX \tag{98}$$

$$= \mathrm{tr}\ (G^{-1}B^TX^{*T}B^T\tilde{P})dX - \mathrm{tr}\ Q^2X^*dX \tag{99}$$

$$\tfrac{1}{2}\ \mathrm{tr}\ dQ^2 = \mathrm{tr}\ \{(G^{-1}B^TX^{*T}B^T\tilde{P} - Q^2X^*)\ dX\} \tag{100}$$

or

$$\tfrac{1}{2}\frac{\partial q_2}{\partial X^T} = G^{-1}Q^TB^T\tilde{P} - Q^2X^* \tag{101}$$

That is,

$$\frac{\partial q_2}{\partial X^T} = 2\ (G^{-1}Q^TB^T\tilde{P} - Q^2X^*) \tag{102}$$

Let n=3 and

$$Q^3 = QQQ \tag{103}$$

with

$$\mathrm{tr}\ Q^3 = q_3 \tag{104}$$

The differential of Equation (103) is

$$dQ^3 = dQQ^2 + QdQQ + Q^2dQ \tag{105}$$

$$= \frac{\partial q_3}{\partial X^T}\ dX \tag{106}$$

and the trace of Equation (105) is

$$\text{tr } dQ^3 = 3 \text{ tr } Q^2 dQ \tag{107}$$

By Equation (83) in Equation (107)

$$\text{tr } dQ^3 = 3 \text{ tr } Q^2 \left[G^{-1} dX^T \tilde{P} B - X^* dX Q\right] \tag{108}$$

Using the commuting and permuting properties of trace one obtains

$$\underset{m\times p}{\frac{\partial q_3}{\partial X^T}} = 3\left[G^{-1} (Q^T)^2 B^T \tilde{P} - Q^3 X^*\right] \tag{109}$$

For n=1,2, and 3 by Equation (88), Equation (101) and Equation (109),

$$\frac{\partial q_1}{\partial X^T} = G^{-1} B^T \tilde{P} - QX^* \tag{110}$$

$$\frac{\partial q_2}{\partial X^T} = 2\left[G^{-1} Q^T B^T \tilde{P} - Q^2 X^*\right] \tag{111}$$

$$\frac{\partial q_3}{\partial X^T} = 3\left[G^{-1} (Q^T)^2 B^T \tilde{P} - Q^3 X^*\right] \tag{112}$$

This gives the inductive step, therefore for

$$\underset{m\times m}{Q} = \underset{m\times p}{X^*} \underset{p\times m}{B} \tag{113}$$

$$\underset{m\times m}{G^{-1}} = \underset{m\times m}{(X^T X)^{-1}} \tag{114}$$

$$\underset{m\times p}{X^*} = G^{-1} X^T \tag{115}$$

$$\underset{p\times p}{P} = XX^*$$ (116)

$$\tilde{P} = I - P$$ (117)

$$q_n = \text{tr } Q^n; \quad n = 1,2,\cdots$$ (118)

we have

$$\frac{\partial q_n}{\partial X^T} = n \left[G^{-1}(Q^T)^{n-1}B^T\tilde{P} - Q^n X^*\right]$$ (119)

A number of special cases follow from Equation (119). Suppose X is square and full rank, then

$$\underset{m\times m}{X^*} = X^{-1}$$ (120)

and

$$Q = X^{-1}B$$ (121)

Then

$$P = XX^{-1} = I$$ (122)

and

$$\tilde{P} = 0$$ (123)

hence

$$\frac{\partial \text{tr}}{\partial X^T} (X^{-1}B)^n = - n (X^{-1}B)^n X^{-1}$$ (124)

Suppose now that B=I then

$$\frac{\partial \text{tr}}{\partial X^T} (X^{-1})^n = - n (X^{-1})^n X^{-1}$$ (125)

or

$$\frac{\partial \, tr}{\partial X^T}(X^{-1})^n = nX^{-(n+1)} \tag{126}$$

E. POWERS OF $(XB)^n$. The powers of

$$\underset{p\times p}{Q^n} = (XB)^n \tag{127}$$

can be obtained in a similar manner. By Equation (20)

$$\frac{\partial q_1}{\partial X^T} = \underset{m\times p}{B} \tag{128}$$

Let

$$Q^2 = (XB)^2 \tag{129}$$

$$q_2 = tr\ Q^2 \tag{130}$$

and

$$dQ^2 = dQQ + QdQ \tag{131}$$

$$tr\ dQ^2 = 2\ tr\ (QdQ) \tag{132}$$

$$= 2\ tr\ QdXB = 2\ tr\ (BQdX) \tag{133}$$

or

$$\frac{\partial q_2}{\partial X^T} = 2\ BQ \tag{134}$$

Repeating the arguments as before one obtains

$$\frac{\partial}{\partial X^T}[tr\ (XB)^n] = n \underset{m\times p}{B} (\underset{p\times p}{X\ B})^{n-1} \tag{135}$$

For X square, full rank, and

$$B = I$$

Equation (135) becomes

$$\frac{\partial \, tr X^n}{\partial X^T} = nX^{n-1} \qquad (136)$$

F. TABLES OF GRADIENTS

1. Linear Forms:

$$\left.\begin{array}{l}\dfrac{\partial tr}{\partial X^T}\left(\underset{\ell\times p}{A}\ \underset{p\times m}{X}\ \underset{m\times\ell}{B}\right)\\[2ex] \dfrac{\partial tr}{\partial X^T}\left(\underset{m\times\ell}{B}\ \underset{\ell\times p}{A}\ \underset{p\times m}{X}\right)\\[2ex] \dfrac{\partial tr}{\partial X^T}\left(\underset{p\times m}{X}\ \underset{m\times\ell}{B}\ \underset{\ell\times p}{A}\right)\end{array}\right\} = BA$$

$$\left.\begin{array}{l}\dfrac{\partial tr}{\partial X^T}\langle aXb\rangle\\[2ex] \dfrac{\partial tr}{\partial X^T}[b(m\rangle\langle p)a\ \underset{p\times m}{X}]\\[2ex] \dfrac{\partial tr}{\partial X^T}[\underset{p\times m}{X}\ b(m\rangle\langle p)a]\end{array}\right\} = b(m\rangle\langle p)a$$

$$\frac{\partial tr}{\partial X^T}[\underset{m\times p}{A}\ \underset{p\times m}{X}] = \underset{m\times p}{A}$$

$$\frac{\partial tr}{\partial X^T}[\underset{p\times m}{X}\ \underset{m\times p}{B}] = \underset{m\times p}{B}$$

$$\frac{\partial tr}{\partial X^T}\left(x(p\rangle\langle p)b\right) = \langle p)b$$

2. <u>Quadratic Forms:</u>

$$\left.\begin{aligned}
&\frac{\partial tr}{\partial X^T}\,[\underset{\ell\times p}{A}\ \underset{p\times m}{X}\ \underset{m\times m}{C}\ \underset{m\times p}{X^T}\ \underset{p\times \ell}{B}] \\
&\frac{\partial tr}{\partial X^T}\,[\underset{m\times m}{C}\ \underset{m\times p}{X^T}\ \underset{p\times \ell}{B}\ \underset{\ell\times p}{A}\ \underset{p\times m}{X}] \\
&\frac{\partial tr}{\partial X^T}\,[\underset{p\times m}{X}\ \underset{m\times m}{C}\ \underset{m\times p}{X^T}\ \underset{p\times \ell}{B}\ \underset{\ell\times p}{A}]
\end{aligned}\right\} = C^T X^T A^T B^T + C X^T B A$$

$$\left.\begin{aligned}
&\frac{\partial tr}{\partial X^T}\,[X X^T B A] \\
&\frac{\partial tr}{\partial X^T}\,[A X X^T B] \\
&\frac{\partial tr}{\partial X^T}\,[X^T B A X]
\end{aligned}\right\} = X^T (A^T B^T + BA)$$

$$\frac{\partial tr}{\partial X^T}\,[X^T X] = 2X^T$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial X^T}\,(\langle a X X^T b\rangle) \\
&\frac{\partial tr}{\partial X^T}\,(X^T b\rangle\langle a X)
\end{aligned}\right\} = X^T\ (a\rangle\langle b + b\rangle\langle a)$$

$$\frac{\partial}{\partial X^T}\,(\langle x B A x\rangle) = \langle x (a\rangle \langle b + b\rangle\langle a)$$

$$\frac{\partial tr}{\partial X^T}\,(X^T X)^{-1} = -\ 2(X^T X)^{-2}\ X^T$$

3. "Cubic" Forms and Others: The generalized inverse

$$X^* = (X^T X)^{-1} X^T = G^{-1} X^T$$

$$\frac{\partial tr}{\partial X^T} (\underset{m\times p}{X^*} \underset{p\times m}{B}) = (X^T X)^{-1} B^T \tilde{P} - X^* B X^*$$

where the projectors are

$$\tilde{P} = I - P$$

$$P = XX^*$$

$$G = X^T X$$

$$\frac{\partial tr}{\partial X^T} (X^* B)^n = n [G^{-1} (Q^T)^{n-1} B^T \tilde{P} - Q^n X^*]$$

$$\frac{\partial tr}{\partial X^T} (X^{-1} B)^n = - n (X^{-1} B)^n X^{-1}$$

$$\frac{\partial tr}{\partial X^T} (X^{-1})^n = - n X^{-(n+1)}$$

$$\frac{\partial tr}{\partial X^T} (XB)^n = nB(XB)^{n-1}$$

$$\frac{\partial tr}{\partial X^T} (X^n) = nX^{n-1}$$

III. SOME FUNCTIONS OF GENERALIZED MATRIX PRODUCTS INVOLVING (X, X^T, X^*, X^{*T}) OF FINITE ORDER. In this section we will be concerned with matrix functions of the form

$$Y = B_1 \cdot Z_1 \cdot B_2 \cdot Z_2 \cdots B_n \cdot Z_n \cdot B_{n+1}$$

$n>o$, n an integer and the B_i scalar matrices. Functions of Y, the argument, will be examined, such as $f(Y)=Y$, and certain transcendentals identified below. The algorithms to be developed are divided into three

APPENDIX G

KRONECKER MATRIX PRODUCTS

If $A=(a_{ij})$ is an m×n matrix and B is an s×t matrix then the Kronecker matrix product $A \otimes B$ is an ms×nt matrix

$$A \otimes B = (a_{ij}B) \tag{1}$$

or in open form

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & & \\ \vdots & & & \\ a_{m1}B & & & a_{mn}B \end{bmatrix} \tag{2}$$

Lancaster in reference (49) calls this operation <u>direct product</u>, thus he says if $A \varepsilon F_{m \times m}$ (note A is square in his treatment) and $B \varepsilon F_{n \times n}$, then the direct product of A and B, written $A \otimes B$ is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & & \\ \vdots & & & \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix} \tag{3}$$

Lancaster states that a certain degree of arbitrariness is apparent in the definition, that we may ask why $A \otimes B$ is not defined to be

$$\begin{bmatrix} Ab_{11} & Ab_{12} & \cdots & Ab_{1n} \\ \vdots & & & \\ Ab_{n1} & \cdots & & Ab_{nn} \end{bmatrix} = [Ab_{ij}] \tag{4}$$

Lancaster states that the answer is (for his purposes) that parallel properties can be obtained using the two possible definitions and so far as the applications are concerned, both definitions would be equally useful.

Chen in his applications to Walsh functions defines the Kronecker product as in Equation (4). Conlisk points out that Neudecker's papers are the only published papers he had found which formalized the operations of $\otimes$ and the "stacking operator" L(A), or as used by many vec A where if A is m×n and partitioned into its column space [note L(a) notation will not be used]

$$\underset{m\times n}{A} = \left[a(m\rangle_1, \cdots a(m\rangle_n \right] \tag{5}$$

then the column of column vectors is

$$\text{vec } A = \begin{bmatrix} a(m\rangle_1 \\ a(m\rangle_2 \\ \vdots \\ a(m\rangle_n \end{bmatrix} = v(mn\rangle \tag{6}$$

Hartwig in his paper calls vec X col X, that is for (note x_i are column vectors)

$$X = [x_1, x_2, \cdots x_n]$$

then

$$\text{col } X = [x_1^T, x_2^T, \cdots x_n^T]^T = \text{vec } X$$

The Kronecker product and the vec X definitions of Neudecker will be used in this report since so much published literature is available which has referenced his work.

The matrix $A_{m\times n}$ can be partitioned into a row of column vectors or a column of row vectors

$$\underset{m\times n}{A} = \left[a(m\rangle_1,\ a(m\rangle_2, \cdots a(m\rangle_n \right] = \begin{bmatrix} {}^{1}\langle n)\alpha \\ {}^{2}\langle n)\alpha \\ \vdots \\ {}^{m}\langle n)\alpha \end{bmatrix} \tag{7}$$

and the transpose of A is

$$\underset{n\times m}{A^T} = \begin{bmatrix} {}^{1}\langle m)a \\ {}^{2}\langle m)a \\ \vdots \\ {}^{n}\langle m)a \end{bmatrix} = \left[\alpha(n\rangle_1,\ \alpha(n\rangle_2, \cdots \alpha(n\rangle_m \right] \tag{8}$$

hence

$$\text{vec } A^T = \begin{bmatrix} \alpha(n\rangle^{1} \\ \alpha(n\rangle^{2} \\ \vdots \\ \alpha(n\rangle^{m} \end{bmatrix} \tag{9}$$

It is clear from the for-going, that how the column vectors of Equation (6) are packaged in terms of the partitioning of A is quite arbitrary, for example one could build up an algebra by working with large row vectors made up as a row of the rows of Equation (7).

Ben Israel and Greville in their book define the matrix Kronecker product as in Equation (2) but define the corresponding vectors by Equation (7) as a row of the row vectors

$$v(X)^T = \left[\overset{1}{\langle}n)\alpha,\ \overset{2}{\langle}n)\alpha, \cdots \overset{m}{\langle}m)\alpha\right]$$

or

$$v(X) \equiv \begin{bmatrix} \alpha(n\overset{1}{\rangle} \\ \alpha(n\overset{2}{\rangle} \\ \vdots \\ \alpha(n\overset{m}{\rangle} \end{bmatrix} \qquad (10)$$

which we see by Equation (9) is vec X^T as used by Neudecker.

In Ben Israel's notation via page (42) of reference (10); for any $X=[x_{ij}]\varepsilon C^{m\times n}$, let the vector $v(X)-[v_k]\varepsilon C^{mn}$ be the transpose of the row vector obtained by placing the rows of X end to end with the first row on the left and the last row on the right. In other words for $X_{\times n}$

$$v_{n(i-1)+j} = x_{ij} \ (i=1,2,\ldots m;j=1,2,\ldots n)$$

Using this system (or partitioning and packaging procedure) Ben Israel and Greville then invite the energetic reader to verify their relation (page 42 of their book)

$$v(AXB) = (A\boxtimes B^T)\ v(X) \qquad (11)$$

Clearly the results of Equation (11) will be different from the Equation 42 results of Neudecker, thus the user should be wary of blindly using formulas from various books and papers.

The relationships between the indices for vec X are ($X_{m\times n}$

$$\text{vec } X = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \\ \vdots \\ v^{mn} \end{bmatrix} = \underset{mn\times 1}{[\, v_k \,]} \tag{12}$$

where

$$x_{ij} = v_{m(j-1)+i} \tag{13}$$

$$(i=1,2,\ldots m;\ j=1,2,\ldots n)$$

The Kronecker product of Equation (3) can be written in terms of indices for the square matrix case of A(m,m) and B(n,n) we have

$$\underset{m\times m}{A} \otimes \underset{n\times n}{B} = \underset{(mn)\times(mn)}{C} \tag{14}$$

$$[a_{ri}] \otimes [b_{sj}] = [c_{k\ell}] \tag{15}$$

where

$$r,i = 1,2,\ldots m$$

$$s,j = 1,2,\ldots n$$

or element-wise

$$c_{k\ell} = a_{ri}\, b_{sj} \tag{16}$$

where

$$k = (r-1)\, n + s \tag{17}$$

$$\equiv s \bmod n$$

and

and

$$\ell = (i-1)\, n + j \qquad (18)$$

$$\equiv j \bmod n$$

The dyadic product of a row and a column vector is a simple example of a Kronecker matrix product, e.g.

$$\underset{(m\times 1)(1\times n)}{A \otimes B} = a(m\rangle\langle n)b = \begin{bmatrix} a^1\langle n)b \\ a^2\langle n)b \\ \vdots \\ a^m\langle n)b \end{bmatrix} = \begin{bmatrix} a^1 b_1 & \cdots & a^1 b_n \\ \vdots & & \vdots \\ a^m b_1 & \cdots & a^m b_n \end{bmatrix} \qquad (19)$$

and we see that the size is m×n.

Consider the usual inner product of a column and row vector

$$\langle n)b\ a(n\rangle = b_1 a_1 + b_2 a_2 + \dots + b_n a_n$$

and the corresponding Kronecker product

$$\underset{(1\times n)}{\langle b} \otimes \underset{(n\times 1)}{a\rangle} = [b_1 a\rangle,\ b_2 a\rangle, \dots b_n a\rangle]$$

$$= [a\rangle b_1,\ a\rangle b_2\ \dots\ a\rangle b_n]$$

$$\langle n)b \otimes a(n\rangle = a(n\rangle\langle n)b \qquad (20)$$

or for m≠n, the inner product is not defined but for Equation (20) we have

$$\langle n)b \otimes a(m\rangle = [b_1 a(m\rangle,\ \dots\ b_n a(m\rangle]$$

$$\underset{(1\times n)}{\langle n)b} \otimes \underset{(m\times 1)}{a(m\rangle} = \underset{m\times n}{a(m\rangle\langle n)b} = a(m\rangle \otimes \langle n)b \qquad (21)$$

which is the same as Equation (19).

The following pages will derive a number of Kronecker product relationships and then summarize the bulk of those to be used in the report. Consider an (m×n) matrix X

$$\underset{m\times n}{X} = \underset{m\times m}{I}\ \underset{m\times n}{X} = [I\ x(m)\rangle_1,\ I\ x\rangle_2 \cdots I\ x(m)\rangle_n] \tag{22}$$

or

$$\text{vec } X = \begin{bmatrix} I\ x\rangle_1 \\ I\ x\rangle_2 \\ \vdots \\ I\ x\rangle_n \end{bmatrix} = \begin{bmatrix} \underset{m\times m}{I} & & & \\ 0 & \underset{m\times m}{I} & & \\ & & \ddots & \\ & & & \underset{m\times m}{I} \end{bmatrix} \begin{bmatrix} x(m)\rangle^1 \\ x(m)\rangle^2 \\ \vdots \\ x(m)\rangle^n \end{bmatrix} \tag{23}$$

or

$$\text{vec } X = \left(\underset{n\times n}{I} \otimes \underset{m\times m}{I} \right) \text{vec } X \tag{24}$$

since

$$\underset{n\times m}{I} \otimes \underset{m\times m}{I} = \begin{bmatrix} 1\ \underset{m\times m}{I} & & & \\ & 1\ \underset{m\times m}{I} & & 0 \\ & & \ddots & \\ 0 & & & 1\ \underset{m\times m}{I} \end{bmatrix} \tag{25}$$

Equation (22) can also be written as

$$\underset{m\times n}{X} = \underset{m\times n}{X}\ \underset{n\times n}{I} \tag{26}$$

or

$$X = X\,[e(n)_1\rangle,\ e(n)_2\rangle \cdots e(n)_n\rangle] \qquad (27)$$

$$= [X\,e(n)_1\rangle,\ X\,e(n)_2\rangle \cdots X\,e(n)_n\rangle]$$

and

$$\operatorname{vec} X = \begin{bmatrix} \underset{m\times n}{X} & & & 0 \\ & X & & \\ & & \ddots & \\ 0 & & & X \end{bmatrix} \begin{bmatrix} e(n)_1\rangle \\ e(n)_2\rangle \\ \vdots \\ e(n)_n\rangle \end{bmatrix} \qquad (28)$$

$$\operatorname{vec} X = \left(\underset{n\times n}{I} \otimes \underset{m\times n}{X} \right) \operatorname{vec} \underset{n\times n}{I} \qquad (29)$$

Consider the product

$$\underset{\ell\times m}{L} = \underset{\ell\times m}{A}\ \underset{m\times n}{X} \qquad (30)$$

or partitioning

$$L = [A\,x(m)_1\rangle,\ A\,x(m)_2\rangle \cdots A\,x(m)_n\rangle] \qquad (31)$$

$$\operatorname{vec} L = \begin{bmatrix} \underset{\ell\times m}{A} & & & \\ & A & 0 & \\ & 0 & \ddots & \\ & & & A \end{bmatrix} \begin{bmatrix} x(m)_1\rangle \\ x(m)_2\rangle \\ \vdots \\ x(m)_n\rangle \end{bmatrix} \qquad (32)$$

or

$$\operatorname{vec} AX = \left(\underset{n\times n}{I} \otimes \underset{\ell\times m}{A} \right) \operatorname{vec} X \tag{33}$$

Another expression for vec AX can be obtained from Equation (32), as follows

$$\operatorname{vec} AX = \begin{bmatrix} A & & & 0 \\ & A & & \\ & & \ddots & \\ 0 & & & A \end{bmatrix} \begin{bmatrix} \underset{m\times m}{I} \; x(m)_{1} \\ \vdots \\ I \; x(m)_{m} \end{bmatrix} \tag{34}$$

$$= \begin{bmatrix} A & & & 0 \\ & A & & \\ & & \ddots & \\ 0 & & & A \end{bmatrix} \begin{bmatrix} \vec{e}_1 x^{1}_{\cdot 1} + \vec{e}_2 x^{2}_{\cdot 1} + \cdots + \vec{e}_m x^{m}_{\cdot 1} \\ \vec{e}_1 x^{1}_{\cdot 2} + \vec{e}_2 x^{2}_{\cdot 2} + \cdots + \vec{e}_m x^{m}_{\cdot 2} \\ \vdots \\ \vec{e}_1 x^{1}_{\cdot m} + \vec{e}_2 x^{2}_{\cdot m} + \cdots + \vec{e}_m x^{m}_{\cdot n} \end{bmatrix} \tag{35}$$

$$= \begin{bmatrix} A[\vec{e}_1 x^{1}_{\cdot 1} + \vec{e}_2 x^{2}_{\cdot 1} + \cdots + \vec{e}_m x^{m}_{\cdot 1}] \\ \vdots \\ A[\vec{e}_1 x^{1}_{\cdot n} + \vec{e}^{\,2} x^{2}_{\cdot n} + \cdots + \vec{e}_m x^{m}_{\cdot n}] \end{bmatrix} \tag{36}$$

$$= \begin{bmatrix} x^{1}_{\cdot 1} A e\rangle_{1} + x^{2}_{\cdot 1} A e\rangle_{2} + \cdots + x^{m}_{\cdot 1} A e\rangle_{m} \\ \vdots \\ x^{1}_{\cdot n} A e\rangle_{1} + x^{2}_{\cdot n} A e\rangle_{2} + \cdots + x^{m}_{\cdot n} A e\rangle_{m} \end{bmatrix} \tag{37}$$

$$= \begin{bmatrix} x^{1}_{\cdot 1} A & x^{2}_{\cdot 1} A & \cdots & x^{m}_{\cdot 1} A \\ x^{1}_{\cdot 2} A & x^{2}_{\cdot 2} A & \cdots & x^{m}_{\cdot 2} A \\ \vdots & & & \\ x^{1}_{\cdot n} A & x^{2}_{\cdot n} A & & x^{m}_{\cdot n} A \end{bmatrix} \begin{bmatrix} e(m\rangle_{1} \\ e(m\rangle_{2} \\ \vdots \\ e\rangle_{m} \end{bmatrix} \tag{38}$$

or

$$\text{vec } (AX) = (X^{T} \otimes A) \text{ vec } I_{m} \tag{39}$$

Consider the product of three matrices

$$\underset{k\times s}{L} = \underset{k\times m}{A} \; \underset{m\times n}{X} \; \underset{n\times s}{B} \tag{40}$$

and partition B into a row of column vectors as

$$L = AX[b(n\rangle_{1}, \; b(n\rangle_{2}, \; \cdots \; b(n\rangle_{s}] \tag{41}$$

or

$$L = [AXb(n\rangle_{1}, \; AXb\rangle_{2}, \; \cdots \; AXb\rangle_{s}] \tag{42}$$

Consider the first column vector of Equation (42)

$$AXb(n\rangle_1 = A[x\rangle_1,\ x\rangle_2 \cdots x(m\rangle_n] \begin{bmatrix} b^1 \\ b^2 \\ \vdots \\ b^n \end{bmatrix}_1 \qquad (43)$$

$$AXb(n\rangle_1 = A[x\rangle_1 b_{11} + x\rangle_2 b_{21} + \cdots + x\rangle_n b_{n1}] \qquad (44)$$

$$= b_{11}Ax\rangle_1 + b_{21}Ax\rangle_2 + b_{31}Ax\rangle_3 + \cdots + b_{n1}Ax\rangle_n \qquad (45)$$

and similarly for each of the other column vectors, we have

$$\text{vec } L = \begin{bmatrix} b_{11}Ax\rangle_1 + b_{21}Ax\rangle_2 + \cdots + b_{n1}Ax\rangle_n \\ b_{12}Ax\rangle_1 + b_{22}Ax\rangle_2 + \cdots + b_{22}Ax\rangle_n \\ \vdots \\ b_{1s}Ax\rangle_1 + b_{2s}Ax\rangle_2 + \cdots + b_{ns}Ax\rangle_n \end{bmatrix} \qquad (46)$$

$$= \begin{bmatrix} b_{11}A & b_{21}A & \cdots & b_{n1}A \\ b_{12}A & b_{22}A & \cdots & b_{n2}A \\ \vdots & & & \\ b_{12}A & b_{22}A & \cdots & b_{n2}A \end{bmatrix} \begin{bmatrix} x(m\rangle_1 \\ x(m\rangle_2 \\ \vdots \\ x(m\rangle_n \end{bmatrix} \qquad (47)$$

and using Equation (2)

$$\text{vec}\,(AXB) = (B^T \otimes A)\,\text{vec}\,X \tag{48}$$

By Equation (48) when B=I we obtain Equation (33).

By Equation (48) for A=I we obtain

$$\text{vec}\,(XB) = (B^T \otimes I)\,\text{vec}\,X \tag{49}$$

Consider next the relations

$$\underset{(k\times m)(m\times r)}{A \quad C} \otimes \underset{(s\times m)(n\times t)}{B \quad D} = \underset{(k\times m)\ s\times n}{(A \otimes B)}\underset{m\times r\ \ n\times t}{(C \otimes D)} \tag{50}$$

or

$$\underset{k\times r\ \ s\times t}{AC \otimes BD} = \underset{(ks\times mn)(mn\times rt)}{(A\otimes B)\ (C\otimes D)} \tag{51}$$

or

$$\underset{ks\times rt}{AC\otimes BD} = \underset{ks\times rt}{(A\otimes B)(C\otimes D)} \tag{52}$$

where the matrix sizes are indicated at each step.

The proof of Equation (50) will be demonstrated for 2×2 matrices by construction

$$(A\otimes B)(C\otimes D) = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} \begin{bmatrix} c_{11}D & c_{12}D \\ c_{21}D & c_{22}D \end{bmatrix} \tag{53}$$

$$= \begin{bmatrix} (a_{11}c_{11}+a_{12}c_{21})BD & (a_{11}c_{12}+a_{12}c_{22})BD \\ (a_{21}c_{11}+a_{22}c_{21})BD & (a_{21}c_{12}+A_{22}c_{22})BD \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}c_{11}+a_{12}c_{21} & a_{11}c_{12}+a_{12}c_{22} \\ a_{21}c_{11}+a_{22}c_{21} & a_{21}c_{12}+a_{22}c_{22} \end{bmatrix} \otimes BD$$

or

$$(A \otimes B)(C \otimes D) = AC \otimes BD \tag{54}$$

The relations of Equation (50) will be used to prove (Lancaster page 258)

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \tag{55}$$

Note the inverse matrices are not commuted as in the standard product inverse where

$$(AB)^{-1} = B^{-1}A^{-1} \tag{56}$$

or the full-rank-factor case for the generalized inverse

$$(\underset{m\times\ell}{A}\ \underset{\ell\times n}{B})^{*} = \underset{n\times\ell}{B^{*}}\ \underset{\ell\times m}{A^{*}} \tag{57}$$

where $*$ means generalized inverse and the rank of A and B is ℓ.

By Equation (55)

$$(A \otimes B)(A \otimes B)^{-1} = (A \otimes B)(A^{-1} \otimes B^{-1}) \tag{58}$$

By Equation (54) for square invertible matrices

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = AA^{-1} \otimes BB^{-1} = I_m \otimes I_n = I_{(mn)(mn)} \tag{59}$$

where A is m×m and B is n×n square matrices.

Using Equation (59) in Equation (58)

$$(A \otimes B)(A \otimes B)^{-1} = I_m \otimes I_n = I_{mn} \tag{60}$$

which establishes Equation (55).

The inverse of the Kronecker product of two matrices of Equation (55) can be derived in other ways, some of which are of interest, e.g. consider

$$A \boxtimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \\ a_{n1}B & & a_{nn}B \end{bmatrix}$$

$$A \boxtimes B = \begin{bmatrix} a_{11}I & \cdots & a_{1n}I \\ \vdots & & \\ a_{n1}I & & a_{nn}I \end{bmatrix} \begin{bmatrix} B & & 0 \\ & \ddots & \\ 0 & & B \end{bmatrix} \tag{61}$$

$$= (A \boxtimes I)(I \boxtimes B) \tag{62}$$

and by Equation (56)

$$(A \boxtimes B)^{-1} = (I \boxtimes B)^{-1} (A \boxtimes I)^{-1} \tag{63}$$

It is obvious that

$$(I \boxtimes B)^{-1} = \begin{bmatrix} B & & & \\ & B & & \\ & & \ddots & 0 \\ 0 & & & B \end{bmatrix}^{-1} = \begin{bmatrix} B^{-1} & & & \\ & B^{-1} & & \\ & & \ddots & 0 \\ 0 & & & B^{-1} \end{bmatrix} \tag{64}$$

It is not obvious what the inverse of $A \boxtimes I$ is where in open form

$$(A \boxtimes I) = \begin{bmatrix} a_{11}I_m & \cdots & a_{1m}I_m \\ \vdots & & \\ a_{m_1}I & & a_{mm}I \end{bmatrix} \tag{65}$$

Define

$$A^{-1} = A^* = \begin{bmatrix} a^*_{11} & \cdots & a^*_{1m} \\ a^*_{m_1} & & a^*_{m_m} \end{bmatrix} \tag{66}$$

Then

$$AA^{*} = I_{m\times m} \tag{67}$$

Applying Equation (55) to Equation (65)

$$(A\boxtimes I)^{-1} = A^{-1}\boxtimes I = \begin{bmatrix} a^{*}_{11}I & \cdots & a^{*}_{1m}I \\ \vdots & & \\ a^{*}_{m_1}I & & a^{*}_{m_m}I \end{bmatrix} \tag{68}$$

Using Equation (68) and Equation (65)

$$(A\boxtimes I)(A\boxtimes I)^{-1} = \begin{bmatrix} a_{11}I & \cdots & a_{1m}I \\ \vdots & & \\ a_{m_1}I & & a_{m_m}I \end{bmatrix}\begin{bmatrix} a^{*}_{1_1}I & \cdots & a^{*}_{1m}I \\ \vdots & & \\ a^{*}_{m_1}I & & a^{*}_{m_m}I \end{bmatrix} \tag{69}$$

$$= \begin{bmatrix} (a_{11}a^{*}_{11}+a^{*}_{21}\ldots+a_{1_m}a^{*}_{m_1})I_{m\times m} & \cdot & \cdot & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & \cdot & \cdot & (a^{*}_{m_1}a^{*}_{1_m}+\ldots)I \end{bmatrix} \tag{70}$$

Note that by Equation (69) the biorthogonal relations are

$$AA^{*} = \begin{bmatrix} ^{1}\langle a \\ ^{2}\langle a \\ \vdots \\ ^{m}\langle a \end{bmatrix}\begin{bmatrix} a^{*}\rangle_{1} & \ldots & a^{*}\rangle_{m} \end{bmatrix} = I \tag{71}$$

$$
= \begin{bmatrix}
{}^{1}\langle a\; a^{*}_{1}\rangle & \langle a_{1}\; a^{*}_{2}\rangle & \cdots & \langle a_{1}\; a^{*}_{m}\rangle \\
 & & & \\
{}^{m}\langle a\; a^{*}_{1}\rangle & \cdot & \cdot & \cdot \;\; {}^{m}\langle a\; a^{*}_{m}\rangle
\end{bmatrix} \tag{72}
$$

Using Equation (72) in Equation (70)

$$
(A\boxtimes I)(A^{*}\boxtimes I) = \begin{bmatrix}
{}^{1}\langle a\; a^{*}_{1}\rangle I & \langle a_{1}\; a^{*}_{2}\rangle I & \cdots & \langle a_{1}\; a^{*}_{m}\rangle I \\
\vdots & & & \\
\langle a_{m}\; a^{*}_{1}\rangle I & \cdot & \cdot & \cdot \;\; \langle a_{m}\; a^{*}_{m}\rangle I
\end{bmatrix} \tag{73}
$$

$$
= \begin{bmatrix}
I & & & \\
 & I & & 0 \\
 & & \ddots & \\
 & 0 & & I
\end{bmatrix} \tag{74}
$$

since by Equation (71)

$$
{}^{i}\langle a\; a^{*}_{j}\rangle = \delta ij = \begin{bmatrix}
1 & i=j \\
 & \\
0 & i\neq j
\end{bmatrix} \tag{75}
$$

Consider next the relation

$$
(A\boxtimes B)^{T} = A^{T}\boxtimes B^{T} \tag{76}
$$

which should be contrasted with the standard matrix product

$$(AB)^T = B^T A^T \tag{77}$$

where commuting takes place as in Equation (56).

By Equation (61) and Equation (62)

$$A \otimes B = (A \otimes I)(I \otimes B) \tag{78}$$

$$= \begin{bmatrix} a_{11}I & a_{12}I & \cdots & a_{1m}I \\ a_{21}I & & & \\ \vdots & & & \\ a_{m1}I & & & a_{mm}I \end{bmatrix} \begin{bmatrix} B & & 0 \\ & \ddots & \\ 0 & & B \end{bmatrix}$$

Using Equation (77) concept in Equation (78)

$$(A \otimes B)^T = \begin{bmatrix} B^T & & & 0 \\ 0 & B^T & & \\ \cdot & & \ddots & \\ \cdot & 0 & & \\ \cdot & & & B^T \end{bmatrix} \begin{bmatrix} a_{11}I & a_{21}I & \cdots & a_{m1}I \\ a_{12}I & & & \\ \vdots & & & \\ a_{1m}I & & & a_{mm}I \end{bmatrix} \tag{79}$$

$$= \begin{bmatrix} a_{11}B^T & a_{21}B^T & \cdots & a_{m1}B^T \\ \vdots & & & \\ a_{1m}B^T & \cdot & \cdot & \cdot & a_{mm}B^T \end{bmatrix} \tag{80}$$

$$= \begin{bmatrix} a_{11}I & a_{21}I & \cdots & a_{m1}I \\ a_{12}I & & & \\ \vdots & & & \\ a_{1m}I & \cdot & \cdot & a_{mm}I \end{bmatrix} \begin{bmatrix} B^T & & & \\ & B^T & & 0 \\ & & \ddots & \\ & 0 & & B^T \end{bmatrix} \quad (81)$$

$$(A \otimes B)^T = (A^T \otimes I_m)(I_n \otimes B^T) \quad (82)$$

and by Equation (54)

$$(A \otimes B)^T = A^T \otimes B^T \quad (83)$$

Now that the flavor of the partitioning is established it is easy to establish the following additional properties.

$$(A+B) \otimes (C+D) = (A \otimes C) + (A \otimes D) + (B \otimes C) + (B \otimes D) \quad (84)$$

and also

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C \quad (85)$$

also by Equation (87)

$$(A+B) \otimes C = (A \otimes C) + (B \otimes C) \quad (86)$$

and

$$B \otimes (C+D) = (B \otimes C) + (B \otimes D) \quad (87)$$

By Equation (78)

$$\underset{(m\times m)}{A} \otimes \underset{n\times n}{B} = (A \otimes I_n)(I_m \otimes B) \quad (88)$$

We will show that $(A \otimes I_n)$ and $(I_m \otimes B)$ commute.

$$\underset{m\times m}{A} \boxtimes \underset{n\times n}{B} = \begin{bmatrix} a_{11}I_n & \cdots & a_{1m}I_n \\ \vdots & & \\ a_{m1}I_n & \cdots & a_{mm}I_n \end{bmatrix} \begin{bmatrix} B_{n\times n} & & & \\ & B_{n\times n} & & \\ & & \ddots & \\ & & & B_{n\times n} \end{bmatrix} \tag{89}$$

$$= \begin{bmatrix} a_{11}I_nB_{n\times n} & \cdots & a_{1m}B_{n\times n} \\ \vdots & & \\ a_{m1}B_{n\times n} & \cdots & a_{mm}B_{n\times n} \end{bmatrix} \tag{90}$$

$$= \begin{bmatrix} B & & & \\ & B & & \\ & & \ddots & \\ & & & B \end{bmatrix} \begin{bmatrix} a_{11}I_n & \cdots & a_{1m}I_n \\ \vdots & & \\ a_{m1}I_n & \cdots & a_{mm}I_n \end{bmatrix} \tag{91}$$

or

$$A \boxtimes B = (I_m \boxtimes B)(A \boxtimes I_n) \tag{92}$$

and by Equation (88) and Equation (92)

$$A \boxtimes B = (A \boxtimes I_n)(I_m \boxtimes B) = (I_m \boxtimes B)(A \boxtimes I_n) \tag{93}$$

Summary of Relations

$$\underset{m\times n}{A} \boxtimes \underset{s\times t}{B} = \underset{ms\times nt}{(a_{ij}B)}$$

$$\underset{m\times n}{X} = \left[x\langle m\rangle_1, \cdots x\langle m\rangle_n \right]$$

$$\underset{n\times m}{X^T} = \left[x\langle n\rangle_1, \cdots x\langle n\rangle_m \right]$$

$$\underset{mn\times 1}{\text{vec } X} = \begin{bmatrix} x\langle m\rangle^1 \\ x\langle m\rangle^2 \\ \vdots \\ x\langle m\rangle^n \end{bmatrix} = x\langle mn\rangle$$

$$\text{vec} \underset{m\times n}{X} = \left(\underset{n\times n}{I} \boxtimes \underset{m\times n}{X} \right) \text{vec} \underset{n\times n}{I}$$

$$\text{vec} \underset{\ell\times m}{A} \underset{m\times n}{X} = \left(\underset{n\times n}{I} \boxtimes \underset{\ell\times m}{A} \right) \text{vec } X$$

$$\text{vec } AX = (X^T \boxtimes A) \text{ vec } I_m$$

$$\text{vec} \underset{m\times n}{X} \underset{n\times s}{B} = \left(\underset{s\times n}{B^T} \boxtimes I_m \right) \text{vec } X$$

$$\text{vec} \underset{k\times m}{A} \underset{m\times n}{X} \underset{n\times s}{B} = \left(\underset{s\times n}{B^T} \boxtimes \underset{k\times m}{A} \right) \text{vec} \underset{m\times n}{X}$$

$$\underset{k\times m}{A} \underset{m\times r}{C} \boxtimes \underset{s\times n}{B} \underset{n\times t}{D} = \left(\underset{k\times m}{A} \boxtimes \underset{s\times n}{B} \right) \left(\underset{m\times n}{C} \boxtimes \underset{n\times t}{D} \right)$$

$$(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$$

$$(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}$$

$$(A \boxtimes B)^{T} = A^{T} \boxtimes B^{T}$$

$$(A+B) \boxtimes (C+D) = A \boxtimes C + A \boxtimes D + B \boxtimes C + B \boxtimes D$$

$$A \boxtimes (B \boxtimes C) = (A \boxtimes B) \boxtimes C$$

$$(A+B) \boxtimes C = (A \boxtimes C) + (B \boxtimes C)$$

$$(C^{T} \boxtimes I)^{*} = C^{T*} \boxtimes I$$

For $C^{*} = (C^{T}C)^{-1} C^{T}$ case.

Appendix (H) Dirac Delta Function

The Dirac Delta Function is used to obtain the variance Riccatti Equations for the continuous case; hence some considerations are given to this function in this appendix.

The mean value theorem and the fundamental theorem of the intergral Calculus are stated without proof from ref (78).

<u>Mean Value Theorem</u>. If the function $f(t)$
(i) is continuous in the closed interval $a \leq t \leq b$
(ii) has a derivative at every interior point;
then at some interior point ξ of the interval

$$\frac{f(b) - f(a)}{b - a} = f'(\xi) \tag{1}$$

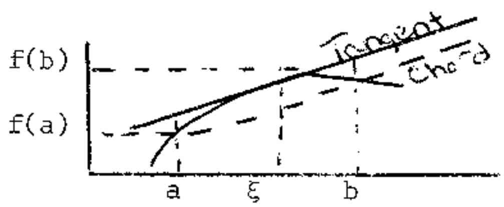

Fig (1) Graph of mean-value Theorem.

The simple geometric interpretation of the mean value theorem is represented by the curve of Fig (1). The straight line chord from the point to the tangent to the curve at same point $(\xi, f(\xi))$.

If

$$b=a+h \tag{2}$$

then one can write

$$\xi=a+\theta h \tag{3}$$

where

$$0<\theta<1$$

the mean value theorem now takes the form

$$f(a+h) - f(a) = h\ f'(a+\theta h) \qquad 0<\theta<1 \tag{4}$$

<u>Fundamental Theorem of The Integral Calculus.</u>
If $f(t)$ is integrable in (a,b) and $F(t)$ is any function having $f(t)$ as derivative, then

$$\int_a^b f(t)dt = F(b) - F(a) = F(t)\Big|_a^b \tag{5}$$

Dirac Function

Consider the Dirac Function (t) defined to be zero if t=0, in such a way that

$$\int_{-\infty}^{\infty}(t)dt = 1 \tag{6}$$

Equation (6) is a concise manner of expressing a function that is very large in a very small region and zero outside this region and has a unit integral.

Consider a function $\delta(t-a)$ that has the following values

$$\delta(t-a) = \begin{cases} 0 & t \leq a \\ \frac{1}{h} & a<t<a+h \\ 0 & a+h<t \end{cases} \tag{7}$$

where h may be made as small as we please. The function $\delta(t-a)$ has the property of Eq (6) for

$$\int_{-\infty}^{\infty}\delta(t-a)dt = \int_{a}^{a+h}\delta(t-a)dt \tag{8}$$

$$= \int_{a}^{a+h}\frac{1}{h}dt = \frac{1}{h}t\Big|_{a}^{a+h} = \frac{h}{h} = 1 \tag{9}$$

Consider next the integral of the function

$$\int_{-\infty}^{\infty}f(t)\ \delta\ (t-a)\ dt$$

$$= \int_{a}^{a+h} f(t)\ \delta\ (t-a)\ dt \tag{10}$$

$$= \frac{1}{h}\int_{a}^{a+h} f(t)dt = \frac{1}{h}\ F(t)\Big|_{a}^{a+h} \tag{11}$$

where

$$F'(t) = \frac{d}{dt}\ F(t) = f(t) \tag{12}$$

Evaluating Eq (11) at the limits

$$\int_{a}^{a+h} f(t)\ \delta(t-a)dt = \frac{1}{h}[F(a+h) - F(a)] \tag{13}$$

By the mean-value theorem of Eq (1)

$$F(a+h) - F(a) = h\,F'(a+\Theta h) \quad 0<\Theta<1 \tag{14}$$

By Eq (12) in Eq (14)

$$\int_a^{a+h} f(t)\,\delta(t-a)dt = \frac{1}{h}\,f(a+\Theta h) \tag{15}$$

in the limit for f(t) continuous

$$\lim_{h\to 0} f(a+\Theta h) = f(a) \tag{16}$$

hence Eq (16) in Eq (10)

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-a)dt = f(a). \tag{17}$$

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